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In memoriam Professor Charles E. Chidume (1947- 2021)

Joint approximation of analytic functions by shifts of the Riemann zeta-function twisted by the Gram function

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ABSTRACT. In the paper, we consider the simultaneous approximation of a collection of analytic functions by a collection of shifts of the Riemann zeta-function $(\zeta(s+it_\tau^{\alpha_1}),\dots,\zeta(s+it_\tau^{\alpha_r}))$, where t_τ is the Gram function and $\alpha_1,\dots,\alpha_r$ are different positive numbers. It is obtained that the set of such shifts has a positive lower density.

1. INTRODUCTION

In this paper, we consider approximation property of the Riemann zeta-function $\zeta(s)$, $s=\sigma+it$. The function $\zeta(s)$ is defined, for $\sigma>1$, by

$$\zeta(s)=\sum_{m=1}^{\infty}\frac{1}{m^s}=\prod_{p\in\mathbb{P}}\left(1-\frac{1}{p^s}\right)^{-1},$$

where $\mathbb{P}$ is the set of all prime numbers. Moreover, $\zeta(s)$ has analytic continuation to the whole complex plane, except for a simple pole at the point $s=1$ with residue 1.

Voronin in [20] discovered an interesting approximation property of the function $\zeta(s)$, and called it the universality. More precisely, he proved that every analytic non-vanishing function $g(s)$ defined in the strip $D=\{s\in\mathbb{C}:1/2<\sigma<1\}$ can be approximated by shifts $\zeta(s+i\tau)$, $\tau\in\mathbb{R}$. The Voronin theorem was observed by the number theorists, and extended in various directions. We recall the last version of the Voronin theorem developed in [3] and [1], see also [13] and [17], and a informative survey paper [15].

Denote by $\mathcal{K}$ the class of compact subsets of the strip D with connected complements, and by $H_0(K)$ with $K\in\mathcal{K}$ the class of continuous non-vanishing functions on K that are analytic in the interior of K. Let $\mathrm{meas}A$ be the Lebesgue measure of a measurable set $A\subset\mathbb{R}$. Then the following statement is valid.

Theorem 1.1. *Suppose that $K\in\mathcal{K}$ and $f(s)\in H_0(K)$. Then, for every $\varepsilon>0$,*

$$\liminf_{T\to\infty}\frac{1}{T}\mathrm{meas}\left\{\tau\in[0,T]:\sup_{s\in K}|\zeta(s+i\tau)-f(s)|<\varepsilon\right\}>0.$$

The Voronin universality theorem is a infinite-dimensional generalization of the Bohr-Courant theorem on the denseness in $\mathbb{C}$ of the set $\{\zeta(\sigma+it):t\in\mathbb{R}\}$ for every fixed $1/2<\sigma\leqslant 1$.

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One of the way of extension of Theorem 1.1 is using of more general shifts than $\zeta(s+i\tau)$. Such a way was proposed in [16]. In this paper, we deal with shifts $\zeta(s+it_\tau)$, where t_τ is the Gram function.

It is well known that the function $\zeta(s)$ for all $s \in \mathbb{C}$ satisfies the functional equation

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s),$$

where $\Gamma(s)$ is the Euler gamma-function. The product $\pi^{-s/2}\Gamma(s/2)$ plays an important role in the theory of the Riemann zeta-function, and this once more was confirmed by Gram [4]. Denote by $\theta(t)$, $t > 0$, the increment of the argument of the function $\pi^{-s/2}\Gamma(s/2)$ along the segment connecting the points $s = 1/2$ and $s = 1/2 + it$. The function $\theta(t)$ is monotonically increasing and unbounded from above for $t > t^* = 6.2898\ldots$, therefore the equation

$$\theta(t) = (n-1)\pi, \quad n \in \mathbb{N},$$

for $t > t^*$, has the unique solution t_n. The latter equation was considered by Gram [4], therefore, the numbers t_n are now called the Gram points. Let γ_n denote the imaginary part of the n-th non-trivial zero of the Riemann zeta-function. Then the Riemann-von Mangoldt formula implies that $t_n \sim \gamma_n$ as $n \to \infty$. Gram calculated [4] the first points t_n and observed that each interval $[t_{n-1}, t_n]$ with $n = 1, \ldots, 15$ contains precisely one zero $\hat{\gamma}_n$ of the function $\zeta(1/2 + it)$. Moreover, he conjectured that this is not true for some large $n > 15$. It turned out that his conjecture is true and it was confirmed by wider calculations, and by a Titchmarsh result [18] that the sequence

$$\frac{\hat{\gamma}_n - t_n}{t_{n+1} - t_n}$$

is unbounded. The Gram points also were considered by Selberg for probabilistic aims. Interesting results connected to Gram points were obtained in [19]. Systematically Gram points were studied by the first author in the series of works [5]–[11]. A discrete universality theorem for the function $\zeta(s)$ on the approximation of analytic functions by shifts $\zeta(s + it_k)$ was obtained in [12]. Denote by $\#A$ the cardinality of the set A. Then the following theorem is true.

Theorem 1.2. *Let $K \in \mathcal{K}$, $f(s) \in H_0(K)$ and h is an arbitrary fixed positive number. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N} \# \left\{ 1 \leqslant k \leqslant N : \sup_{s \in K} |\zeta(s + iht_k) - f(s)| < \varepsilon \right\} > 0.$$

Moreover "$\liminf$*" can be replaced by "*$\lim$*" for all but at most countably many $\varepsilon > 0$.*

Paralelly to the numbers t_n, the Gram function t_u of continuous variable u, $u \geqslant 0$, which is a solution of the equation

$$\theta(t) = (u-1)\pi, \tag{1.1}$$

can be considered, see [11]. This paper is devoted to a joint universality theorem for the function $\zeta(s)$ with shifts involving powers of the Gram function t_τ.

Theorem 1.3. *Suppose that $\alpha_1, \ldots, \alpha_r$ are fixed different positive numbers. For $j = 1, \ldots, r$, let $K_j \in \mathcal{K}$ and $f_j(s) \in H_0(K_j)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0,T] : \sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} |\zeta(s + it_\tau^{\alpha_j}) - f_j(s)| < \varepsilon \right\} > 0.$$

Moreover "$\liminf$*" can be replaced by "*$\lim$*" for all but at most countably many $\varepsilon > 0$.*

The proof of Theorem 1.3 is based on a probabilistic limit theorem in the space of analytic functions.

2. Mean square estimates

It is well known, that, for fixed $1/2 < \sigma < 1$,

$$\int_0^T |\zeta(\sigma + it)|^2 \, \mathrm{d}t \ll_\sigma T. \tag{2.2}$$

For us, a more general result is needed. For this, the following lemma will be useful.

Lemma 2.1. *Suppose that* t_τ, $\tau \geqslant 0$, *is the unique solution of the equation* (1.1) *satisfying* $\theta(t_\tau) > 0$, *and that* $\tau \to \infty$. *Then*

$$t_\tau = \frac{2\pi\tau}{\log \tau}\left(1 + \frac{\log\log\tau}{\log\tau}(1 + o(1))\right)$$

and

$$t'_\tau = \frac{2\pi}{\log \tau}\left(1 + \frac{\log\log\tau}{\log\tau}(1 + o(1))\right).$$

Proof of the lemma is given in [11].

Lemma 2.2. *Suppose that* $1/2 < \sigma < 1$ *and* $\alpha > 0$ *are fixed. Then, for* $t \in \mathbb{R}$,

$$\int_0^T |\zeta(\sigma + it_\tau^\alpha + it)|^2 \, \mathrm{d}\tau \ll_\sigma T(1 + |t|).$$

Proof. Let $X \geqslant \log T$. We have

$$I_\sigma(X, t) \stackrel{def}{=} \int_X^{2X} |\zeta(\sigma + it_\tau^\alpha + it)|^2 \, \mathrm{d}\tau = \int_X^{2X} \frac{1}{(t_\tau^\alpha)'} |\zeta(\sigma + it_\tau^\alpha + it)|^2 \, \mathrm{d}t_\tau^\alpha.$$

Therefore, Lemma 2.1 gives

$$\begin{aligned} I_\sigma(X, t) &\ll \frac{(\log X)^\alpha}{X^{\alpha-1}} \int_X^{2X} \mathrm{d}\left(\int_X^{t_\tau^\alpha + t} |\zeta(\sigma + iu)|^2 \, \mathrm{d}u\right) \\ &\ll \frac{(\log X)^\alpha}{X^{\alpha-1}} \int_{-t_{2X}^\alpha - |t|}^{t_{2X}^\alpha + |t|} |\zeta(\sigma + iu)|^2 \, \mathrm{d}u. \end{aligned}$$

Hence, in view of (2.2),

$$I_\sigma(X, t) \ll_\sigma \frac{(\log X)^\alpha}{X^{\alpha-1}}\left(\frac{X^\alpha}{(\log X)^\alpha} + |t|\right) \ll_\sigma X(1 + |t|).$$

Thus,

$$\int_{\log T}^T |\zeta(\sigma + it_\tau^\alpha + it)|^2 \, \mathrm{d}\tau \ll_\sigma T(1 + |t|).$$

Since the function t_τ is increasing and $\zeta(\sigma + it) \ll_\sigma 1 + |t|^{1/6}$, this proves the lemma. □

Define the set

$$\Omega = \prod_{p \in \mathbb{P}} \gamma_p,$$

where $\gamma_p = \{s \in \mathbb{C} : |s| = 1\}$ for all $p \in \mathbb{P}$. With the product topology and pointwise multiplication, the torus Ω is a compact topological Abelian group. Therefore, denoting by $\mathcal{B}(\mathbb{X})$ the Borel σ-field of the space $\mathbb{X}$, we obtain that on $(\Omega, \mathcal{B}(\Omega))$, the probability Haar measure m_H can be defined. This gives the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by

$\omega(p)$ the pth component of the element $\omega \in \Omega$, $p \in \mathbb{P}$, and extend the function $\omega(p)$ to the $\mathbb{N}$ by formula

$$\omega(m) = \prod_{\substack{p^l | m \\ p^{l+1} \nmid m}} \omega^l(p), \quad m \in \mathbb{N}.$$

Let $H(D)$ stand for the space of analytic functions on D endowed with the topology of uniform convergence on compacta. Now, on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, define the $H(D)$-valued random element $\zeta(s, \omega)$ by

$$\zeta(s, \omega) = \sum_{m=1}^{\infty} \frac{\omega(m)}{m^s} = \prod_{p \in \mathbb{P}} \left(1 - \frac{\omega(p)}{p^s}\right)^{-1}.$$

Note that the series and product are uniformly convergent on compact subsets of the strip D, and the equality holds for almost all $\omega \in \Omega$, see, for example, [13].

To study the functions $\zeta(s)$ and $\zeta(s, \omega)$, define the auxiliary functions. Let $\theta > 1/2$ be fixed, and, for $m, n \in \mathbb{N}$,

$$v_n(m) = \exp\left\{-\left(\frac{m}{n}\right)^{\theta}\right\}.$$

Define

$$\zeta_n(s) = \sum_{m=1}^{\infty} \frac{v_n(m)}{m^s}$$

and

$$\zeta_n(s, \omega) = \sum_{m=1}^{\infty} \frac{v_n(m)\omega(m)}{m^s}.$$

The latter series are absolutely convergent in the half-plane $\sigma > 1/2$.

Recall the metric in the space $H(D)$. There exists a sequence $\{K_l : l \in \mathbb{N}\} \subset D$ of compact sets such that

$$D = \bigcup_{l=1}^{\infty} K_l,$$

$K_l \subset K_{l+1}$ for all $l \in \mathbb{N}$, and if $K \subset D$ is a compact set, then $K \subset K_l$ for some l. For $g_1, g_2 \in H(D)$, define

$$\rho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s \in K_l} |g_1(s) - g_2(s)|}{1 + \sup_{s \in K_l} |g_1(s) - g_2(s)|}.$$

Then ρ is a metric in $H(D)$ that includes the topology of uniform convergence on compacta.

Now we will consider the distance between $\zeta(s)$ and $\zeta_n(s)$.

Lemma 2.3. *Suppose that $\alpha > 0$ is fixed. Then*

$$\lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \rho\left(\zeta(s + iht_\tau^\alpha), \zeta_n(s + iht_\tau^\alpha)\right) \mathrm{d}\tau = 0.$$

Proof. By the definition of the metric ρ, it suffices to show that, for arbitrary compact set $K \subset D$,

$$\lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \sup_{s \in K} |\zeta(s + iht_\tau^\alpha) - \zeta_n(s + iht_\tau^\alpha)| \, \mathrm{d}\tau = 0. \tag{2.3}$$

Let θ be from the definition $v_n(m)$, $\Gamma(s)$ denote the Euler gamma-function, and

$$l_n(s) = \frac{s}{\theta} \Gamma\left(\frac{s}{\theta}\right) n^s.$$

Then the Mellin formula implies the representation, see, for example, [13],

$$\zeta_n(s) = \frac{1}{2\pi i}\int_{\theta-i\infty}^{\theta+i\infty} \zeta(s+z) l_n(z) \frac{\mathrm{d}z}{z}.$$

Hence, with $\hat{\theta} > 0$,

$$\zeta_n(s) - \zeta(s) = \frac{1}{2\pi i}\int_{-\hat{\theta}-i\infty}^{\hat{\theta}+i\infty} \zeta(s+z) l_n(z) \frac{\mathrm{d}z}{z} + \frac{l_n(1-s)}{1-s}. \tag{2.4}$$

Let K be an arbitrary subset of the strip D. Then there exists $\varepsilon > 0$ such that, for all $s \in K$, $s = \sigma + it$, we have $1/2 + 2\varepsilon < \sigma < 1 - \varepsilon$. Now choose

$$\hat{\theta} = \sigma - \frac{1}{2} - \varepsilon, \qquad \theta = \frac{1}{2} + \varepsilon.$$

Then, in view of (2.3), for $s \in K$,

$$\begin{aligned}\zeta_n(s + it_\tau^\alpha) - \zeta(s + it_\tau^\alpha) \ll \int_{-\infty}^{\infty} \left|\zeta\left(\frac{1}{2} + \varepsilon + it_\tau^\alpha + iu\right)\right| \sup_{s\in K}\left|\frac{l_n(1/2+\varepsilon-s+iu)}{1/2+\varepsilon-s+iu}\right| \mathrm{d}u \\ + \left|\frac{l_n(1-s-it_\tau^\alpha)}{1-s-it_\tau^\alpha}\right|.\end{aligned}$$

Therefore,

$$\frac{1}{T}\int_0^T \sup_{s\in K} |\zeta(s+it_\tau^\alpha) - \zeta_n(s+it_\tau^\alpha)| \,\mathrm{d}\tau \ll I_1 + I_2, \tag{2.5}$$

where

$$I_1 = \int_{-\infty}^{\infty} \left(\frac{1}{T}\int_0^T \left|\zeta\left(\frac{1}{2} + \varepsilon + it_\tau^\alpha + iu\right)\right| \mathrm{d}\tau\right) \sup_{s\in K}\left|\frac{l_n(1/2+\varepsilon-s+iu)}{1/2+\varepsilon-s+iu}\right| \mathrm{d}u$$

and

$$I_2 = \frac{1}{T}\int_0^T \sup_{s\in K}\left|\frac{l_n(1-s-it_\tau^\alpha)}{1-s-it_\tau^\alpha}\right| \mathrm{d}\tau.$$

For the function $\Gamma(s)$, the estimate

$$\Gamma(\sigma+it) \ll \exp\{-c|t|\}, \quad c > 0, \tag{2.6}$$

is valid uniformly in $\sigma_1 \leqslant \sigma \leqslant \sigma_2$ with arbitrary $\sigma_1 < \sigma_2$. Therefore,

$$\begin{aligned}\frac{l_n(1/2+\varepsilon-s+iu)}{1/2+\varepsilon-s+iu} &\ll \frac{n^{1/2+\varepsilon-\sigma}}{\theta}\Gamma\left(\frac{1/2+\varepsilon-\sigma}{\theta} + \frac{i(u-t)}{\theta}\right) \\ &\ll_\varepsilon n^{-\varepsilon}\exp\left\{-\frac{c}{\theta}|u-t|\right\} \ll_{\varepsilon,K} n^{-\varepsilon}\exp\{-c_1|u|\}\end{aligned}$$

with $c_1 > 0$. Hence, in view of Lemma 2.2, we obtain

$$I_1 \ll_{\varepsilon,K} n^{-\varepsilon}\int_{-\infty}^{\infty}(1+|u|)\exp\{-c_1|u|\}\,\mathrm{d}u \ll_{\varepsilon,K} n^{-\varepsilon}. \tag{2.7}$$

Using (2.6), we find similarly as above that, for all $s \in K$,

$$\frac{l_n(1-s-it_\tau^\alpha)}{1-s-it_\tau^\alpha} \ll_\varepsilon n^{1-\sigma}\exp\{-c_2|t+t_\tau^\alpha|\} \ll_{\varepsilon,K} n^{1/2-2\varepsilon}\exp\{-c_3 t_\tau^\alpha\}, \quad c_2 > 0,\ c_3 > 0.$$

Therefore,

$$I_2 \ll_{\varepsilon,K} \frac{n^{1/2-2\varepsilon}}{T}\left(\int_0^{\log T} + \int_{\log T}^{T}\right)\exp\{-c_4 t_\tau^\alpha\}\,\mathrm{d}\tau$$
$$\ll_{\varepsilon,K} \frac{n^{1/2-2\varepsilon}\log T}{T} + n^{1/2-2\varepsilon}\exp\{-c_4 t_{\log T}^\alpha\}, \quad c_4>0.$$

By Lemma 2.1, $t_{\log T}^\alpha \to \infty$ as $T\to\infty$. Thus, $I_2 = o(1)$ as $T\to\infty$. This, (2.5) and (2.7) prove equality (2.3). □

Now we generalize Lemma 2.3 for the r-dimensional space

$$H^r(D) = \underbrace{H(D)\times\cdots\times H(D)}_{r}.$$

For $\underline{g}_1 = (g_{11}\dots,g_{1r}), \underline{g}_2 = (g_{21}\dots,g_{2r}) \in H^r(D)$, define

$$\underline{\rho}(\underline{g}_1,\underline{g}_2) = \max_{1\leqslant j\leqslant r}\rho(g_{1j},g_{2j}).$$

Then $\underline{\rho}$ is a metric in $H^r(D)$ inducing its product topology.

Let

$$\Omega^r = \Omega_1\times\cdots\times\Omega_r,$$

where $\Omega_j=\Omega$ for all $j=1,\dots,r$. Then Ω^r, as Ω, is a compact topological Abelian group, therefore, on $(\Omega^r,\mathcal{B}(\Omega^r))$ the probability Haar measure $\underline{m}_H$ can be defined. This leads to the probability space $(\Omega^r,\mathcal{B}(\Omega^r),\underline{m}_H)$. Define by $\omega_j(p)$ the pth component of an element $\omega_j\in\Omega_j$, $j=1,\dots,r$, $p\in\mathbb{P}$, and by $\underline{\omega}=(\omega_1,\cdots,\omega_r)$ the elements of Ω^r. Let m_{jH} be the Haar measure on $(\Omega_j,\mathcal{B}(\Omega_j))$, $j=1,\dots,r$. We note that $\underline{m}_H$ is the product of the measures $m_{1H},\dots,m_{rH}$.

Let, for brevity, $\underline{\alpha}=(\alpha_1,\dots,\alpha_r)$, $\underline{t}_\tau^{\underline{\alpha}} = (t_\tau^{\alpha_1},\dots,t_\tau^{\alpha_r})$,

$$\underline{\zeta}(s+i\underline{t}_\tau^{\underline{\alpha}}) = (\zeta(s+it_\tau^{\alpha_1}),\dots,\zeta(s+it_\tau^{\alpha_r}))$$

and

$$\underline{\zeta}_n(s+i\underline{t}_\tau^{\underline{\alpha}}) = (\zeta_n(s+it_\tau^{\alpha_1}),\dots,\zeta_n(s+it_\tau^{\alpha_r}))\,.$$

Lemma 2.4. *Suppose that $\alpha_1,\dots,\alpha_r$ are fixed positive numbers. Then*

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \underline{\rho}\left(\underline{\zeta}(s+i\underline{t}_\tau^{\underline{\alpha}}),\underline{\zeta}_n(s+i\underline{t}_\tau^{\underline{\alpha}})\right)\mathrm{d}\tau = 0.$$

Proof. The lemma follows from Lemma 2.3 and the definition of the metric $\underline{\rho}$. □

3. The main lemma

For $A\in\mathcal{B}(H^r(D)$, define

$$P_{T,\underline{\alpha}}(A) = \frac{1}{T}\mathrm{meas}\left\{\tau\in[0,T]: \underline{\zeta}(s+i\underline{t}_\tau^{\underline{\alpha}})\in A\right\}.$$

Our aim is to prove a limit theorem on weak convergence of $P_{T,\underline{\alpha}}$ as $T\to\infty$. We start the proof of such a theorem with a limit measure on Ω^r.

For $A\in\mathcal{B}(\Omega^r(D)$, define

$$Q_{T,\underline{\alpha}}(A) = \frac{1}{T}\mathrm{meas}\left\{\tau\in[0,T]: \left(\left(p^{-it_\tau^{\alpha_1}}:p\in\mathbb{P}\right),\dots,\left(p^{-it_\tau^{\alpha_r}}:p\in\mathbb{P}\right)\right)\in A\right\}.$$

Lemma 3.5. *Suppose that $\alpha_1,\dots,\alpha_r$ are fixed different positive numbers. Then $Q_{T,\underline{\alpha}}$ converges weakly to the Haar measure $\underline{m}_H$ as $T\to\infty$.*

Proof. Let $g_{T,\underline{\alpha}}(\underline{k}_1,\dots,\underline{k}_r)$, $\underline{k}_j=(k_{jp}:k_{jp}\in\mathbb{Z},p\in\mathbb{P})$, $j=1,\dots,r$, denote the Fourier transform of $Q_{T,\underline{\alpha}}$, i. e.,

$$g_{T,\underline{\alpha}}(\underline{k}_1,\dots,\underline{k}_r)=\int_{\Omega^r}\prod_{j=1}^{r}{\prod_{p\in\mathbb{P}}}'\,\omega_j^{k_{jp}}(p)\,\mathrm{d}Q_{T,\underline{\alpha}},$$

where the sign "$'$" shows that only a finite number of integers k_{jp} are distinct from zero. Thus, by the definition of $Q_{T,\underline{\alpha}}$,

$$\begin{aligned} g_{T,\underline{\alpha}}(\underline{k}_1,\dots,\underline{k}_r)&=\frac{1}{T}\int_0^T\prod_{j=1}^{r}{\prod_{p\in\mathbb{P}}}'\,p^{-ik_{jp}t_\tau^{\alpha_j}}\,\mathrm{d}\tau\\ &=\frac{1}{T}\int_0^T\exp\left\{-i\sum_{j=1}^{r}t_\tau^{\alpha_j}{\sum_{p\in\mathbb{P}}}'\,k_{jp}\log p\right\}\mathrm{d}\tau. \end{aligned}\tag{3.8}$$

Clearly, we have

$$g_{T,\underline{\alpha}}(\underline{0},\dots,\underline{0})=1.\tag{3.9}$$

Now suppose that $(\underline{k}_1,\dots,\underline{k}_r)\neq(\underline{0},\dots,\underline{0})$. Then we have

$$a_j\overset{def}{=}{\sum_{p\in\mathbb{P}}}'\,k_{jp}\log p\neq 0$$

for some $j=\{1,\dots,r\}$. Without loss a generality, we may suppose that $\alpha_1<\alpha_2<\dots<\alpha_r$. Let $j_0=\max\{j\leqslant r:a_j\neq 0\}$. Then

$$A_{\underline{\alpha}}(\tau)\overset{def}{=}\sum_{j=1}^{r}t_\tau^{\alpha_j}a_j=\sum_{j=1}^{j_0}t_\tau^{\alpha_j}a_j$$

and hence

$$\begin{aligned} g_{T,\underline{\alpha}}(\underline{k}_1,\dots,\underline{k}_r)&=\frac{1}{T}\int_0^T\big(\cos A_{\underline{\alpha}}(\tau)-i\sin A_{\underline{\alpha}}(\tau)\big)\,\mathrm{d}\tau=\\ &=\frac{1}{T}\int_{\log T}^T\big(\cos A_{\underline{\alpha}}(\tau)-i\sin A_{\underline{\alpha}}(\tau)\big)\,\mathrm{d}\tau+O\left(\frac{\log T}{T}\right). \end{aligned}$$

To estimate two last integrals (which we denote by j_1 and j_2, respectively), we note that

$$A'_{\underline{\alpha}}(\tau)=\sum_{j=1}^{j_0}\alpha_ja_jt_\tau^{\alpha_j-1}t'_\tau=\alpha_{j_0}a_{j_0}t_\tau^{\alpha_{j_0}-1}t'_\tau\left(1+\sum_{j=1}^{j_0-1}c_jt_\tau^{-\beta_j}\right),$$

where $c_j=\alpha_ja_j/(\alpha_{j_0}a_{j_0})$ and $\beta_j=\alpha_{j_0}-\alpha_j>0$. In view of Lemma 2.1,

$$A'_{\underline{\alpha}}(\tau)=\alpha_{j_0}a_{j_0}\left(\frac{2\pi}{\log\tau}\right)^{\alpha_{j_0}}\tau^{\alpha_{j_0}-1}\big(1+o(1)\big).$$

Hence, the function $(A'_{\underline{\alpha}}(\tau))^{-1}$ is monotonic for $\tau\geqslant\log T$ and for sufficiently large T. Setting $a=A_{\underline{\alpha}}(\log T)$, $b=A_{\underline{\alpha}}(T)$ and using the second mean value theorem, we get

$$j_1=\int_a^b\frac{\cos u}{A'_{\underline{\alpha}}(\tau)}\,\mathrm{d}u=\frac{1}{A'_{\underline{\alpha}}(\log T)}\int_a^\xi\cos u\,\mathrm{d}u+\frac{1}{A'_{\underline{\alpha}}(T)}\int_\xi^b\cos u\,\mathrm{d}u$$

for some ξ, $a\leqslant\xi\leqslant b$. Thus we find

$$|j_1|\leqslant 2\left(\frac{1}{|A'_{\underline{\alpha}}(\log T)|}+\frac{1}{|A'_{\underline{\alpha}}(T)|}\right)\ll\left(\frac{\log\log T}{\log T}\right)^{\alpha_{j_0}-1}+\left(\frac{\log T}{T}\right)^{\alpha_{j_0}-1}.$$

The same bound holds for j_2. Obviously, $|j_1|, |j_2| = o(T)$ in both cases $\alpha_{j_0} \geqslant 1$ and $0 < \alpha_{j_0} < 1$. Thus we obtain that

$$\lim_{T\to\infty} g_{T,\underline{\alpha}}(\underline{k}_1, \dots, \underline{k}_r) = 0.$$

This together with (3.9) proves that

$$\lim_{T\to\infty} g_{T,\underline{\alpha}}(\underline{k}_1, \dots, \underline{k}_r) = \begin{cases} 1 & \text{if} \quad (\underline{k}_1, \dots, \underline{k}_r) = (\underline{0}, \dots, \underline{0}), \\ 0 & \text{if} \quad (\underline{k}_1, \dots, \underline{k}_r) \neq (\underline{0}, \dots, \underline{0}). \end{cases}$$

Since the right-hand side of the latter equality is the Fourier transform of the Haar measure $\underline{m}_H$, the lemma is proved. □

Lemma 3.5 is very important for probabilistic investigation of $\underline{\zeta}(s+i\underline{t}_\tau^{\underline{\alpha}})$, it implies limit lemmas in the space $H^r(D)$ for $\underline{\zeta}(s+i\underline{t}_\tau^{\underline{\alpha}})$ and $\underline{\zeta}_n(s+i\underline{t}_\tau^{\underline{\alpha}})$.

Define $u_n : \Omega^r \to H^r(D)$ by the formula

$$\underline{u}_n(\underline{\omega}) = \underline{\zeta}_n(s, \underline{\omega}),$$

where

$$\underline{\zeta}_n(s, \underline{\omega}) = (\zeta_n(s, \omega_1), \dots, \zeta_n(s, \omega_r)).$$

Since the series for $\zeta_n(s, \omega_j)$, $j = 1, \dots, r$, are absolutely convergent for $\sigma > 1/2$, the mapping $\underline{u}_n$ is continuous. Moreover,

$$\underline{u}_n\left(\left(p^{-it_\tau^{\alpha_1}} : p \in \mathbb{P}\right), \dots, \left(p^{-it_\tau^{\alpha_r}} : p \in \mathbb{P}\right)\right) = \underline{\zeta}_n(s+i\underline{t}_\tau^{\underline{\alpha}}).$$

For $A \in \mathcal{B}(H^r(D))$, define

$$P_{T,n,\underline{\alpha}}(A) = \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \underline{\zeta}_n(s+i\underline{t}_\tau^{\underline{\alpha}}) \in A\right\},$$

and $\underline{V}_n = \underline{m}_H u_n^{-1}$, where

$$\underline{V}_n(A) = \underline{m}_H \underline{u}_n^{-1}(A) = \underline{m}_H(\underline{u}_n^{-1}A).$$

Then the above remarks, Lemma 3.5, and the preservation of weak convergence under continuous mappings (Theorem 5.1 of [2]) lead to the following lemma.

Lemma 3.6. *Suppose that $\alpha_1, \dots, \alpha_r$ are fixed different positive numbers. Then $P_{T,n,\underline{\alpha}}$ converges weakly to the measure $\underline{V}_n$ as $T \to \infty$.*

4. Tightness

We recall that a family of probability measures $\{P\}$ on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ is called tight if, for every $\varepsilon > 0$, there exists a compact set $K = K(\varepsilon) \subset \mathbb{X}$ such that

$$P(K) > 1 - \varepsilon$$

for all $P \in \{P\}$.

For proving the weak convergence for $P_{T,\underline{\alpha}}$, the tightness of the sequence $\{\underline{V}_n : n \in \mathbb{N}\}$ is applied.

Lemma 4.7. *The sequence $\{\underline{V}_n\}$ is tight.*

Proof. Define the function $u_n : \Omega \to H(D)$ by $u_n(\omega) = \zeta(s, \omega)$, and put

$$V_n = m_H u_n^{-1}.$$

It is well known, see, for example, [13], that the sequence $\{V_n : n \in \mathbb{N}\}$ is tight. Thus, for $j = 1, \dots, r$, the sequence $\{V_{jn} : n \in \mathbb{N}\}$, where $V_{jn} = m_{jH} u_{jn}^{-1}$, $u_{jn}(\omega_j) = \zeta(s, \omega_j)$, is tight. Hence, for every $\varepsilon > 0$, there exists compact sets $K_j \subset H(D)$ such that

$$V_{jn}(K_j) > 1 - \frac{\varepsilon}{r}, \quad j = 1, \dots, r,$$

for all $n \in \mathbb{N}$. Let $K = K_1 \times \cdots \times K_r$, Then K is a compact set in the space $H^r(D)$, Moreover,

$$\begin{aligned}
\underline{V}_n(H^r(D) \setminus K) &= \sum_{j=1}^{r} \underline{V}_n(\underbrace{H(D) \times \cdots \times H(D)}_{j-1} \times (H(D) \setminus K_j) \times H(D) \times \cdots \times H(D)) \\
&= \sum_{j=1}^{r} \underline{m}_H(\underline{u}_n(\underbrace{H(D) \times \cdots \times H(D)}_{j-1} \times (H(D) \setminus K_j) \times H(D) \times \cdots \times H(D))) \\
&= \sum_{j=1}^{r} \underline{m}_H(\Omega_1 \times \cdots \times \Omega_{j-1} \times (u_{jn}^{-1}(H(D) \setminus K_j)) \times \Omega_{j+1} \times \cdots \times \Omega_r) \\
&= \sum_{j=1}^{r} (m_{1H}(\Omega_1) \cdots m_{j-1H}(\Omega_{j-1}) m_{jH} u_{jn}^{-1}(\Omega_j \setminus K_j) m_{j+1H}(\Omega_{j+1}) \cdots m_{rH}(\Omega_r)) \\
&= \sum_{j=1}^{r} V_{jn}(H(D) \setminus K_j) \leqslant \sum_{j=1}^{r} \left(1 - 1 + \frac{\varepsilon}{r}\right) = \varepsilon
\end{aligned}$$

for all $n \in \mathbb{N}$. Thus

$$\underline{V}_n(K) \geqslant 1 - \varepsilon$$

for all $n \in \mathbb{N}$, i. e., the sequence $\{\underline{V}_n\}$ is tight. □

5. A LIMIT THEOREM

In this section we will apply Lemmas 3.6 and 4.7 to prove limit theorem for $P_{T,\underline{\alpha}}$. Define the $H^r(D)$-valued random element

$$\underline{\zeta}(s, \underline{\omega}) = (\zeta(s, \omega_1), \dots, \zeta(s, \omega_r)),$$

where

$$\zeta(s, \omega_j) = \prod_{p \in \mathbb{P}} \left(1 - \frac{\omega_j(p)}{p^s}\right)^{-1}, \quad j = 1, \dots, r,$$

and denote by $P_{\underline{\zeta}}$ its distribution.

Theorem 5.4. *Suppose that $\alpha_1, \dots, \alpha_r$ are fixed different positive numbers. Then $P_{T,\underline{\alpha}}$ converges weakly to $P_{\underline{\zeta}}$ as $T \to \infty$.*

Proof. Suppose that θ_T is a random variable defined on a certain probability space with measure μ and uniformly distributed on $[0, T]$. Define the $H^r(D)$-valued random element

$$\underline{X}_{T,n,\underline{\alpha}} = X_{T,n,\underline{\alpha}}(s) = \underline{\zeta}_n(s + i\underline{t}^{\alpha}_{\theta_T}).$$

Then, denoting by $\xrightarrow{\mathcal{D}}$ the convergence in distribution, by Lemma 3.6 we have

$$\underline{X}_{T,n,\underline{\alpha}} \xrightarrow[T\to\infty]{\mathcal{D}} \underline{X}_n, \tag{5.10}$$

where $\underline{X}_n$ is the $H^r(D)$-valued random element with the distribution $\underline{V}_n$. Since, in view of Lemma 4.7, the sequence $\{\underline{V}_n\}$ is tight, by the classical Prokhorov theorem, see, for example, [2], it is relatively compact. Therefore, there exists a subsequence $\{\underline{X}_{n_l}\}$, $n_l \to \infty$ as $l \to \infty$, and a probability measure P on $(H^r(D), \mathcal{B}(H^r(D)))$ such that

$$\underline{X}_{n_l} \xrightarrow[l\to\infty]{\mathcal{D}} P. \tag{5.11}$$

Define

$$\underline{X}_{T,\underline{\alpha}} = X_{T,\underline{\alpha}}(s) = \underline{\zeta}(s + i\underline{t}^{\alpha}_{\theta_T}).$$

Then Lemma 2.4 shows that

$$\lim_{n\to\infty} \limsup_{T\to\infty} \mu\left\{\underline{\rho}\left(\underline{X}_{T,\underline{\alpha}}, \underline{X}_{T,n,\underline{\alpha}}\right) \geqslant \varepsilon\right\}$$
$$\leqslant \lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T\varepsilon}\int_0^T \underline{\rho}\left(\underline{\zeta}(s + i\underline{t}^{\alpha}_{\tau}), \underline{\zeta}_n(s + i\underline{t}^{\alpha}_{\tau})\right)\, \mathrm{d}\tau = 0.$$

This together with relations (5.10) and (5.11) allows to apply Theorem 4.2 of [2], thus we have

$$\underline{X}_{T,\underline{\alpha}} \xrightarrow[T\to\infty]{\mathcal{D}} P. \tag{5.12}$$

The latter relation also show that the measure P is independent of the sequence $\{\underline{X}_{n_l}\}$. Therefore, the relative compactness of $\{\underline{V}_n\}$ implies the relation

$$\underline{X}_n \xrightarrow[n\to\infty]{\mathcal{D}} P. \tag{5.13}$$

Thus, by (5.12) we have that $P_{T,\underline{\alpha}}$ converges weakly to P.

It remains to identify the measure P. Usually, for this, arguments of the ergodic theory are applied. However, in our case, such a method does not work. Therefore, we will reduce the problem to a known case.

In [14], the joint approximation of analytic functions by shifts

$$(\zeta(s + ia_1\tau), \ldots, \zeta(s + ia_r\tau)),$$

where $a_1, \ldots, a_r$ are certain algebraic numbers, was consider. For this, the weak convergence of

$$P_T(A) \stackrel{def}{=} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : (\zeta(s + ia_1\tau), \ldots, \zeta(s + ia_r\tau)) \in A\right\}, \quad A \in \mathcal{B}(H^r(D)),$$

as $T \to \infty$ was investigated, and it was obtained that P_T converges weakly to $P_{\underline{\zeta}}$ as $T \to \infty$. In the proof of the latter limit theorem, the mapping $\underline{u}_n$ and the measure $\underline{V}_n$ defined in Section 3 are also involved. Moreover, it was obtained that the limit measure of P_T as $T \to \infty$ coincides with the limit measure of $\underline{V}_n$ as $n \to \infty$. Thus, we have that, in view of (5.12) and (5.13), $P_{T,\underline{\alpha}}$ converges weakly to $P_{\underline{\zeta}}$ as $T \to \infty$. □

6. Support

The proof of the Theorem 1.3 requires the explicit form of the support of the measure $P_{\underline{\zeta}}$. We recall that the support of $P_{\underline{\zeta}}$ is a minimal closed set $S_{\underline{\zeta}} \subset \mathcal{B}(H^r(D))$ such that $P_{\underline{\zeta}}(S_{\underline{\zeta}}) = 1$. Let

$$S = \{g \in H(D) : g(s) \neq 0 \,\text{or}\, g(s) \equiv 0\}.$$

Lemma 6.8. *The support of the measure $P_{\underline{\zeta}}$ is the set S^r.*

Proof. The space $H(D)$ is separable. Therefore, see, for example, [2],

$$\mathcal{B}(H^r(D)) = \underbrace{\mathcal{B}(H(D)) \times \cdots \times \mathcal{B}(H(D))}_{r}.$$

From this, it follows that it is sufficient to consider the measure $P_{\underline{\zeta}}$ on rectangular sets

$$A = A_1 \times \cdots \times A_r, \quad A_1, \dots, A_r \in \mathcal{B}(H(D)).$$

It is well known that the support of the measures

$$m_{jH}\{\omega_j \in \Omega_j : \zeta(s, \omega_j) \in A_j\}, \quad A_j \in \mathcal{B}(H(D)),\ j = 1, \dots, r, \tag{6.14}$$

is the set S. Since the measure $\underline{m}_H$ is the product of m_{jH}, we have

$$\begin{aligned} P_{\underline{\zeta}}(A) &= \underline{m}_H \left\{\underline{\omega} \in \Omega^r : \underline{\zeta}(s, \underline{\omega}) \in A\right\} \\ &= \underline{m}_H \left\{\omega_1 \in \Omega_1, \dots, \omega_r \in \Omega_r : \zeta(s, \omega_1) \in A_1, \dots, \zeta(s, \omega_r) \in A_r\right\} \\ &= m_{1H} \left\{\omega_1 \in \Omega_1 : \zeta(s, \omega_1) \in A_1\right\} \cdots m_{rH} \left\{\omega_r \in \Omega_r : \zeta(s, \omega_r) \in A_r\right\}. \end{aligned}$$

This equality, the minimality of the support and that the support of (6.14) is the set S prove the lemma. □

7. Proof of Theorem 1.3

Theorem 1.3 is a consequence of Theorem 5.4, Lemma 6.8 and the Mergelyan theorem on an approximation of analytic functions on compact sets with connected complements by polynomials.

Proof of Theorem 1.3. Let $p_1(s), \dots, p_r(s)$ be polynomials and

$$G_\varepsilon = \left\{ g \in H(D) : \sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} \left| g_j(s) - \mathrm{e}^{p_j(s)} \right| < \frac{\varepsilon}{2} \right\}.$$

In view of Lemma 6.8, G_ε is an open neighbourhood of an element $\left(\mathrm{e}^{p_1(s)}, \dots, \mathrm{e}^{p_1(s)}\right)$ of the support of the measure $P_{\underline{\zeta}}$. Therefore,

$$P_{\underline{\zeta}}(G_\varepsilon) > 0. \tag{7.15}$$

Hence, Theorem 5.4 and the equivalent of weak convergence of probability measures in terms of open sets imply

$$\liminf_{T \to \infty} \frac{1}{T} \mathrm{meas} \left\{ \tau \in [0, T] : \sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} \left| \zeta(s + it_\tau^{\alpha_j}) - \mathrm{e}^{p_j(s)} \right| < \frac{\varepsilon}{2} \right\} \geqslant P_{\underline{\zeta}}(G_\varepsilon) > 0. \tag{7.16}$$

By the Mergelyan theorem, we may chose the polynomials $p_1(s), \dots, p_r(s)$ such that

$$\sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} \left| f_j(s) - \mathrm{e}^{p_j(s)} \right| < \frac{\varepsilon}{2}. \tag{7.17}$$

This together with (7.16) prove the firs part of the theorem.

2. Define one more set

$$\hat{G}_\varepsilon = \left\{ g \in H(D) : \sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} |g_j(s) - f_j(s)| < \varepsilon \right\}.$$

We have that the boundaries $\partial \hat{G}_{\varepsilon_1}$ and $\partial \hat{G}_{\varepsilon_2}$ do not intersect for different positive ε_1 and ε_2. Hence, the set $\hat{G}_\varepsilon$ is a continuity set of the measure $P_{\underline{\zeta}}$ ($P_{\underline{\zeta}}(\partial \hat{G}_\varepsilon) = 0$) for all but at most countably many $\varepsilon > 0$. Therefore, Theorem 5.4 and the equivalent of weak convergence of probability measures in terms of continuity sets imply

$$\lim_{T \to \infty} \frac{1}{T} \mathrm{meas} \left\{ \tau \in [0, T] : \sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} |\zeta(s + it_\tau^{\alpha_j}) - f_j(s)| < \varepsilon \right\} = P_{\underline{\zeta}}(\hat{G}_\varepsilon). \tag{7.18}$$

In view of (7.17) and the definitions of the sets G_ε and $\hat{G}_\varepsilon$, we have the inclusion $G_\varepsilon \subset \hat{G}_\varepsilon$. Therefore, by (7.15), the inequality $P_{\underline{\zeta}}(\hat{G}_\varepsilon) > 0$ is true. This and (7.18) prove the second part of the theorem. □

8. Concluding remarks

A method of the paper of using different powers $t_\tau^{\alpha_j}$ in shifts of $\zeta(s)$ for approximation of collections of analytic functions is flexible, it can be applied for the investigation of joint universality for Dirichlet L-functions $L(s,\chi)$ not only with non-equivalent Dirichlet characters, for L-functions of modular forms, etc. An application of the paper method for zeta-functions having no Euler's product over primes, for example, for Hurwitz zeta-functions, should give some new estimates involving the Gram function for the number of zeros. It is well known that universality theorems are the main ingredients for the proof of the functional independence of zeta-functions which comes back to Hilbert. We hope that Theorem 1.3 can be applied to extend the results in the latter field. Also, Theorem 1.3 has generalizations for some compositions and approximation by absolutely convergent Dirichlet series.

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