CARPATHIAN J. MATH.
Volume **39** (2023), No. 1,
Pages 189 - 200

Online version at https://semnul.com/carpathian/
Print Edition: ISSN 1584 - 2851; Online Edition: ISSN 1843 - 4401
DOI: https://doi.org/10.37193/CJM.2023.01.12

In memoriam Professor Charles E. Chidume (1947- 2021)

A parallel method for common variational inclusion and common fixed point problems with applications

THANASAK MOUKTONGLANG[1,2], KANYUTA POOCHINAPAN[1,2] and RAWEEROTE SUPARATULATORN[2]

ABSTRACT. In this paper, we construct a new parallel method to solve common variational inclusion and common fixed point problems in a real Hilbert space. We obtain a weak convergence theorem by using this method. Besides, numerical results on the signal recovery problem consisting of various blurred filters present that our proposed method outperforms the two previous methods.

1. INTRODUCTION

Throughout this article, let $\mathcal{H}$ be a real Hilbert space equipped with their own inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, and define $\mathcal{K} = \{1, 2, \ldots, K\}$, where K is positive integer. The problem of identifying a point $\bar{x} \in \mathcal{H}$ such that

$$0 \in (F + G)\bar{x} \tag{1.1}$$

is called the variational inclusion problem, where $F : \mathcal{H} \to \mathcal{H}$ is a single valued mapping and $G : \mathcal{H} \to 2^{\mathcal{H}}$ is a multivalued mapping. The solution set of the problem (1.1) is represented as $(F + G)^{-1}(0)$. The problem (1.1) can be interpreted as a model of numerous issues in different research fields, such as machine learning [8, 21], signal processing [7, 26] and image recovery [17, 20]. Many splitting algorithms have been introduced and improved to find a solution to the variational inclusion problem (1.1), one of the most famous splitting algorithms is the forward-backward splitting algorithm, see in [12, 18] for more details. In 2015, O'Donoghue and Candès [16] showed the forward-backward splitting algorithm, which is reduced to the proximal gradient algorithm for convex optimization problems. It is well known that the problem (1.1) is equivalent to the following fixed point equation $x = J_\gamma^G(I - \gamma F)x$, where J_γ^G is the resolvent operator of G defined by $J_\gamma^G = (I + \gamma G)^{-1}$ such that $\gamma > 0$. Before that in 1964, the inertial extrapolation technique was proposed by Polyak [19] to speed up the convergence of iterative algorithms which is called the heavy ball method. Moudafi and Oliny [15] in 2003 introduced the inertial proximal algorithm to solve the problem (1.1), which was developed from the forward-backward splitting algorithm with the inertial extrapolation technique. Some very recent results on the modified forward-backward splitting method have also been in [1, 5, 6, 14].

Many real-world problems necessitate finding a solution that satisfies several constraints. These constraints can be reformulated using a nonlinear functional model. We are motivated to study common variational inclusion and common fixed point problems in this paper because the problem can be utilized to solve real-world problems such as

Received: 11.02.2022. In revised form: 12.04.2022.. Accepted: 10.07.2022
2020 *Mathematics Subject Classification.* 65Y05, 68W10, 47H05, 47J25, 49M37.
Key words and phrases. *parallel method, common variational inclusion problem, common fixed point problem, signal recovery problem.*
Corresponding author: R. Suparatulatorn; raweerote.s@gmail.com

signal recovery and image recovery problems with various blurred filters, see [23, 25]. We also show that our problem can be applied for solving signal recovery with various blurred filters as shown in Section 4. This problem consists of finding a point $\bar{x} \in \mathcal{H}$ such that

$$0 \in (F_i + G_i)\bar{x} \text{ and } \bar{x} = S_i\bar{x}, \tag{1.2}$$

where F_i, S_i are single valued mappings on $\mathcal{H}$ and $G_i : \mathcal{H} \to 2^{\mathcal{H}}$ is a multivalued mapping for all $i \in \mathcal{K}$. For finding a common fixed point of a finite family of G-nonexpansive mappings $\{S_i\}_{i\in\mathcal{K}}$, Suantai et al. [23] introduced Algorithm 1 in $\mathcal{H}$ with directed graphs. This algorithm is defined as follows:

Algorithm 1 : Parallel monotone hybrid algorithm

Initialization: Select an arbitrary element $v_1 \in C_1 \subseteq \mathcal{H}$ and set $k := 1$.
Iterative Steps : Construct $\{v_k\}$ by using the following steps:
Step 1. For any $i \in \mathcal{K}$, set $u_k^i = \rho_k^i v_k + (1 - \rho_k^i)S_i v_k$, where $\{\rho_k^i\} \subset [0, 1]$, and compute

$$\bar{u}_k = \text{argmax}\left\{\|u_k^i - v_k\| : i \in \mathcal{K}\right\}.$$

Step 2. Compute

$$v_{k+1} = P_{C_{k+1}} v_1,$$

where $C_{k+1} = \{c \in C_k : \|c - \bar{u}_k\| \leq \|c - v_k\|\}$.
Replace k by $k + 1$ and then repeat **Step 1.**

Suparatulatorn et al. [24] recently proposed Algorithm 2 to solve a common variational inclusion problem under Lipschitz continuity and monotonicity of F_i, and maximal monotonicity of G_i, for all $i \in \mathcal{K}$. This algorithm is defined as follows:

Algorithm 2 : Parallel inertial Tseng type algorithm

Initialization: Given $\lambda_i \in (0, 1)$ and $\gamma_1^i > 0$ for all $i \in \mathcal{K}$. Select arbitrary elements $v_0, v_1 \in \mathcal{H}$ and set $k := 1$.
Iterative Steps : Construct $\{v_k\}$ by using the following steps:
Step 1. Set $r_k = v_k + \xi_k(v_k - v_{k-1})$, where $\{\xi_k\} \subset [0, \infty)$, and compute, for all $i \in \mathcal{K}$,

$$s_k^i = J_{\gamma_k^i}^{G_i}(I - \gamma_k^i F_i)r_k.$$

If $r_k = s_k^i$ for all $i \in \mathcal{K}$, then stop and $r_k \in \bigcap_{i\in\mathcal{K}} (F_i + G_i)^{-1}(0)$. Otherwise
Step 2. Compute, for all $i \in \mathcal{K}$,

$$t_k^i = s_k^i - \gamma_k^i(F_i s_k^i - F_i r_k) \text{ and } \bar{t}_k = \text{argmax}\left\{\|t_k^i - r_k\| : i \in \mathcal{K}\right\}.$$

Step 3. Compute

$$v_{k+1} = a_k\varphi(v_k) + (1 - a_k - b_k)v_k + b_k\bar{t}_k,$$

where $\{a_k\}, \{b_k\} \subset (0, 1)$, φ is a contractive on $\mathcal{H}$, and update, for all $i \in \mathcal{K}$,

$$\gamma_{k+1}^i = \begin{cases} \min\left\{\lambda_i \frac{\|r_k - s_k^i\|}{\|F_i r_k - F_i s_k^i\|}, \gamma_k^i\right\} & \text{if } F_i r_k \neq F_i s_k^i; \\ \gamma_k^i & \text{otherwise.} \end{cases}$$

Replace k by $k + 1$ and then repeat **Step 1.**

Furthermore, Suparatulatorn and Chaichana [25] studied an image recovery problem in which several blurred filters are considered and the mathematical model used there is the common variational inclusion problem. Several interesting outcomes for the problem (1.2) and related problems were published, see [9, 10, 22, 27].

Inspired by the previous works, we develop a novel parallel algorithm based on the inertial Mann iteration process to prove a weak convergence result for solving the problem (1.2) under some control conditions in $\mathcal{H}$. Additionally, we compare our algorithm with Algorithm 1 and Algorithm 2 in order to solve the signal recovery problem involving multiple blurring filters.

2. PRELIMINARIES

In this section, we collect some necessary definitions and lemmas for proving our main result. We denote $\rightharpoonup$ and $\to$ as weak and strong convergence, respectively. Denote the set of the fixed point of the mapping S by $Fix(S)$. For each $x, y \in \mathcal{H}$, we have the following facts:

$$\|x+y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2 \tag{2.3}$$

and

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle. \tag{2.4}$$

Definition 2.1. A self-mapping $S : \mathcal{H} \to \mathcal{H}$ is said to be

(i) $\mathcal{L}$-Lipschitz continuous if there is $\mathcal{L} > 0$ such that for all $x, y \in \mathcal{H}$,

$$\|Sx - Sy\| \leq \mathcal{L}\|x - y\|,$$

(ii) nonexpansive if S is $\mathcal{L}$-Lipschitz continuous when $\mathcal{L} = 1$,

(iii) μ-demicontractive if $Fix(S) \neq \emptyset$ and there is $\mu \in [0, 1)$ such that for all $x \in \mathcal{H}$ and all $p \in Fix(S)$,

$$\|Sx - p\|^2 \leq \|x - p\|^2 + \mu\|x - Sx\|^2,$$

this is equivalent to

$$\langle x - p, Sx - x\rangle \leq \frac{\mu - 1}{2}\|x - Sx\|^2$$

for all $x \in \mathcal{H}$ and all $p \in Fix(S)$.

Definition 2.2. Let $G : \mathcal{H} \to 2^{\mathcal{H}}$ be a multivalued mapping. Then G is said to be

(i) monotone if for all $(x, u), (y, v) \in graph(G)$ (the graph of mapping G),

$$\langle u - v, x - y\rangle \geq 0,$$

(ii) maximal monotone if for every $(x, u) \in \mathcal{H} \times \mathcal{H}$, $\langle u - v, x - y\rangle \geq 0$ for all $(y, v) \in graph(G)$ if and only if $(x, u) \in graph(G)$.

Definition 2.3. [28]Assume that $S : \mathcal{H} \to \mathcal{H}$ is a mapping with $Fix(S) \neq \emptyset$. Then, $I - S$ is said to be demiclosed at zero if for any $\{v_k\} \in \mathcal{H}$, the following implication holds:

$$v_k \rightharpoonup v \text{ and } (I - S)v_k \to 0 \Longrightarrow v \in Fix(S).$$

Lemma 2.1. [4] *If $G : \mathcal{H} \to 2^{\mathcal{H}}$ is a maximal monotone mapping and $F : \mathcal{H} \to \mathcal{H}$ is a Lipschitz continuous and monotone mapping, then the mapping $F + G$ is maximal monotone.*

Lemma 2.2. [2] *Let $\{a_k\}$ and $\{b_k\}$ be nonnegative sequences of real numbers satisfying $\sum_{k=1}^{\infty} b_k < \infty$ and $a_{k+1} \leq a_k + b_k$. Then, $\{a_k\}$ is a convergent sequence.*

Lemma 2.3. [3, Opial] *Let Φ be a nonempty set of $\mathcal{H}$ and $\{v_k\}$ be a sequence in $\mathcal{H}$. Suppose the following assertions hold.*
(i) For every $v \in \Phi$, the sequence $\{\|v_k - v\|\}$ converges.
(ii) Every weak sequential cluster point of $\{v_k\}$ belongs to Φ.
Then $\{v_k\}$ weakly converges to a point in Φ.

3. Main results

In this section, we propose a new method for solving the problem (1.2). For the convergence analysis of the proposed method, we assume the following assumptions, for all $i \in \mathcal{K}$.

Assumption 1. $\mathcal{H}$ is a real Hilbert space, $F_i : \mathcal{H} \to \mathcal{H}$ is $\mathcal{L}_i$-Lipschitz continuous and monotone mapping and $G_i : \mathcal{H} \to 2^{\mathcal{H}}$ is maximal monotone operator.

Assumption 2. $\{p_k^i\} \subset [0, \infty)$, $\{q_k^i\} \subset [1, \infty)$ such that $\sum_{k=1}^{\infty} p_k < \infty$ and $\lim_{k\to\infty} q_k = 1$.

Assumption 3. $S_i : \mathcal{H} \to \mathcal{H}$ is μ_i-demicontractive mapping such that $I - S_i$ is demiclosed at zero.

Assumption 4. $\{\xi_k\} \subset [0, \xi)$, $\{\alpha_k^i\} \subset (\mu_i, \bar{\alpha}_i) \subset (0, 1)$, for some $\xi, \bar{\alpha}_i > 0$.

Assumption 5. $\Phi := \bigcap_{i\in\mathcal{K}} (F_i + G_i)^{-1}(0) \cap \bigcap_{i\in\mathcal{K}} Fix(S_i)$ is nonempty.

Algorithm 3

Initialization: Given $\lambda_i \in (0, 1)$ and $\gamma_1^i > 0$ for all $i \in \mathcal{K}$. Select arbitrary elements $v_0, v_1 \in \mathcal{H}$ and set $k := 1$.

Iterative Steps : Construct $\{v_k\}$ by using the following steps:

Step 1. Set $r_k = v_k + \xi_k(v_k - v_{k-1})$ and compute, for all $i \in \mathcal{K}$,

$$s_k^i = J_{\gamma_k^i}^{G_i}(I - \gamma_k^i F_i) r_k.$$

Step 2. Compute, for all $i \in \mathcal{K}$,

$$t_k^i = s_k^i - \gamma_k^i (F_i s_k^i - F_i r_k) \text{ and } u_k^i = \alpha_k^i t_k^i + (1 - \alpha_k^i) S_i t_k^i.$$

Step 3. Compute

$$v_{k+1} = \text{argmax}\left\{\|u_k^i - r_k\| : i \in \mathcal{K}\right\}$$

and update, for all $i \in \mathcal{K}$,

$$\gamma_{k+1}^i = \begin{cases} \min\left\{\frac{\lambda_i q_k^i \|r_k - s_k^i\|}{\|F_i r_k - F_i s_k^i\|}, \gamma_k^i + p_k^i\right\} & \text{if } F_i r_k \neq F_i s_k^i; \\ \gamma_k^i + p_k^i & \text{otherwise.} \end{cases}$$

Replace k by $k + 1$ and then repeat **Step 1.**

Lemma 3.4. *Suppose that Assumptions 1-2 hold. Then the sequence $\{\gamma_k^i\}$ generated by Algorithm 3 is well defined and converges to $\gamma_i \in \left[\min\left\{\gamma_1^i, \frac{\lambda_i}{\mathcal{L}_i}\right\}, \gamma_1^i + p_i\right]$, where $p_i = \sum_{k=1}^{\infty} p_k^i$.*

Proof. Since F_i is an $\mathcal{L}_i$-Lipschitz continuous mapping for all $i \in \mathcal{K}$, if $F_i s_k^i \neq F_i r_k$, then

$$\frac{\lambda_i q_k^i \|r_k - s_k^i\|}{\|F_i r_k - F_i s_k^i\|} \geq \frac{\lambda_i q_k^i \|r_k - s_k^i\|}{\mathcal{L}_i \|r_k - s_k^i\|} = \frac{\lambda_i q_k^i}{\mathcal{L}_i} \geq \frac{\lambda_i}{\mathcal{L}_i}.$$

By using the same technique as in the proof of [13, Lemma 3.1], we obtain that $\lim_{k\to\infty} \gamma_k^i = \gamma_i \in \left[\min\left\{\gamma_1^i, \frac{\lambda_i}{\mathcal{L}_i}\right\}, \gamma_1^i + p_i\right]$. □

Lemma 3.5. *Let $v \in \Phi$. Then under Assumptions 1-5, we have, for all $i \in \mathcal{K}$,*

$$\|u_k^i - v\|^2 \leq \|r_k - v\|^2 - \left[1 - \left(\varrho_k^i\right)^2\right] \|r_k - s_k^i\|^2 - (1-\alpha_k^i)(\alpha_k^i - \mu_i)\|S_i t_k^i - t_k^i\|^2 \tag{3.5}$$

and

$$\|t_k^i - r_k\| \leq \left(1 + \varrho_k^i\right) \|r_k - s_k^i\|, \tag{3.6}$$

where $\varrho_k^i = \lambda_i q_k^i \frac{\gamma_k^i}{\gamma_{k+1}^i}$.

Proof. By the definitions of t_k^i and γ_k^i, we obtain that for all $i \in \mathcal{K}$,

$$\|t_k^i - s_k^i\| = \gamma_k^i \|F_i s_k^i - F_i r_k\| \leq \varrho_k^i \|s_k^i - r_k\|, \tag{3.7}$$

which together with (2.3) indicates that for all $i \in \mathcal{K}$,

$$\begin{aligned}
\|t_k^i - v\|^2 &= \|s_k^i - v\|^2 - 2\gamma_k^i \langle s_k^i - v, F_i s_k^i - F_i r_k\rangle + \left(\gamma_k^i\right)^2 \|F_i s_k^i - F_i r_k\|^2 \\
&= \|r_k - v\|^2 + \|s_k^i - r_k\|^2 - 2\langle s_k^i - r_k, s_k^i - r_k\rangle + 2\langle s_k^i - r_k, s_k^i - v\rangle \\
&\quad - 2\gamma_k^i \langle s_k^i - v, F_i s_k^i - F_i r_k\rangle + \left(\varrho_k^i\right)^2 \|s_k^i - r_k\|^2 \\
&\leq \|r_k - v\|^2 - \left[1 - \left(\varrho_k^i\right)^2\right] \|r_k - s_k^i\|^2 - 2\langle s_k^i - v, r_k - s_k^i - \gamma_k^i (F_i r_k - F_i s_k^i)\rangle. \qquad (3.8)
\end{aligned}$$

From the definition of s_k^i, we have that $(I - \gamma_k^i F_i) r_k \in (I + \gamma_k^i G_i) s_k^i$ for all $i \in \mathcal{K}$. This implies that there exists $g_k^i \in G_i s_k^i$ such that $g_k^i = \frac{1}{\gamma_k^i}\left(r_k - s_k^i - \gamma_k^i F_i r_k\right)$ for all $i \in \mathcal{K}$. Since $F_i + G_i$ is maximal monotone, we obtain that $\langle F_i s_k^i + g_k^i, s_k^i - v\rangle \geq 0$ for all $i \in \mathcal{K}$, implying that $\langle s_k^i - v, r_k - s_k^i - \gamma_k^i (F_i r_k - F_i s_k^i)\rangle \geq 0$ for all $i \in \mathcal{K}$. This combined with (3.8) yields that $\|t_k^i - v\|^2 \leq \|r_k - v\|^2 - \left[1 - \left(\varrho_k^i\right)^2\right] \|r_k - s_k^i\|^2$ for all $i \in \mathcal{K}$. This follows from the equivalence of demicontractive mapping S_i and (2.3) that for all $i \in \mathcal{K}$,

$$\begin{aligned}
\|u_k^i - v\|^2 &= \|\alpha_k^i t_k^i + (1-\alpha_k^i) S_i t_k^i - v\|^2 \\
&= \|t_k^i - v\|^2 + (1-\alpha_k^i)^2 \|S_i t_k^i - t_k^i\|^2 + 2(1-\alpha_k^i)\langle t_k^i - v, S_i t_k^i - t_k^i\rangle \\
&\leq \|t_k^i - v\|^2 + (1-\alpha_k^i)^2 \|S_i t_k^i - t_k^i\|^2 + (1-\alpha_k^i)(\mu_i - 1)\|S_i t_k^i - t_k^i\|^2 \\
&\leq \|r_k - v\|^2 - \left[1 - \left(\varrho_k^i\right)^2\right] \|r_k - s_k^i\|^2 - (1-\alpha_k^i)(\alpha_k^i - \mu_i)\|S_i t_k^i - t_k^i\|^2.
\end{aligned}$$

Further, using Cauchy-Schwarz and by (3.7), we obtain that the inequality (3.6) holds. □

Lemma 3.6. *Suppose that $\lim_{k\to\infty} \|r_k - s_k^i\| = \lim_{k\to\infty} \|S_i t_k^i - t_k^i\| = 0$ for all $i \in \mathcal{K}$. If there exists a weakly convergent subsequence $\{r_{k_j}\}$ of $\{r_k\}$, then under Assumptions 1-5, we have that the weak limit of $\{r_{k_j}\}$ belongs to Φ.*

Proof. Let $\bar{r} \in \mathcal{H}$ such that $r_{k_j} \rightharpoonup \bar{r}$. Since $\lim_{k\to\infty} \varrho_k^i = \lambda_i > 0$ and by (3.6), we have $\lim_{k\to\infty} \|t_k^i - r_k\| = 0$. It follows that $t_{k_j}^i \rightharpoonup \bar{r}$. This together with $\lim_{k\to\infty} \|S_i t_k^i - t_k^i\| = 0$, by the demiclosedness at zero of $I - S_i$, $\bar{r} \in \bigcap_{i\in\mathcal{K}} Fix(S_i)$. Next, we show that $\bar{r} \in \bigcap_{i\in\mathcal{K}} (F_i + G_i)^{-1}(0)$. Let $(v_i, u_i) \in graph\,(F_i + G_i)$ for all $i \in \mathcal{K}$, that is, $u_i - F_i v_i \in G_i v_i$ for all $i \in \mathcal{K}$. It implies by the definition of s_k^i that for all $i \in \mathcal{K}$, $\frac{1}{\gamma_{k_j}^i}\left(r_{k_j} - s_{k_j}^i - \gamma_{k_j}^i F_i r_{k_j}\right) \in G_i s_{k_j}^i$. By the maximal monotonicity of G_i, we have that $\left\langle v_i - s^i_{k_j}, u_i - F_i v_i - \frac{1}{\gamma^i_{k_j}}\left(r_{k_j} - s^i_{k_j} - \gamma^i_{k_j} F_i r_{k_j}\right)\right\rangle \geq 0$ for all $i \in \mathcal{K}$. Thus, for all $i \in \mathcal{K}$,

$$\begin{aligned}\left\langle v_i - s^i_{k_j}, u_i\right\rangle &\geq \left\langle v_i - s^i_{k_j}, F_i v_i + \frac{1}{\gamma^i_{k_j}}\left(r_{k_j} - s^i_{k_j} - \gamma^i_{k_j} F_i r_{k_j}\right)\right\rangle \\ &= \left\langle v_i - s^i_{k_j}, F_i v_i - F_i s^i_{k_j}\right\rangle + \left\langle v_i - s^i_{k_j}, F_i s^i_{k_j} - F_i r_{k_j}\right\rangle \\ &\quad + \frac{1}{\gamma^i_{k_j}}\left\langle v_i - s^i_{k_j}, r_{k_j} - s^i_{k_j}\right\rangle \\ &\geq \left\langle v_i - s^i_{k_j}, F_i s^i_{k_j} - F_i r_{k_j}\right\rangle + \frac{1}{\gamma^i_{k_j}}\left\langle v_i - s^i_{k_j}, r_{k_j} - s^i_{k_j}\right\rangle.\end{aligned}$$

This follows from the Lipschitz continuity of F_i, $\lim_{k\to\infty}\|r_k - s^i_k\| = 0$ and $\lim_{k\to\infty}\gamma^i_k = \gamma_i > 0$ that $\langle v_i - \bar{r}, u_i\rangle = \lim_{j\to\infty}\left\langle v_i - s^i_{k_j}, u_i\right\rangle \geq 0$ for all $i \in \mathcal{K}$, which, together with the maximal monotonicity of $F_i + G_i$, we get that $0 \in (F_i + G_i)\,\bar{r}$ for all $i \in \mathcal{K}$, that is, $\bar{r} \in \bigcap_{i\in\mathcal{K}} (F_i + G_i)^{-1}(0)$. Therefore, $\bar{r} \in \Phi$. □

Theorem 3.1. *Suppose that $\sum_{k=1}^{\infty}\xi_k\|v_k - v_{k-1}\| < \infty$, then under Assumptions 1-5, we have that the sequence $\{v_k\}$ generated by Algorithm 3 weakly converges to a solution of Φ.*

Proof. Let $v \in \Phi$. Since $\lim_{k\to\infty}\left[1 - \left(\varrho^i_k\right)^2\right] = 1 - \lambda_i^2 > 0$, one can find $m_i \in \mathbb{N}$ such that $1 - \left(\varrho^i_k\right)^2 > 0$ for all $i \in \mathcal{K}$ and all $k \geq k_0$, where $k_0 = \max_{i\in\mathcal{K}} m_i$. From Assumption 4, by the definition of r_k and using (3.5), we have $\|u^i_k - v\| \leq \|r_k - v\| = \|v_k + \xi_k(v_k - v_{k-1}) - v\| \leq \|v_k - v\| + \xi_k\|v_k - v_{k-1}\|$ for all $i \in \mathcal{K}$ and all $k \geq k_0$. It implies by the definition of i that $\|v_{k+1} - v\| \leq \|r_k - v\| \leq \|v_k - v\| + \xi_k\|v_k - v_{k-1}\|$ for all $k \geq k_0$. This follows that $\{\|v_k - v\|\}$ is convergent because of using Lemma 2.2 and $\sum_{k=1}^{\infty}\xi_k\|v_k - v_{k-1}\| < \infty$. In particular, $\{v_k\}$ is bounded and also $\{r_k\}$. Next, applying (2.4) and (3.5), we have,

$$\begin{aligned}\|u^i_k - v\|^2 &\leq \|v_k - v\|^2 + 2\xi_k\langle v_k - v_{k-1}, r_k - v\rangle \\ &\quad - \left[1 - \left(\varrho^i_k\right)^2\right]\|r_k - s^i_k\|^2 - (1 - \alpha^i_k)(\alpha^i_k - \mu_i)\|S_i t^i_k - t^i_k\|^2\end{aligned}$$

for all $i \in \mathcal{K}$. It follows that there are $i_k \in \mathcal{K}$ and $M_1 > 0$ such that

$$\begin{aligned}\left[1 - \left(\varrho^{i_k}_k\right)^2\right]\|r_k - s^{i_k}_k\|^2 &\leq \|v_k - v\|^2 - \|v_{k+1} - v\|^2 + M_1\xi_k\|v_k - v_{k-1}\| \\ &\quad - (1 - \alpha^{i_k}_k)(\alpha^{i_k}_k - \mu_{i_k})\|S_{i_k} t^{i_k}_k - t^{i_k}_k\|^2.\end{aligned} \tag{3.9}$$

From Assumption 4, $\lim_{k\to\infty}\left[1 - \left(\varrho^{i_k}_k\right)^2\right] > 0$ and $\sum_{k=1}^{\infty}\xi_k\|v_k - v_{k-1}\| < \infty$, and using $\lim_{k\to\infty}\|v_k - v\|$ exists, we obtain

$$\lim_{k\to\infty}\|r_k - s^{i_k}_k\| = 0 \tag{3.10}$$

and so

$$\lim_{k\to\infty}\|S_{i_k} t^{i_k}_k - t^{i_k}_k\| = 0. \tag{3.11}$$

Indeed, using (3.6), (3.10) and (3.11), we deduce

$$\begin{aligned}\|v_{k+1}-r_k\| &\le \|v_{k+1}-t_k^{i_k}\|+\|t_k^{i_k}-r_k\|\\ &= \|\alpha_k^{i_k}t_k^{i_k}+(1-\alpha_k^{i_k})S_{i_k}t_k^{i_k}-t_k^{i_k}\|+\|t_k^{i_k}-r_k\|\\ &\le (1-\alpha_k^{i_k})\|S_{i_k}t_k^{i_k}-t_k^{i_k}\|+\left(1+\varrho_k^{i_k}\right)\|r_k-s_k^{i_k}\|\to 0 \text{ as } k\to\infty.\end{aligned}$$

This implies by the definition of v_{k+1} that

$$\lim_{k\to\infty}\|r_k-u_k^i\|=0 \tag{3.12}$$

for all $i\in\mathcal{K}$. Again, applying (3.5), we have

$$\begin{aligned}\left[1-\left(\varrho_k^i\right)^2\right]\|r_k-s_k^i\|^2+(1-\alpha_k^i)(\alpha_k^i-\mu_i)\|S_it_k^i-t_k^i\|^2 &\le \|r_k-v\|^2-\|u_k^i-v\|^2\\ &\le M_2\|r_k-u_k^i\|\end{aligned}$$

for all $i\in\mathcal{K}$ and for some $M_2>0$. Combining this to (3.12) with Assumption 4 and $\lim\limits_{k\to\infty}\left[1-\left(\varrho_k^i\right)^2\right]>0$, we obtain that for all $i\in\mathcal{K}$,

$$\lim_{k\to\infty}\|r_k-s_k^i\|=\lim_{k\to\infty}\|S_it_k^i-t_k^i\|=0. \tag{3.13}$$

Finally, let $\bar{v}$ be a weak sequential cluster point of $\{v_k\}$, that is, it has a subsequence $\{v_{k_j}\}$ fulfilling $v_{k_j}\rightharpoonup\bar{v}$ as $j\to\infty$. Since $\lim\limits_{k\to\infty}\xi_k\|v_k-v_{k-1}\|=0$, we get $r_{k_j}\rightharpoonup\bar{v}$ as $j\to\infty$. Applying Lemma 3.6 to (3.13), we deduce that $\bar{v}\in\Phi$. Using Opial's lemma (Lemma 2.3), we can conclude that $\{v_k\}$ weakly converges to an element in Φ. $\square$

4. Application to signal recovery problem

The signal recovery problem consisting of various blurring filters can be expressed as:

$$b_i=A_ix+\varepsilon_i,$$

where $x\in\mathbb{R}^N$ is the original signal, $b_i\in\mathbb{R}^M$ is the observed signal with noise ε_i and $A_i\in\mathbb{R}^{M\times N}$ $(M<N)$ is filter matrix for all $i\in\mathcal{K}$. Then, we focus on the following problem:

$$\begin{aligned}&\min_{x\in\mathbb{R}^N}\frac{1}{2}\|A_1x-b_1\|_2^2+\eta_1\|x\|_1,\\ &\min_{x\in\mathbb{R}^N}\frac{1}{2}\|A_2x-b_2\|_2^2+\eta_2\|x\|_1,\\ &\min_{x\in\mathbb{R}^N}\frac{1}{2}\|A_3x-b_3\|_2^2+\eta_3\|x\|_1,\\ &\qquad\vdots\\ &\min_{x\in\mathbb{R}^N}\frac{1}{2}\|A_Kx-b_K\|_2^2+\eta_K\|x\|_1,\end{aligned} \tag{4.14}$$

where $\eta_i>0$ for all $i\in\mathcal{K}$. By Proposition 3.1 (iii) of [7], this problem can be seen as the problem (1.2) through the following settings: $\mathcal{H}=\mathbb{R}^N$, $F_i=\nabla h_i$, $G_i(\cdot)=\partial\ell_i(\cdot)$ and $S_i(\cdot)=\operatorname{prox}_{\zeta_i\ell_i}(I-\zeta_i\nabla h_i)(\cdot)$, where $\zeta_i>0$, $h_i(\cdot)=\frac{1}{2}\|A_i(\cdot)-b_i\|_2^2$ and $\ell_i(\cdot)=\eta_i\|\cdot\|_1$ for all $i\in\mathcal{K}$. It is known that the mapping F_i is monotone and $\|A_i\|_2^2$-Lipschitz continuous, and G_i is maximal monotone mapping. Besides, the mapping S_i is nonexpansive for $\zeta_i\in\left(0,\frac{2}{\|A_i\|_2^2}\right)$ and hence 0-demicontractive. Numerical experiments are performed by Matlab R2021a and run on an iMac (Apple M1 chip with 16GB of RAM). Set the original signal x is generated by the uniform distribution in $[-2, 2]$ with m nonzero elements. Let A_i be the Gaussian matrix generated by command $randn(M, N)$.

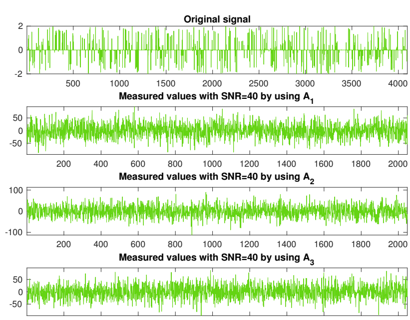
In the first part of the experiment, we investigate the behavior of our algorithm and then compare it with Algorithm 1 of Suantai et al. [23] and Algorithm 2 of Suparatulatorn et al. [24]. We select the signal size to be $N = 4096$ and $M = 2048$. Let the observation b_i be generated by white Gaussian noise with signal-to-noise ratio SNR=40, $\eta_i = 1$ and $\zeta_i = \frac{1}{\|A_i\|_2^2}$ for all $i \in \{1, 2, 3\}$. Let v_0, v_1 be the vectors generated randomly. For Algorithm 1, we set $\rho_k^i = \frac{3}{4}$ for all $k \in \mathbb{N}$ and all $i \in \{1, 2, 3\}$. For Algorithm 2, let $a_k = \frac{1}{k+1}$ and $b_k = \frac{99k}{100(k+1)}$ for all $k \in \mathbb{N}$, and define $\varphi(\cdot) = \frac{\cos(\cdot)}{10}$. For Algorithm 3, let $\alpha_k^i = \frac{1}{4}$, $p_k^i = \frac{1}{(k+1)^{1.4}}$ and $q_k^i = 1 + \frac{1}{k+1}$ for all $k \in \mathbb{N}$ and all $i \in \{1, 2, 3\}$. Further, for Algorithm 2 and Algorithm 3, we suppose $\lambda_i = \frac{95}{100}$, $\gamma_1^i = \frac{1}{100}$ and

$$\xi_k = \begin{cases} \min\left\{\frac{1}{(k+1)^{1.1}\max\{\|v_k - v_{k-1}\|_2, \|v_k - v_{k-1}\|_2^2\}}, \frac{1}{4}\right\} & \text{if } v_k \neq v_{k-1}; \\ \frac{1}{4} & \text{otherwise} \end{cases}$$

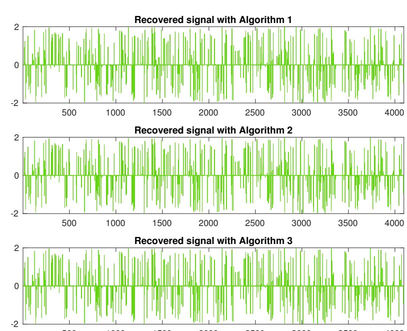
for all $k \in \mathbb{N}$ and all $i \in \{1, 2, 3\}$. We use the mean-squared error to measure the restoration accuracy defined as follows: $MSE_k = \frac{1}{N}\|v_k - x\|_2^2 < 5 \times 10^{-5}$. The results are presented next.

		m Nonzero Elements			
		$m = 64$	$m = 128$	$m = 256$	$m = 512$
Algorithm 1	Iter	2386	2461	2702	2867
	CPU Time	20.4741	21.1546	23.2616	24.6196
Algorithm 2	Iter	749	950	1254	2449
	CPU Time	12.2045	15.3795	21.4745	39.8037
Algorithm 3	Iter	215	230	232	251
	CPU Time	5.1991	5.5572	5.8327	6.0672

TABLE 1. Numerical comparison of three algorithms.



(A) The original signal and the measurements.



(B) The reconstructed signals

FIGURE 1. The original signal, the measurements and the reconstructed signals by three algorithms for $m = 512$.

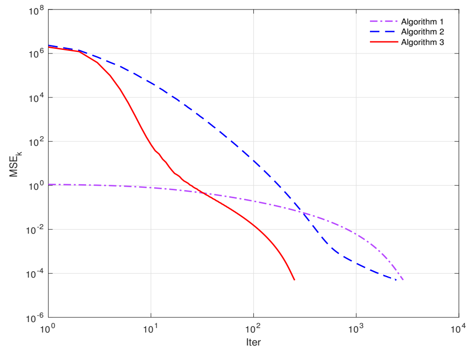


FIGURE 2. Plots of MSE$_k$ over Iter when $m = 512$.

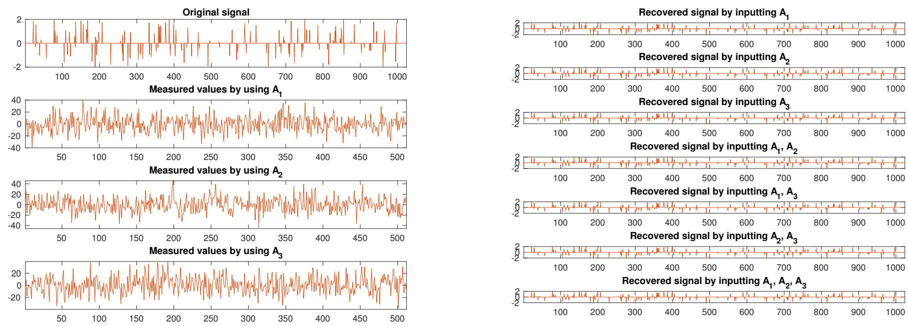
Based on Table 1, Algorithm 3 requires fewer iterations and takes less time than Algorithm 1 and Algorithm 2.

The last part of the experiment considers Algorithm 3 for solving the problem (4.14) with multiple inputs A_i. We select the signal size to be $N = 1024$ and $M = 512$. For any $i \in \{1, 2, 3\}$, let the observation b_i be generated by the white Gaussian noise ε_i of the variance σ_i^2. Set $v_0, v_1, \gamma_1^i, \lambda_i, \eta_i, \zeta_i, \alpha_k^i, p_k^i, q_k^i$ and ξ_k are the same as in the first part of the experiment for all $k \in \mathbb{N}$ and all $i \in \{1, 2, 3\}$. Further, we select $\sigma_i = \frac{i}{100}$ for all $i \in \{1, 2, 3\}$. The results are presented next.

Inputting		m Nonzero Elements			
		$m = 16$	$m = 32$	$m = 64$	$m = 128$
A_1	Iter	1085	1062	1463	2986
	CPU Time	0.7912	0.7415	1.0678	2.1037
A_2	Iter	1049	1027	1402	2460
	CPU Time	0.8564	0.8177	0.9194	1.6673
A_3	Iter	1088	1063	1506	2311
	CPU Time	0.7830	0.7432	0.8780	1.5621
A_1, A_2	Iter	379	384	449	456
	CPU Time	0.7045	0.5749	0.5747	0.7831
A_1, A_3	Iter	364	400	875	477
	CPU Time	0.7210	0.5797	1.2161	0.7123
A_2, A_3	Iter	382	401	435	490
	CPU Time	0.7831	0.6120	0.6199	0.6830
A_1, A_2, A_3	Iter	125	125	123	126
	CPU Time	0.4139	0.2938	0.2731	0.2618

TABLE 2. Numerical results of Algorithm 3.

From Table 2, we can observe that incorporating all 3 Gaussian matrices (A_1, A_2 and A_3) into Algorithm 3 is more effective with respect to time and number of iterations than involving only one or two of them.



(A) The original signal and the measurements. (B) The reconstructed signals

FIGURE 3. The original signal, the measurements and the reconstructed signals by using each input for $m = 128$.

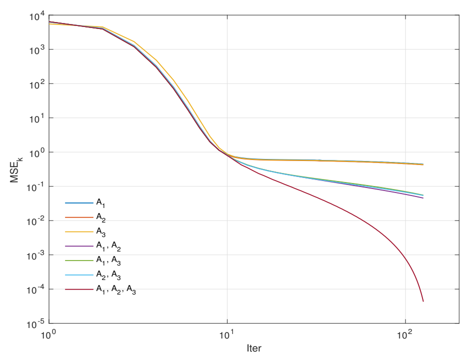


FIGURE 4. Plots of MSE_k over Iter when $m = 128$.

5. NUMERICAL EXAMPLE

We utilize Algorithm 3 to solve the problem (1.2) with $\mathcal{K} = \{1, 2\}$ in the finite-dimensional Hilbert space. Suppose that $\mathcal{H} = \mathbb{R}^2$ with the the Euclidean norm. For any $i \in \{1, 2\}$, define $F_i z = (x+y+\sin x, -x+y+\sin y)^t$ for all $z = (x, y)^t \in \mathcal{H}$, and set $G_i = \partial \iota_{C_i}$, where ι_{C_i} is the indicator function of C_i and $C_i = [-i, i]^2$. It is not hard to show that the mapping F_i is 3-Lipschitz continuous and monotone on $\mathcal{H}$, and the mapping G_i is maximal monotone for all $i \in \{1, 2\}$. For any $z = (x, y)^t \in \mathcal{H}$, define $S_1 z = -\frac{3}{2} z$ and $S_2 z = \|A\|^{-1} A z$, where $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. Then S_1 is $\frac{1}{5}$-demicontractive and S_2 is 0-demicontractive. Furthermore, S_i is continuous and so $I - S_i$ is demiclosed at zero for all $i \in \{1, 2\}$. The solution of our main problem is $x^* = 0$.

In this experiment, we compare Algorithm 3 with Algorithm 1 and Algorithm 2. Let $v_0 = (10^5, 10^5)^t$ and $v_1 = (10^4, 10^4)^t$. For Algorithm 1, we set $\rho_k^i = \frac{3}{4}$ for all $k \in \mathbb{N}$ and all $i \in \{1, 2\}$. For Algorithm 2, select a_k and b_k as in Section 4, and define $\varphi(\cdot) = \frac{1}{10}$. For Algorithm 3, let $\alpha_k^i = \frac{1}{2}$, $p_k^i = \frac{1}{(k+1)^{1.4}}$ and $q_k^i = 1 + \frac{1}{k+1}$ for all $k \in \mathbb{N}$ and all $i \in \{1, 2\}$. Further, for Algorithm 2 and Algorithm 3, we suppose $\lambda_i = \frac{95}{100}$ and $\gamma_1^i = \frac{7}{100}$ for all $i \in$

$\{1, 2\}$, and we set ξ_k as in Section 4. We let the stopping criterion $E_k := \|v_k - x^*\| < 10^{-5}$. The numerical result is presented in Figure 5.

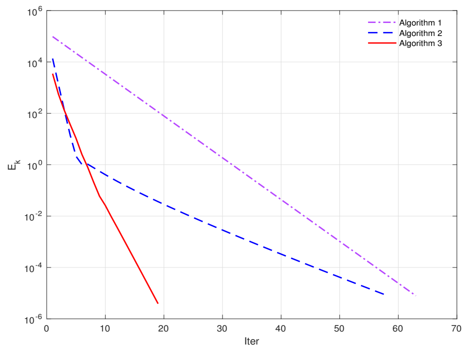


FIGURE 5. Plots of E_k over Iter.

From Figure 5, we can see that the number of iterations of Algorithm 1 is 63, the number of iterations of Algorithm 2 is 58 and the number of iterations of Algorithm 3 is 19, that is, the sequence generated by Algorithm 3 improves the number of iterations.

Acknowledgments. This research was partially supported by Chiang Mai University. R. Suparatulatorn was supported by Post-Doctoral Fellowship of Chiang Mai University, Thailand.

REFERENCES

[1] Adamu, A.; Kitkuan, D.; Padcharoen, A.; Chidume, C. E.; Kumam, P. Inertial viscosity-type iterative method for solving inclusion problems with applications. *Math. Comput. Simul.* **194** (2022), 445–459.

[2] Auslender, A.; Teboulle, M.; Ben-Tiba, S. A logarithmic-quadratic proximal method for variational inequalities. *Computational Optimization*, 31–40, Springer, Boston, MA, 1999.

[3] Bauschke, H. H.; Combettes, P. L. *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*. CMS Books in Mathematics, 408. Springer, New York, 2011.

[4] Brézis, H. Operateurs maximaux monotones. *North-Holland Math. Stud.* **5** (1973), 1–183.

[5] Chidume, C. E.; Adamu, A.; Kumam, P.; Kitkuan, D. Generalized hybrid viscosity-type forward-backward splitting method with application to convex minimization and image restoration problems. *Numer. Funct. Anal. Optim.* **42** (2021), no. 13, 1586–1607.

[6] Cholamjiak, P.; Hieu, D. V.; Cho, Y. J. Relaxed forward-backward splitting methods for solving variational inclusions and applications. *J. Sci. Comput.* **88** (2021), Article no.: 85. https://doi.org/10.1007/s10915-021-01608-7.

[7] Combettes, P. L.; Wajs, V. R. Signal recovery by proximal forward-backward splitting. *Multiscale Model Simul.* **4** (2005), 1168–1200.

[8] Dixit, A.; Sahu, D. R.; Singh, A. K.; Som, T. Application of a new accelerated algorithm to regression problems. *Soft Comput.* **24** (2020), 1539–1552.

[9] Eslamian, M. Strong convergence theorem for common zero points of inverse strongly monotone mappings and common fixed points of generalized demimetric mappings. *Optimization* (2021). https://doi.org/10.1080/02331934.2021.1939341.

[10] Eslamian, M.; Kamandi, A. A novel algorithm for approximating common solution of a system of monotone inclusion problems and common fixed point problem. *J. Ind. Manag. Optim.* (2021). https://doi.org/10.3934/jimo.2021210.

[11] Gibali, A.; Thong, D. V. Tseng type methods for solving inclusion problems and its applications. *Calcolo* **55** (2018), Article no.: 49. https://doi.org/10.1007/s10092-018-0292-1.

[12] Lions, P.-L.; Mercier, B. Splitting algorithms for the sum of two nonlinear operators. *SIAM J. Numer. Anal.* **16** (1979), 964–979.

[13] Liu, H.; Yang, J. Weak convergence of iterative methods for solving quasimonotone variational inequalities. *Comput. Optim. Appl.* **77** (2020), 491–508.

[14] Lorenz, D.; Pock, T. An inertial forward-backward algorithm for monotone inclusions. *J. Math. Imaging Vis.* **51** (2015), 311–325.

[15] Moudafi, A.; Oliny, M. Convergence of a splitting inertial proximal method for monotone operators. *J. Comput. Appl. Math.* **155** (2003), 447–454.

[16] O'Donoghue, B.; Candès, E. J. Adaptive restart for accelerated gradient schemes. *Found. Comput. Math.* **15** (2015), 715–732.

[17] Padcharoen, A.; Kitkuan, D.; Kumam, W.; Kumam, P. Tseng methods with inertial for solving inclusion problems and application to image deblurring and image recovery problems. *Comput. Math. Methods* **3** (2021). https://doi.org/10.1002/cmm4.1088.

[18] Passty, G. B. Ergodic convergence to a zero of the sum of monotone operators in Hilbert space. *J. Math. Anal. Appl.* **72** (1979), 383–390.

[19] Polyak, B.T. Some methods of speeding up the convergence of iteration methods. *USSR Comput. Math. Math. Phys.* **4** (1964), 1–17.

[20] Raguet, H.; Fadili, J.; Peyré, G. A generalized forward-backward splitting. *SIAM J. Imaging Sci.* **6** (2013), 1199–1226.

[21] Sahu, D. R.; Yao, J. C.; Verma, M.; Shukla, K. K. Convergence rate analysis of proximal gradient methods with applications to composite minimization problems. *Optimization* **70** (2021), 75–100.

[22] Suantai, S.; Cholamjiak, W.; Cholamjiak, P. Strong convergence theorems of a finite family of quasi-nonexpansive and Lipschitz multi-valued mappings. *Afr. Mat.* **26** (2015), 345–355.

[23] Suantai, S.; Kankam, K.; Cholamjiak, P.; Cholamjiak, W. A parallel monotone hybrid algorithm for a finite family of G-nonexpansive mappings in Hilbert spaces endowed with a graph applicable in signal recovery. *Comp. Appl. Math.* **40** (2021), Article no.: 145. https://doi.org/10.1007/s40314-021-01530-6.

[24] Suparatulatorn, R.; Cholamjiak, W.; Gibali, A.; Mouktonglang, T. A parallel Tseng's splitting method for solving common variational inclusion applied to signal recovery problems. *Adv. Differ. Equ.* **2021** (2021), Article no.: 492. https://doi.org/10.1186/s13662-021-03647-8.

[25] Suparatulatorn, R.; Chaichana, K. A strongly convergent algorithm for solving common variational inclusion with application to image recovery problems. *Appl. Numer. Math.* **173** (2022), 239–248.

[26] Suparatulatorn, R.; Khemphet, A. Tseng type methods for inclusion and fixed point problems with applications. *Mathematics* **7** (2019), no. 12, Article no.: 1175. https://doi.org/10.3390/math7121175.

[27] Yambangwai, D.; Khan, S. A.; Dutta, H.; Cholamjiak, W. Image restoration by advanced parallel inertial forward-backward splitting methods. *Soft Comput.* **25** (2021), 6029–6042.

[28] Zhou, H.; Qin, X. *Fixed Points of Nonlinear Operators: Iterative Methods*. De Gruyter, Berlin, 2020.

[1] ADVANCED RESEARCH CENTER FOR COMPUTATIONAL SIMULATION, CHIANG MAI UNIVERSITY
CHIANG MAI 50200, THAILAND
Email address: thanasak.m@cmu.ac.th
Email address: kanyuta@hotmail.com

[2] DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, CHIANG MAI UNIVERSITY
CHIANG MAI 50200, THAILAND
Email address: raweerote.s@gmail.com