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In memoriam Professor Charles E. Chidume (1947- 2021)

Some applications of idempotent elements in MV algebras

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ABSTRACT. In this paper we provide some properties and applications of MV-algebras. We prove that a Fibonacci stationary sequence in an MV-algebra gives us an idempotent element. Moreover, taking into account of the representation of a finite MV-algebra, by using Boolean elements of this algebra, we prove that a Fibonacci sequence in an MV-algebra is always stationary. This result is interesting comparing with the behavior of such a sequence on the group $(\mathbb{Z}_n, +)$, where the Fibonacci sequences are periodic, with the period given by the Pisano period. We also give some examples of finite MV-algebras and the number of their idempotent elements. As an application in Coding Theory, to a Boolean algebra it is attached a binary block code and it is proved that, under some conditions, the converse is also true.

1. INTRODUCTION

C. C. Chang, in the paper [2], introduced MV-algebras as a generalization of Boolean algebras. In the last decades, numerous papers have been devoted to the study of the properties and the applications of MV-algebras, Wajsberg algebras and Boolean algebras. Some applications of these algebras were provided in [5], where was presented an algorithm which can built Wajsberg algebras starting from a binary block codes.

In this paper, some new properties and applications of MV-algebras are given.

In [9] was defined a Fibonacci sequence on an MV-algebra and some examples and properties of this notion were given. Starting from these ideas, in the present paper, we improved some of the above mentioned results and we give other new ones. We proved that a Fibonacci stationary sequence on MV-algebras gives us an idempotent element. Moreover, taking into account of the representation of a finite MV-algebra, by using Boolean elements of this algebra, we prove that such a sequence is always stationary. This result is interesting, since in the group $(\mathbb{Z}_n, +)$, the group of integers modulo n, the Fibonacci sequences are periodic, with period given by the Pisano period. From here, we can remark that the representation of a finite MV-algebra can gives us a method for finding new interesting results.

In Section 3, some examples of finite MV-algebras and the number of their idempotent elements are given. In Section 4, starting from a Boolean algebra of order two, it was provided an algorithm to build a Boolean algebra of order 2^{k+1}. Moreover, as an application in Coding Theory, to a Boolean algebra it is attached a binary block code. Under some conditions, the converse of this statement is also true.

Definition 1.1. The following ordered set $(\mathcal{L}, \leq)$ is called *lattice* if for all elements $x, y \in \mathcal{L}$ there are their supremum and infimum elements, $sup\{x, y\}$ and $inf\{x, y\}$, denoted by

$$sup\{x, y\} = x \vee y \text{ and } inf\{x, y\} = x \wedge y.$$

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The lattice $(\mathcal{L}, \leq)$ is called a *distributive lattice* if for each elements x, y, z, we have the following relations:

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z),$$

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

A lattice $(\mathcal{L}, \leq)$ is called a *bounded lattice* if there are an element 0 being the least element in $\mathcal{L}$ and an element 1 being the greatest element in $\mathcal{L}$.

In a lattice $(\mathcal{L}, \leq)$ an element $x \in \mathcal{L}$ has a *complement* if there is an element $y \in \mathcal{L}$ satisfying the following relations:

$$x \vee y = 1 \text{ and } x \wedge y = 0.$$

An element having a complement is called *complemented*. We remark that a complement of an element is not unique. If $(\mathcal{L}, \leq)$ is distributive, then each element has at most a complement.

A lattice $(\mathcal{L}, \leq)$ is called a *complemented lattice* if it is a bounded lattice and each element $x \in \mathcal{L}$ has a complement.

Definition 1.2. ([2]) We consider an abelian monoid $(X, \oplus, 0)$ equipped with an unary operation "$\lceil$", such that the following conditions are satisfied:

i) $x \oplus \lceil 0 = \lceil 0$;

ii) $\lceil(\lceil x) = x$;

iii) $\lceil(\lceil x \oplus y) \oplus y = \lceil(\lceil y \oplus x) \oplus x$, for all elements $x, y \in X$. This abelian monoid is called an *MV-algebra* and we denote it by $(X, \oplus, \lceil, 0)$.

Remark 1.1. 1) ([11]) With the above notations, in an MV-algebra we denote the constant element $\lceil 0$ with 1, therefore

$$1 = \lceil 0.$$

Considering the following multiplications

$$x \odot y = \lceil(\lceil x \oplus \lceil y)$$

and

$$x \ominus y = x \odot \lceil y = \lceil(\lceil x \oplus y),$$

we have that

$$x \oplus y = \lceil(\lceil x \odot \lceil y).$$

2) ([3], Lemma 1.1.3) For each $x \in X$, the relations $x \oplus \lceil x = 1$ and $x \odot \lceil x = 0$ are satisfied.

Proposition 1.1. ([11]) *For the MV-algebra $(X, \oplus, \lceil, 0)$ and $x, y \in X$, the following conditions are equivalent:*

a) $x \odot \lceil y = 0$;

b) $\lceil x \oplus y = \lceil 0 = 1$;

c) $y = x \oplus (y \ominus x) = x \oplus \lceil(\lceil y \oplus x)$;

d) An element $w \in X$ such that $x \oplus w = y$ can be found.

Definition 1.3. ([11]) We consider MV-algebra $(X, \oplus, \lceil, 0)$. For $x, y \in X$, the following order relation are defined on X:

$$x \leq y \text{ if } \lceil x \oplus y = \lceil 0 = 1.$$

Remark 1.2. i) From the above, we have that the definition of the order relation on the MV-algebra $(X, \oplus, \lceil, 0)$ can be done by using one of the equivalent conditions a)-d) from the above proposition.

ii) ([3], Proposition 1.1.5) The order relation defined above gives us a lattice structure on an MV-algebra:

$a)$ $x \vee y = (x \odot \lceil y) \oplus y = (x \ominus y) \oplus y = \lceil(\lceil x \oplus y) \oplus y;$

$b)$ $x \wedge y = \lceil(\lceil x \vee \lceil y) = x \odot (\lceil x \oplus y)$. We will denote this lattice with $\mathcal{L}(X)$.

Definition 1.4. ([2]) We consider an algebra $(W, *, ^-, 1)$ equipped with a binary operation " $*$ " and a unary operation "$^-$" satisfying the following conditions, for every $x, y, z \in W$:

i) $(x * y) * [(y * z) * (x * z)] = 1;$

ii) $(x * y) * y = (y * x) * x;$

iii) $(\overline{x} * \overline{y}) * (y * x) = 1;$

iv) $1 * x = x.$

This algebra is called a *Wajsberg algebra.*

Remark 1.3. ([3], Lemma 4.2.2 and Theorem 4.2.5)

a) For a Wajsberg algebra $(W, *, ^-, 1)$, if we define the following multiplications

$$x \odot y = \overline{(x * \overline{y})}$$

and

$$x \oplus y = \overline{x} * y,$$

for all $x, y \in W$, the obtained algebra $(W, \oplus, \odot, ^-, 0, 1)$ is an MV-algebra with $0 = \overline{1}$.

b) If on the MV-algebra $(X, \oplus, \odot, \lceil, 0, 1)$ we define the operation

$$x * y = \lceil x \oplus y,$$

it results that $(X, *, \lceil, 1)$ is a Wajsberg algebra.

Definition 1.5. For the finite MV-algebras $(X, *, ^-, 0_1)$ and $(Y, \cdot, ', 0_2)$, we define on their Cartesian product $Z = X \times Y$ the following multiplication "Δ",

$$(x_1, y_1) \Delta (x_2, y_2) = (x_1 * x_2, y_1 \cdot y_2), \tag{1.1.}$$

The complement of the element (x_1, y_1) is $\rfloor (x_1, y_1) = (\overline{x}_1, x_2')$ and $0 = (0_1, 0_2)$. Therefore, by straightforward calculation, we obtain that $(Z, \Delta, \rfloor, 0)$ is also an MV-algebra.

Definition 1.6. The algebra $(\mathcal{B}, \vee\wedge, \rceil, 0, 1)$, equipped with two binary operations $\vee$ and $\wedge$ and a unary operation $\rceil$, is called a *Boolean algebra* if $(\mathcal{B}, \vee\wedge)$ is a distributive and a complemented lattice with

$$b \vee \rceil b = 1,$$
$$b \wedge \rceil b = 0,$$

for all elements $b \in \mathcal{B}$. The elements 0 and 1 are the least and the greatest elements from the algebra $\mathcal{B}$.

Remark 1.4. Boolean algebras represent a particular case of MV-algebras. Indeed, if $(\mathcal{B}, \vee\wedge, \rceil, 0, 1)$ is a Boolean algebra, then can be easily checked that $(\mathcal{B}, \vee, \rceil, 0)$ is an MV-algebra.

Remark 1.5. ([3], p.25)

1) With the above notations, for each MV-algebra $(X, \oplus, \lceil, 0)$, we have that $\mathcal{L}(X)$ is a distributive lattice. For the algebra X we will denote $\mathcal{B}(X)$ or $\mathcal{B}(\mathcal{L}(X))$ the set of all complemented elements in X. The elements from $\mathcal{B}(X)$ are called *Boolean* or *idempotent elements*.

2) Let $(X, \oplus, \lceil, 0)$ be an MV-algebra. Therefore $x \in \mathcal{B}(X)$ if $x \oplus y = x \vee y$, for all $y \in X$.

Definition 1.7. Let $(\mathcal{L}, \vee \wedge)$ be a lattice with 0 and 1, the least and the greatest elements from $\mathcal{L}$. A nonempty subset $\mathcal{I} \subseteq \mathcal{L}$ is called *an ideal* of the lattice $\mathcal{L}$ if the following conditions are satisfied:

a) $0 \in \mathcal{I}$;

b) If $x \in \mathcal{I}$ and $y \leq x$, then $y \in \mathcal{I}$;

c) If $x, y \in \mathcal{I}$, therefore $x \vee y \in \mathcal{I}$.

If $x \in \mathcal{L}$,the set

$$\{z \in \mathcal{L} \,/\, z \leq x\}$$

is called *the principal ideal generated by* x and will be denoted by $(-\infty, x]$.

Remark 1.6. i) ([3], Theorem 6.4.1) We consider $(X, \oplus, \lceil, 0)$ an MV-algebra. For an element $\beta \in \mathcal{B}(X)$, we have that $\left((-\infty, \beta], \oplus, \lceil^{\beta}, 0\right)$ is an MV-algebra, where $\lceil^{\beta} x = \beta \wedge \lceil x$.

ii) ([3], Lemma 6.4.5) For the MV-algebra $(X, \oplus, \lceil, 0)$, we consider the elements $x_1, x_2, ..., x_k \in \mathcal{B}(X) - \{0, e\}, k \geq 2$, such that

a) $x_1 \vee x_2 \vee ... \vee x_k = 1$;

b) For $i \neq j$, we have $x_i \wedge x_j = 0, i, j \in \{1, 2, ..., k\}$.

Therefore, we have that

$$X \simeq (-\infty, x_1] \times (-\infty, x_2] \times ... \times (-\infty, x_k].$$

iii) If $x_1, x_2, ..., x_k \in \mathcal{B}(X) - \{0, e\}$, the above decomposition is proper.

2. Connections between Fibonacci sequences and indempotent elements in an MV-algebra

In the last decades, a lot of papers have been devoted to the study of the properties and applications of Fibonacci sequences in various algebraic structure, as for example [7], [9], [10], [12], etc. In the following, we will prove that a Fibonacci sequence defined on a finite MV-algebra is stationary and is not periodic. This result is interesting comparing with the behavior of such a sequence on the group $(\mathbb{Z}_n, +)$, the group of integers modulo n, where the Fibonacci sequences are periodic, with period given by the Pisano period.

Let $X = \{x_0 \leq x_1 \leq ... \leq x_n\}$ be a finite totally ordered set, with x_0 the minimum element and x_n the maximum element. The following multiplication " $*$ " is defined on X:

$$\left\{ \begin{array}{c} x_i * x_j = 1, \text{ if } x_i \leq x_j; \\ x_i * x_j = x_{n-i+j}, \text{ otherwise}; \\ x_0 = 0, x_n = 1, x \circ 0 = \rfloor x. \end{array} \right. \tag{2.1.}$$

It results that $(X, *, \rfloor, 1)$ is a Wajsberg algebra. As was remarked in [6], Theorem 19, relation (2.1) gives us the only modality in which a Wajsberg algebra structure can be defined on a finite totally ordered set, such that, on this algebra, the induced order relation is given by (2.1). Moreover, the relation $\rfloor x_i = x_{n-i}$ is fulfilled.

Remark 2.7. 1) Theorem 5.2, p. 43, from [8] tells us that an MV-algebra is finite if and only if it is isomorphic to a finite product of finite totally ordered MV-algebras. Using connections between MV-algebras and Wajsberg algebras, it results that if $M = (X, \oplus, \odot, \lceil, 0, 1)$ is a totally ordered MV-algebra, then the obtained Wajsberg algebra, $W = (X, *, \lceil, 1)$, is also totally ordered. The converse of this statement is also true, since $x * y = \lceil x \oplus y$ implies that $x \leq_M y$ if and only $x \leq_W y$.

If the number of elements in a finite MV-algebra or in a finite Wajsberg algebra is a prime number, therefore these algebras are totally ordered algebras. Remark 1.6, ii) gives us a similar result and, additionally, we obtain that the sets $(-\infty, x_i], i \in \{1, 2, ..., k\}$, from that decomposition are totally ordered.

Example 2.1. ([13], Example 3.3) We consider the following MV-algebra $(X, \oplus, \lceil, 0)$, with the multiplication $\oplus$ and the operation "$\lceil$" given in the below tables:

$\oplus$	0	α	β	γ	δ	ε
0	0	α	β	γ	δ	ε
α	α	γ	δ	γ	ε	ε
β	β	δ	β	ε	δ	ε
γ	γ	γ	ε	γ	ε	ε
δ	δ	ε	δ	ε	ε	ε
ε	ε	ε	ε	ε	ε	ε

$\lceil$	θ	α	β	γ	δ	ε
	ε	δ	γ	β	α	0

.

We remark that $\beta \oplus \beta = \beta$ and $\gamma \oplus \gamma = \gamma$. We have
$\beta \vee \gamma = \lceil(\gamma \oplus \gamma) \oplus \gamma = \beta \oplus \gamma = \varepsilon$ and
$\beta \wedge \gamma = \lceil(\lceil\beta \vee \lceil\gamma) = \lceil(\gamma \vee \beta) = \lceil\varepsilon = 0$. Now, we compute $(-\infty, \beta]$ and $(-\infty, \gamma]$. By using Remark 1.6, ii), it results: $(-\infty, \beta] = \{0, \beta\}, (-\infty, \gamma] = \{0, \alpha, \gamma\}$. Therefore, $X \simeq (-\infty, \beta] \times (-\infty, \gamma]$.

Definition 2.8. We consider $(X, \oplus, \lceil, 0)$ an MV-algebra. For $x, y \in X$, the following sequence

$$< x, y >= \{x, y, x \oplus y, y \oplus (x \oplus y), ..., u_n, u_{n+1}, u_{n+2}, ...\}$$

are defined, where $u_0 = x, u_1 = y$ and $u_{n+2} = u_n \oplus u_{n+1}$, for $n \in \mathbb{N}$. This sequence is called the *Fibonacci sequence attached to the elements* x, y, (see [9], Definition 3.1). If there is a number $k \in \mathbb{N}$ such that $u_n = a$, for all $n \geq k$, $a \in X$,then the sequence $< x, y >$ is called *k-stationary*.

Proposition 2.2. *For the MV-algebra $(X, \oplus, \lceil, 0)$ we consider $x, y \in X$. If the sequence $< x, y >$ is k-stationary then u_k is idempotent.*

Proof. Let $k \in \mathbb{N}$ be a natural number such that the sequence $< x, y >$ is k-stationary. Therefore, $u_k = u_{k+1} = u_{k+2} = ...$ We have $u_2 = x \oplus y, u_3 = y \oplus (x \oplus y) = x \oplus 2y, 2y = y \oplus y, u_4 = 2x \oplus 3y$, etc. It results that

$$u_n = f_{n-1}x \oplus f_n y,$$

where $(f_n)_{n \in \mathbb{N}}$ is the Fibonacci sequence

$$f_0 = 0, f_1 = 1, f_{n+1} = f_n + f_{n-1}, n \in \mathbb{N}, n \geq 1.$$

Since $u = u_k = u_{k+1} = u_{k+2} = ...$, we have
$u = f_{k-1}x \oplus f_k y = f_k x \oplus f_{k+1} y = f_{k+1} x \oplus f_{k+2} y$.
Therefore, $u = u_{k+2} = f_{k+1}x \oplus f_{k+2}y = (f_{k-1}x \oplus f_k y) \oplus (f_k x \oplus f_{k+1} y) = u \oplus u$. It results that $u = u_k$ is an idempotent (Boolean) element. □

Proposition 2.3. *Let $(X, \oplus, \lceil, 0)$ be an MV-algebra and $x, y \in X$. If the sequence $< x, y >$ is 2-stationary for all $x, y \in X$, therefore X is a Boolean algebra.*

Proof. We have that $x \oplus y$ is idempotent, therefore $x \oplus y = (x \oplus y) \oplus (x \oplus y)$, for all $x, y \in X$. If we take $y = 0$, therefore $x = x \oplus x$, for all $x \in X$. It results that X is a Boolean algebra. □

The above proposition is a kind of generalization of the Proposition 4.18 from [9] with a proof included.

Proposition 2.4. *Let $(X, \oplus, \lceil, 0)$ be a finite MV-algebra and $x, y \in X$. Therefore the sequence $< x, y >$ is stationary for all $x, y \in X$.*

Proof. Since a finite MV-algebra is isomorphic to a finite product of finite totally ordered MV-algebras, it is enough to prove this result in the case of the finite totally ordered algebras. Let $X = \{x_0, x_1, ..., x_n\}$ be a finite totally ordered MV-algebra. We denote $x_0 = 0$

and $x_n = 1$. From relation (2.1) and since $\lceil x_i = x_{n-i}$, we have that the multiplication " $\oplus$ " is given by the following formulae:

$$\begin{cases} x_i \oplus x_j = 1, \text{if } i+j > n; \\ x_i \oplus x_j = x_{i+j}, \text{if } i+j \leq n; \\ x_0 = 0, x_n = 1, x \oplus 0 = x. \end{cases}.$$

Case 1. Let $x, y \in X, x \neq 0, y \neq 0$, and the sequence

$$< x, y >= \{x, y, x \oplus y, y \oplus (x \oplus y), ..., u_n, u_{n+1}, u_{n+2}, ...\},$$

where $u_0 = x, u_1 = y$ and $u_{n+2} = u_n \oplus u_{n+1}$, for $k \in \mathbb{N}$. It is clear that $u_2 = x \oplus y > x$ and $u_2 = x \oplus y > y$. If $x \oplus y = x$ or $x \oplus y = y$, therefore $y = 0$ or $x = 0$, false. Assuming $u_2 = x \oplus y > x$, we have that
$u_3 = y \oplus (x \oplus y) = (x \oplus y) \oplus y > x \oplus y = u_2$,
$u_4 = (x \oplus y) \oplus (y \oplus (x \oplus y)) > y \oplus (x \oplus y) = u_3$,
$u_5 = (x \oplus y) \oplus ((x \oplus y) \oplus (y \oplus (x \oplus y))) > ((x \oplus y) \oplus (y \oplus (x \oplus y))) = u_4$, etc. It results that the obtained increased sequence is stationary, since the set X is finite. Therefore, there is $k \in \mathbb{N}$ such that $u_k = 1 = u_{k+1} = u_{k+2} = ...$ We get

$$< x, y >= \{x, y, x \oplus y, y \oplus (x \oplus y), ..., u_{k-1}, 1, 1, 1, 1, ...\}.$$

Case 2. Let $x, y \in X, y = 0$. We obtain the sequence $< x, y >$ with $u_0 = x, u_1 = 0, u_2 = x, u_3 = x, u_4 = x \oplus x > x$. We apply the Case 1, obtaining a stationary sequence, that means a number $k \in \mathbb{N}$ such that $u_k = 1 = u_{k+1} = u_{k+2} = ...$

Now, by using Definition 1.5, it is clear that in a finite MV-algebra the sequence $< x, y >$ is stationary for all $x, y \in X$. □

Remark 2.8. 1) The above result is not true for infinite MV-algebras, as can be easily seen by using the famous Chang's MV-algebra.

2) Let $(X, \oplus, \lceil, 0)$ be a finite MV-algebra and $x, y \in X$. From the above, it results that the algebra X is k-stationary and we have that $u_n = a$, for all $n \geq k$. We consider the following map

$$\lambda : X \times X \to X, \lambda(x, y) = u_k.$$

In MV-algebra $(X, \oplus, \lceil, 0)$, given by the Example 2.1, we have that $\lambda(\gamma, 0) = \gamma$, $\lambda(\gamma, \delta) = \varepsilon$, $\lambda(\alpha, \beta) = \varepsilon$. Indeed,
-the sequence $[\gamma, 0] = \gamma, 0, \gamma, \gamma, ...$ is 2-stationary;
-the sequence $[\gamma, \delta] = \gamma, \delta, \varepsilon, \varepsilon, ...$ is 2-stationary;
-the sequence $[\alpha, \beta] = \alpha, \beta, \delta, \delta, \varepsilon, \varepsilon, ...$ is 4-stationary.

Remark 2.9. We consider $(X, \oplus, \lceil, 0)$ a finite MV-algebra such that $X \simeq (-\infty, x_1] \times (-\infty, x_2] \times ... \times (-\infty, x_k]$, with the sets $(-\infty, x_i] = \{0, x_i\}$, for all $i \in \{1, 2, ..., k\}$. It results that X is a Boolean algebra. Indeed, using above results, we have that all elements in X have the form $(\alpha_1, \alpha_2, ..., \alpha_k)$, where $\alpha_i \in (-\infty, x_i]$. From here, we get the known result that a finite Boolean algebra has 2^k elements.

3. Examples

In [1], [4] was presented classification of MV-algebras by using different algorithms. In [5] was presented an application of these algebras in Coding Theory. In the following, by using examples from [4], Sections 4.1-4.3 and the above Remark 1.6, ii), which give an alternative method to characterize MV-algebras and, as a consequence, Boolean algebras, we will give some examples of MV-algebras and Boolean algebras.

Example 3.2. We consider $W = \{0 \leq \alpha \leq \beta \leq \varepsilon\}$, a totally ordered set on which we define two multiplications Δ_0^4 and $\oplus_0^4$, given in the below tables. Multiplication Δ_0^4, with $\overline{\alpha} = \beta$ and $\overline{\beta} = \alpha$, gives us a Wajsberg algebra structure on W. The associated MV-algebra is obtained with multiplication $\oplus_0^4$. Therefore, in the obtained MV-algebra the only idempotent elements are 0 and ε.

Δ_0^4	0	α	β	ε
0	ε	ε	ε	ε
α	β	ε	ε	ε
β	α	β	ε	ε
ε	0	α	β	ε

$\oplus_0^4$	0	α	β	ε
0	0	α	β	ε
α	α	β	ε	ε
β	β	ε	ε	ε
ε	ε	ε	ε	ε

.

We consider now partially ordered set $W = \{0, \alpha, \beta, \varepsilon\}$. On W we define two multiplications Δ_{11}^4 and $\oplus_{11}^4$, given in the below tables. With multiplication Δ_{11}^4, W becomes a Wajsberg algebra and, with multiplication $\oplus_{11}^4$, an MV-algebra structure is obtained.

(3.1.)

Δ_{11}^4	0	α	β	ε
0	ε	ε	ε	ε
α	β	ε	β	ε
β	α	α	ε	ε
ε	0	α	β	ε

$\oplus_{11}^4$	0	α	β	ε
0	0	α	β	ε
α	α	α	ε	ε
β	β	ε	β	ε
ε	ε	ε	ε	ε

.

In this algebra all elements are idempotent. We have that $\alpha \vee \beta = \varepsilon$ and $\alpha \wedge \beta = 0$. Therefore, as MV-algebra, $W \simeq (-\infty, \alpha] \times (-\infty, \beta]$, where $(-\infty, \alpha] = \{0, \alpha\}$ and $(-\infty, \beta] = \{0, \beta\}$. From here, we obtain that there exist only two non-isomorphic MV-algebras of order 4. Thus, we can remark that in an MV-algebra of order 4 we can have only 0 or 2 proper idempotents. The MV-algebra $\left(W, \oplus_{11}^4\right)$ is the only Boolean algebra of order 4. We denote this algebra with $\mathcal{B}_4$.

Example 3.3. We consider $W = \{0 \leq \alpha \leq \beta \leq \gamma \leq \delta \leq \varepsilon\}$, a totally ordered set on which we define two multiplications Δ_0^6 and $\oplus_0^6$, given in the below tables. With multiplication Δ_0^6, with $\overline{\alpha} = \delta$, $\overline{\beta} = \gamma$, $\overline{\gamma} = \beta$, $\overline{\delta} = \alpha$, W becomes a Wajsberg algebra and with multiplication $\oplus_0^6$an MV-algebra is obtained. We can see that, in this structure, the only idempotent elements are 0 and ε.

Δ_0^6	0	α	β	γ	δ	ε
0	ε	ε	ε	ε	ε	ε
α	δ	ε	ε	ε	ε	ε
β	γ	δ	ε	ε	ε	ε
γ	β	γ	δ	ε	ε	ε
δ	α	β	γ	δ	ε	ε
ε	0	α	β	γ	δ	ε

$\oplus_0^6$	0	α	β	γ	δ	ε
0	0	α	β	γ	δ	ε
α	α	β	γ	δ	ε	ε
β	β	γ	δ	ε	ε	ε
γ	γ	δ	ε	ε	ε	ε
δ	δ	ε	ε	ε	ε	ε
ε	ε	ε	ε	ε	ε	ε

.

We consider now the partially ordered set $W = \{0, \alpha, \beta, \gamma, \delta, \varepsilon\}$ on which we define two multiplications Δ_{11}^6 and $\oplus_{11}^6$, given in the below tables. With multiplication Δ_{11}^6, W becomes a Wajsberg algebra and with multiplication $\oplus_{11}^6$the associated MV-algebra is obtained.

Δ_{11}^{6}	0	α	β	γ	δ	ε
0	ε	ε	ε	ε	ε	ε
α	δ	ε	ε	δ	ε	ε
β	γ	δ	ε	γ	δ	ε
γ	β	β	β	ε	ε	ε
δ	α	β	β	δ	ε	ε
ε	0	α	β	γ	δ	ε

$\oplus_{11}^{6}$	0	α	β	γ	δ	ε
0	0	α	β	γ	δ	ε
α	α	β	β	δ	ε	ε
β	β	β	β	ε	ε	ε
γ	γ	δ	ε	γ	δ	ε
δ	δ	ε	ε	δ	ε	ε
ε	ε	ε	ε	ε	ε	ε

.

We remark that in this structure the idempotent elements are $\{0, \beta, \gamma, \varepsilon\}$ and we have $\beta \vee \gamma = \varepsilon$ and $\beta \wedge \gamma = 0$. Therefore, as MV-algebra, $W \simeq (-\infty, \beta] \times (-\infty, \gamma]$, where $(-\infty, \beta] = \{0, \alpha, \beta\}$, $(-\infty, \gamma] = \{0, \gamma\}$. From here, we get that there are only two non-isomorphic MV-algebras of order 6. Thus, we can remark that in an MV-algebra of order 6 we can have only 0 or 2 proper idempotents.

Example 3.4. We consider now a totally ordered set
$W = \{0 \leq \alpha \leq \beta \leq \gamma \leq \tau \leq \upsilon \leq \rho \leq \varepsilon\}$ on which we define two multiplications Δ_0^8 and $\oplus_0^8$, given in the below tables. With multiplication Δ_0^8, having the properties $\overline{\alpha} = \rho$, $\overline{\beta} = \upsilon$, $\overline{\gamma} = \tau$, W becomes a Wajsberg algebra and with multiplication $\oplus_0^8$the associated MV-algebra is obtained. We remark that in the MV-algebra structure, the only idempotent elements are 0 and ε.

Δ_{0}^{8}	0	α	β	γ	τ	υ	ρ	ε
0	ε	ε	ε	ε	ε	ε	ε	ε
α	ρ	ε	ε	ε	ε	ε	ε	ε
β	υ	ρ	ε	ε	ε	ε	ε	ε
γ	τ	υ	ρ	ε	ε	ε	ε	ε
τ	γ	τ	υ	ρ	ε	ε	ε	ε
υ	β	γ	τ	υ	ρ	ε	ε	ε
ρ	α	β	γ	τ	υ	ρ	ε	ε
ε	0	α	β	γ	τ	υ	ρ	ε

$\oplus_{0}^{8}$	0	α	β	γ	τ	υ	ρ	ε
0	0	α	β	γ	τ	υ	ρ	ε
α	α	β	γ	τ	υ	ρ	ε	ε
β	β	γ	τ	υ	ρ	ε	ε	ε
γ	γ	τ	υ	ρ	ε	ε	ε	ε
τ	τ	υ	ρ	ε	ε	ε	ε	ε
υ	υ	ρ	ε	ε	ε	ε	ε	ε
ρ	ρ	ε	ε	ε	ε	ε	ε	ε
ε	ε	ε	ε	ε	ε	ε	ε	ε

We consider now the partially ordered set $W = \{0, \alpha, \beta, \gamma, \tau, \upsilon, \rho, \varepsilon\}$. On W we define two multiplications Δ_{11}^8 and $\oplus_{11}^8$, given in the below tables. With multiplication Δ_{11}^8, W becomes a Wajsberg algebra and with multiplication $\oplus_{11}^8$the associated MV-algebra is obtained .

Δ_{11}^{8}	0	α	β	γ	τ	υ	ρ	ε
0	ε	ε	ε	ε	ε	ε	ε	ε
α	ρ	ε	ρ	ε	ρ	ε	ρ	ε
β	υ	υ	ε	ε	ε	ε	ε	ε
γ	τ	υ	ρ	ε	ρ	ε	ρ	ε
τ	γ	γ	υ	υ	ε	ε	ε	ε
υ	β	γ	τ	υ	τ	ε	ρ	ε
ρ	α	α	γ	γ	υ	υ	ε	ε
ε	0	α	β	γ	τ	υ	ρ	ε

$\oplus_{11}^{8}$	0	α	β	γ	τ	υ	ρ	ε
0	0	α	β	γ	τ	υ	ρ	ε
α	α	α	γ	γ	υ	υ	ε	ε
β	β	γ	τ	υ	τ	ε	ρ	ε
γ	γ	γ	υ	υ	ε	ε	ε	ε
τ	τ	υ	ρ	ε	ρ	ε	ρ	ε
υ	υ	υ	ε	ε	ε	ε	ε	ε
ρ	ρ	ε	ρ	ε	ρ	ε	ρ	ε
ε	ε	ε	ε	ε	ε	ε	ε	ε

In this structure the idempotent elements are $\{0, \alpha, \rho, \varepsilon\}$. For α and ρ we have that $\alpha \vee \rho = \varepsilon$ and $\alpha \wedge \rho = 0$. Therefore, as MV-algebra, $W \simeq (-\infty, \alpha] \times (-\infty, \rho]$, where $(-\infty, \alpha] = \{0, \alpha\}$, $(-\infty, \rho] = \{0, \beta, \tau, \rho\}$.

In the following, we consider the partially ordered set $W = \{0, \alpha, \beta, \gamma, \tau, \upsilon, \rho, \varepsilon\}$, where we define two multiplications Δ_{21}^{8} and $\oplus_{21}^{8}$, given in the below tables. With multiplication Δ_{21}^{8}, W becomes a Wajsberg algebra and with multiplication $\oplus_{21}^{8}$ the associated MV-algebra is obtained.

(3.2.)

Δ_{21}^{8}	0	α	β	γ	τ	υ	ρ	ε
0	ε	ε	ε	ε	ε	ε	ε	ε
α	ρ	ε	ρ	ε	ρ	ε	ρ	ε
β	υ	υ	ε	ε	υ	υ	ε	ε
γ	τ	υ	ρ	ε	τ	υ	ρ	ε
τ	γ	γ	γ	γ	ε	ε	ε	ε
υ	β	γ	β	γ	ρ	ε	ρ	ε
ρ	α	α	γ	γ	υ	υ	ε	ε
ε	0	α	β	γ	τ	υ	ρ	ε

$\oplus_{21}^{8}$	0	α	β	γ	τ	υ	ρ	ε
0	0	α	β	γ	τ	υ	ρ	ε
α	α	α	γ	γ	υ	υ	ε	ε
β	β	γ	β	γ	ρ	ε	ρ	ε
γ	γ	γ	γ	γ	ε	ε	ε	ε
τ	τ	υ	ρ	ε	τ	υ	ρ	ε
υ	υ	υ	ε	ε	υ	υ	ε	ε
ρ	ρ	ε	ρ	ε	ρ	ε	ρ	ε
ε	ε	ε	ε	ε	ε	ε	ε	ε

In this structure all elements are idempotent, therefore it is a Boolean algebra. For α and ρ we have that $\alpha \vee \beta \vee \tau = \varepsilon$, $\alpha \wedge \beta \wedge \tau = 0$ and $\upsilon \vee \rho \vee \gamma = \varepsilon$, $\upsilon \wedge \rho \wedge \gamma = 0$. Therefore, as MV-algebra, $W \simeq (-\infty, \alpha] \times (-\infty, \beta] \times (-\infty, \tau]$ or $W \simeq (-\infty, \upsilon] \times (-\infty, \rho] \times (-\infty, \gamma]$, where $(-\infty, \alpha] = \{0, \alpha\}$, $(-\infty, \beta] = \{0, \beta\}$, $(-\infty, \gamma] = \{0, \gamma\}$, $(-\infty, \tau] = \{0, \tau\}$, $(-\infty, \upsilon] = \{0, \upsilon\}$, $(-\infty, \rho] = \{0, \rho\}$.

From here, we get that there are only three non-isomorphic MV-algebras of order 8. Thus, it results that in an MV-algebra of order 8 we can have only 0, 2 or 6 proper idempotents.

4. Binary block codes associated to a Boolean algebra

In this section we will denote Boolean algebras of order $2^k, k \geq 1$, with $\mathcal{B}_{2^k}$.

In [5], to an MV-algebra and to a Wajsberg algebra were associated binary block codes and, in some circumstances, it was proved that the converse is also true. Using some of these ideas, to algebra $\mathcal{B}_{2^{k+1}}$ we will associate a binary block code and we will prove that the converse of this statement is also true, namely to such a binary block code a Boolean algebra $\mathcal{B}_{2^{k+1}}$ can be associated.

Definition 4.9. Two Boolean algebras $(\mathcal{B}, \vee \wedge, \rceil, 0, 1)$ and $(\mathcal{B}', \curlyvee, \curlywedge, \widetilde{\ }, \mathbf{0}, \mathbf{1})$ are said to be *isomorphic* if there is a bijective function $f : \mathcal{B} \to \mathcal{B}'$ satisfying the following conditions:

i) $f(x \vee y) = f(x) \curlyvee f(y)$,for all $x, y \in \mathcal{B}$;
ii) $f(x \wedge y) = f(x) \curlywedge f(y)$, for all $x, y \in \mathcal{B}$;
iii) $f(\rceil x) = \widetilde{f(x)}$, for all $x \in \mathcal{B}$;
iv) $f(0) = \mathbf{0}$;
v) $f(1) = \mathbf{1}$.

Let $\mathcal{B}_2$ be a Boolean algebra with multiplication given in the following table

(4.1.)

$\oplus_{11}^{2}$	β	ε
β	β	ε
ε	ε	ε

.

and the map

$$\varphi_2 : \mathcal{B}_2 \times \mathcal{B}_2 \to \mathcal{B}_2, \varphi_2(x, y) = x \oplus_{11}^{2} y. \tag{4.2.}$$

In the following, we will use for the table

β	ε
ε	ε

the same notation $\mathcal{B}_2$.

We consider $\mathcal{C}_2$ a Boolean algebra of order 2 isomorphic to $\mathcal{B}_2$, which has the following multiplication table

(4.3.)
$$\begin{array}{|c|c|c|}\hline \oplus_{11}^{\prime 2} & 0 & \alpha \\ \hline 0 & 0 & \alpha \\ \hline \alpha & \alpha & \alpha \\ \hline \end{array}$$

and the map

(4.4.)
$$\theta_2 : \mathcal{C}_2 \times \mathcal{C}_2 \to \mathcal{C}_2, \theta_2 (x, y) = x \oplus_{11}^{\prime 2} y.$$

Similar as above, for the table $\begin{array}{|c|c|}\hline 0 & \alpha \\ \hline \alpha & \alpha \\ \hline \end{array}$, we use the same notation $\mathcal{C}_2$. Let $f_2 : \mathcal{C}_2 \to \mathcal{B}_2$ be an isomorphism of Boolean algebras.

Therefore, the multiplication table of $\mathcal{B}_4$, given in (3.1), can be written under the form

$$\mathcal{B}_4 = \begin{array}{|c|c|}\hline \mathcal{C}_2 & \mathcal{B}_2 \\ \hline \mathcal{B}_2 & \mathcal{B}_2 \\ \hline \end{array}.$$

We remark that $\mathcal{B}_4 = \mathcal{C}_2 \cup \mathcal{B}_2$ and $\mathcal{C}_2 \cap \mathcal{B}_2 = \emptyset$. To Boolean algebra $\mathcal{B}_4$ we will attach the map

(4.5.)
$$\varphi_4 : \mathcal{B}_4 \times \mathcal{B}_4 \to \mathcal{B}_4, \varphi_4 (x, y) = \begin{cases} \theta_2 (x, y), \text{ for } x, y \in \mathcal{C}_2 \\ \varphi_2 (f_2(x), y), \text{ for } x \in \mathcal{C}_2, y \in \mathcal{B}_2 \\ \varphi_2 (x, f_2(y)), \text{ for } x \in \mathcal{B}_2, y \in \mathcal{C}_2 \\ \varphi_2 (x, y), \text{ for } x \in \mathcal{B}_2, y \in \mathcal{B}_2 \end{cases}.$$

From the above, it is easy to see that $\varphi_4 (x, y) = x \oplus_{11}^{4} y$.

Continuing the above idea, with the above notations, we remark that the multiplication table of the Boolean algebra $\mathcal{B}_8$ can be written under the form

$$\mathcal{B}_8 = \begin{array}{|c|c|}\hline \mathcal{C}_4 & \mathcal{B}_4 \\ \hline \mathcal{B}_4 & \mathcal{B}_4 \\ \hline \end{array},$$

where $(\mathcal{C}_4, \oplus_{11}^{\prime 4})$ is a Boolean algebra isomorphic to $\mathcal{B}_4$. Let $f_4 : \mathcal{C}_4 \to \mathcal{B}_4$ be an isomorphism of Boolean algebras. To Boolean algebra $\mathcal{C}_4$ we will attach the map

$$\theta_4 : \mathcal{C}_4 \times \mathcal{C}_4 \to \mathcal{C}_4, \theta_4 (x, y) = x \oplus_{11}^{\prime 4} y.$$

We remark that $\mathcal{B}_8 = \mathcal{C}_4 \cup \mathcal{B}_4$ and $\mathcal{C}_4 \cap \mathcal{B}_4 = \emptyset$. To Boolean algebra $\mathcal{B}_4$ we will attach the following map

$$\varphi_8 : \mathcal{B}_8 \times \mathcal{B}_8 \to \mathcal{B}_8, \varphi_8 (x, y) = \begin{cases} \theta_4 (x, y), \text{ for } x, y \in \mathcal{C}_4 \\ \varphi_4 (f_4(x), y), \text{ for } x \in \mathcal{C}_4, y \in \mathcal{B}_4 \\ \varphi_4 (x, f_4(y)), \text{ for } x \in \mathcal{B}_4, y \in \mathcal{C}_4 \\ \varphi_4 (x, y), \text{ for } x, y \in \mathcal{B}_4 \end{cases}.$$

From here, it is easy to see that $\varphi_8 (x, y) = x \oplus_{21}^{8} y$.

Therefore, by using the above ideas, we obtain an algorithm with which the multiplication table of the Boolean algebra $\mathcal{B}_{2^k}$ can be written under the form

(4.6.)
$$\mathcal{B}_{2^k} = \begin{array}{|c|c|}\hline \mathcal{C}_{2^{k-1}} & \mathcal{B}_{2^{k-1}} \\ \hline \mathcal{B}_{2^{k-1}} & \mathcal{B}_{2^{k-1}} \\ \hline \end{array},$$

where $\mathcal{C}_{2^{k-1}}$ is a Boolean algebra isomorphic to $\mathcal{B}_{2^{k-1}}$.

We can remark that if we will interpret Boolean algebras as Boolean rings the above results can be also and easily obtained. But, to describe the way to attach a binary block code to a Boolean algebra and reciprocally we need another algorithm. This algorithm can be described as follows.

Algorithm 1

Assuming that we built Boolean algebra $(\mathcal{B}_{2^k}, \oplus_{21}^{2^k})$ with $\varphi_{2^k} : \mathcal{B}_{2^k} \times \mathcal{B}_{2^k} \to \mathcal{B}_{2^k}, \varphi_{2^k}(x, y) = x \oplus_{21}^{2^k} y$ and, considering $(\mathcal{C}_{2^k}, \oplus_{11}^{\prime 2^k})$ a Boolean algebra isomorphic to $\mathcal{B}_{2^k}$, let $f_{2^k} : \mathcal{C}_{2^k} \to \mathcal{B}_{2^k}$ be an isomorphism of these two Boolean algebras.

To Boolean algebra $\mathcal{C}_{2^k}$, we will attach the map

$$\theta_{2^k} : \mathcal{C}_{2^k} \times \mathcal{C}_{2^k} \to \mathcal{C}_{2^k}, \theta_{2^k}(x, y) = x \oplus_{11}^{\prime 2^k} y. \tag{4.7.}$$

We consider the set $\mathcal{B}_{2^{k+1}} = \mathcal{C}_{2^k} \cup \mathcal{B}_{2^k}$, with $\mathcal{C}_{2^k} \cap \mathcal{B}_{2^k} = \emptyset$ and to Boolean algebra $\mathcal{B}_{2^{k+1}}$ we will attach the map

$$\varphi_{2^{k+1}}:\mathcal{B}_{2^{k+1}} \times \mathcal{B}_{2^{k+1}} \to \mathcal{B}_{2^{k+1}}, \varphi_{2^{k+1}}(x, y) = \begin{cases} \theta_{2^k}(x, y), \text{ for } x, y \in \mathcal{C}_{2^k} \\ \varphi_{2^k}(f_{2^k}(x), y), \text{ for } x \in \mathcal{C}_{2^k}, y \in \mathcal{B}_{2^k} \\ \varphi_{2^k}(x, f_{2^k}(y)), \text{ for } x \in \mathcal{B}_{2^k}, y \in \mathcal{C}_{2^k} \\ \varphi_{2^k}(x, y), \text{ for } x \in \mathcal{B}_{2^k}, y \in \mathcal{B}_{2^k} \end{cases}. \tag{4.8.}$$

By defining the following multiplication

$$x \oplus_{21}^{2^{k+1}} y = \varphi_{2^{k+1}}(x, y),$$

it results that $(\mathcal{B}_{2^{k+1}}, \oplus_{21}^{2^{k+1}})$ is a Boolean algebra of order 2^{k+1}, as can easily be checked.

We consider now $(X, \oplus, \lceil, 0)$ a finite MV-algebra of order n, with $X = \{0 = \alpha_0, \alpha_1, \alpha_2, ..., \alpha_{n-2}, \varepsilon = \alpha_{n-1}\}$. Let $C_n = \{w_0, w_1, ..., w_{n-2}, w_\varepsilon\}$ be a binary block code, with codewords of length n. In [5], to an MV-algebra and to its associated Wajsberg algebra binary block codes were associated. Summarizing these methods, we give the following definition.

Definition 4.10. 1) The block code C is attached to MV-algebra X, if for a codeword $w_j \in X$, $w_j = i_0 i_1 ... i_{n-2} i_\varepsilon$, $i_0, i_1, ..., i_{n-2}, i_\varepsilon \in \{0, 1\}$,
$j \in \{0, 1, 2, ..., n-2, \varepsilon\}$, we have that $i_s = 1$ if $\alpha_j \oplus \alpha_s = \varepsilon$ and $i_s = 0$, otherwise,
$s \in \{0, 1, 2, ..., n-2, \varepsilon\}$.

2) A matrix attached to the code C, is a quadratic matrix $M_C = (m_{i,j})_{i,j \in \{1,2,...,n\}} \in \mathcal{M}_n(\{0, 1\})$ such that its rows are formed by the codewords of C.

Remark 4.10. Since a Boolean algebra is an MV-algebra, we have that:

- the code C_2 attached the the algebra $\mathcal{B}_2$ is $C_2 = \{01, 11\} = \{w_0, w_1\}$ and the attached matrix is

$$M_{C_2} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

- the code C_4 attached to the algebra $\mathcal{B}_4$ is $C_4 = \{0001, 0011, 0101, 1111\} = \{w_0, w_1, w_2, w_3\}$ and the attached matrix is

$$M_{C_4} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

- the code attached to the algebra $\mathcal{B}_8$ is $C_8 = \{00000001, 00000011, 00000111, 00001111, 00010001, 00110011, 01110111, 11111111\}$ and the attached matrix is

$$M_{C_8} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

If we denote with $\mathbf{0}_n$ the zero matrix with n elements, we remark that

$$M_{C_4} = \begin{pmatrix} \mathbf{0}_2 & M_{C_2} \\ M_{C_2} & M_{C_2} \end{pmatrix}, M_{C_8} = \begin{pmatrix} \mathbf{0}_4 & M_{C_4} \\ M_{C_4} & M_{C_4} \end{pmatrix}.$$

Therefore, we have

$$M_{C_{2^{k+1}}} = \begin{pmatrix} \mathbf{0}_{2^k} & M_{C_{2^k}} \\ M_{C_{2^k}} & M_{C_{2^k}} \end{pmatrix}, \tag{4.9.}$$

with $C_{2^{k+1}}$ the attached binary block code having as codewords the rows of the matrix $M_{C_{2^{k+1}}}$.

In the following, we provide a method to attach to a binary block code $C_{2^{k+1}}$ a Boolean algebra.

First of all, we consider the binary block code $C_2 = \{w_0 \preceq w_1\}$, with $\preceq$ the lexicographic order. We define the following multiplication:

$$w_0 *_2 w_1 = w_1 *_2 w_0 = w_1 *_2 w_1 = w_1,\ w_0 *_2 w_0 = w_0. \tag{4.10.}$$

It results that $(C_2, *)$ is a Boolean algebra, isomorphic to $\mathcal{B}_2$.

If we consider $C_4 = \{w_0 \preceq w_1 \preceq w_2 \preceq w_3\}$ with the lexicographic order $\preceq$, let $C_4' = \{w_0 \preceq w_1\}$ and $C_4'' = \{w_2 \preceq w_3\}$ be two disjoint subsets of C_4. We have that $(C_4', *_2)$ is a Boolean algebra of order 2. On C_4'' we define the multiplication

$$w_2 *_2' w_3 = w_3 *_2' w_2 = w_3 *_2' w_3 = w_3,\ w_2 *_2' w_2 = w_2.$$

It results that $(C_4'', *_2')$ is a Boolean algebra isomorphic to $(C_4', *_2)$. On C_4 we define the following multiplication

$$w_i *_4 w_j = \begin{cases} w_i *_2 w_j, \text{ for } w_i, w_j \in C_4', \\ w_i *_2' w_j, \text{ for } w_i \in C_4', w_j \in C_4'', \\ w_i *_4' w_j, \text{ for } w_i \in C_4'', w_j \in C_4', \\ w_i *_4' w_j, \text{ for } w_i, w_j \in C_4''. \end{cases}.$$

From here, we have that $(C_4, *_4)$ is a Boolean algebra isomorphic to $\mathcal{B}_4$.

Algorithm 2

Assuming that we have defined the Boolean algebra $(C_{2^k}, *_{2^k})$ isomorphic to $\mathcal{B}_{2^k}$, let $C_{2^{k+1}} = \{w_0 \preceq w_1 \preceq, ... \preceq w_{2^k-1} \preceq ... \preceq w_{2^{k+1}-1}\}$ be the binary block code defined by the matrix $M_{C_{2^{k+1}}}$, given by the relation (4.9), with the codewords lexicographically ordered. We consider the sets $C'_{2^{k+1}} = \{w_0 \preceq w_1 \preceq ... \preceq w_{2^k-1}\}$ and $C''_{2^{k+1}} = \{w_{2^k} \preceq ... \preceq w_{2^{k+1}-1}\}$. We have that $(C'_{2^{k+1}}, *_{2^k})$ is isomorphic to $\mathcal{B}_{2^k}$ and on $C''_{2^{k+1}}$ we define a multiplication $*'_{2^k}$ such that $(C''_{2^{k+1}}, *'_{2^k})$ is isomorphic to $\mathcal{B}_{2^k}$.

On $C_{2^{k+1}}$ we define the following multiplication

$$w_i *_{2^{k+1}} w_j = \begin{cases} w_i *_{2^{k+1}} w_j, \text{ for } w_i, w_j \in C'_{2^{k+1}}, \\ w_i *'_{2^{k+1}} w_j, \text{ for } w_i \in C'_{2^{k+1}}, w_j \in C''_{2^{k+1}}, \\ w_i *'_{2^{k+1}} w_j, \text{ for } w_i \in C''_{2^{k+1}}, w_j \in C'_{2^{k+1}}, \\ w_i *'_{2^{k+1}} w_j, \text{ for } w_i, w_j \in C''_{2^{k+1}}. \end{cases} \tag{4.11.}$$

It results that $(C_{2^{k+1}}, *_{2^{k+1}})$ is a Boolean algebra isomorphic to $\mathcal{B}_{2^{k+1}}$.

From the above results, we proved the following Theorem.

Theorem 4.1. *1) To each Boolean algebra of order 2^{k+1}, $\mathcal{B}_{2^{k+1}}$, we can associate a binary block code $C_{2^{k+1}}$, with associated matrix given by the relation* (4.9)*.*

2) On binary block code $C_{2^{k+1}}$ we can define a multiplication $*_{2^{k+1}}$such that $(C_{2^{k+1}}, *_{2^{k+1}})$ is a Boolean algebra isomorphic to $\mathcal{B}_{2^{k+1}}$.

5. Conclusions

The results obtained in this paper can be considered as a new starting point in the study of MV-algebras or other logical algebras (such as BCK-algebras, due to the connection between MV-algebras and bounded commutative BCK-algebras). The obtained connections can rise the question "who influence who?". We have a partial answer to this question, in the sense that the study of Fibonacci elements or the study of the binary block codes defined on these algebras provides us new and interesting properties of these algebras. Will be interesting to find the answer at the converse question: if the study of properties of MV-algebras, Boolean algebras, bounded commutative BCK-algebras can gives us new opportunity to find new results regarding Fibonacci sequences or binary block codes attached.

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Compliance with Ethical Standards

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1) There are no conflicts of interest;

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References

[1] Belohlavek, R.; Vychodil, V. Residuated Lattices of Size ≤ 12. *Order* **27** (2010), 147–161.

[2] Chang, C. C. Algebraic analysis of many-valued logic. *Trans. Amer. Math. Soc.* **88** (1958), 467–490.

[3] Cignoli, R. L. O.; Ottaviano, I. M. L. D.; Mundici, D. *Algebraic foundations of many-valued reasoning*. Trends in Logic, Studia Logica Library, Dordrecht, Kluwer Academic Publishers, **7** (2000).

[4] Flaut, C.; Hoskova-Mayerova, S.; Saeid, A. B.; Vasile, R. Wajsberg algebras of order n $(n \leq 9)$. accepted in *Neural Computing and Applications*, DOI: 10.1007/s00521-019-04676-x.

[5] Flaut, C.; Vasile, R. Wajsberg algebras arising from binary block codes. accepted in *Soft Computing*, DOI: 10.1007/s00500-019-04653-5.

[6] Font, J., M.; Rodriguez, A., J.; Torrens, A. Wajsberg Algebras. *Stochastic* **8** (1984), no. 1, 5-30.

[7] Han, J. S.; Kim, H. S.; Neggers, J. Fibonacci sequences in groupoids. *Advances in Difference Equations* 2012, 2012:19.

[8] Höhle, U.; Rodabaugh, S., E. *Mathematics of Fuzzy Sets: Logic, Topology and Measure Theory*. Springer Science and Business Media, LLC, 1999.

[9] Jahanshahi, M. A.; Saeid, A. B. Fibonacci sequences on MV-algebras. *J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math.* **25** (2018), no. 4.

[10] Kim, H. S.; Neggers, J.; So, K. S. Generalized Fibonacci sequences in groupoids. *Advances in Difference Equations* 2013, 2013:26.

[11] Mundici, D. *MV-algebras-a short tutorial*. Department of Mathematics *Ulisse Dini*, University of Florence, 2007.

[12] Renault, M. The Period, Rank, and Order of the (a, b)-Fibonacci Sequence Mod m. *Mathematics Magazine* **86** (2013), no. 5, 372-380.

[13] Wang, J. T.; Davvaz, B.; He, P. F. *On derivations of MV-algebras*. https://arxiv.org/pdf/1709.04814.pdf.

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