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In memoriam Professor Charles E. Chidume (1947- 2021)

Inertial Iteration Scheme for Approximating Fixed Points of Lipschitz Pseudocontractive Maps in Arbitrary Real Banach Spaces

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ABSTRACT. We study a perturbed inertial Krasnoselskii-Mann-type algorithm and prove that the algorithm is an approximate fixed point sequence for Lipschitz pseudocontractive maps in arbitrary real Banach spaces. Strong convergence results are then established for our inertial iteration scheme for approximation of fixed points of Lipschitz pseudocontractive maps and solutions of certain important accretive-type operator equations in certain real Banach spaces. Implementation of our algorithm is illustrated using numerical examples in both finite and infinite dimensional Banach spaces. Our results improve rate of convergence and extend several related recent results.

1. INTRODUCTION

Let X be a real Banach with dual X^*. Let $J : X \to X^*$ denote the normalized duality mapping given by

$$Jx := \{f^* \in X^* : \langle x, f^*\rangle = \|x\|^2 = \|f^*\|^2\},$$

where $\langle .,.\rangle$ denotes the generalized duality pairing (see for example [10]). J is single-valued if X^* is strictly convex (see for example [10]) and in what follows we denote single-valued normalized duality mapping by j. A mapping $T : D(T) \subseteq X \to R(T) \subseteq X$ is said to be *L-Lipschitzian*(see for example [10]) if there exists $L \geq 0$ such that

$$||Tx - Ty|| \leq L||x - y||, \ \forall x, y \in D(T). \tag{1.1}$$

T is said to be a *contraction* if $L \in [0, 1)$ and T is said to be *nonexpansive* if $L = 1$ (see for example [3, 6, 10, 28]). T is said to be *pseudocontractive* (see for example [10]) if

$$\|x - y\| \leq \|x - y + \lambda((x - Tx) - (y - Ty))\|, \ \forall x, y \in D(T), \ \lambda > 0. \tag{1.2}$$

An operator $A : D(T) \subseteq X \to R(T) \subseteq X$ is said to be *accretive* (see for example [10]) if for all $t > 0$ and every $x, y \in D(A)$ we have

$$\|x - y\| \leq \|x - y + t(Ax - Ay)\|. \tag{1.3}$$

It follows from Lemma 1.1 of Kato [15] that T is pseudocontractive if and only if for all $x, y \in D(T)$ there exists $j(x - y) \in J(x - y)$ such

$$\langle (I - T)x - (I - T)y, j(x - y)\rangle \geq 0. \tag{1.4}$$

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Furthermore, A is accretive if and only if for each $x, y \in D(A)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Ax - Ay, j(x-y)\rangle \geq 0. \tag{1.5}$$

An accretive operator A is said to be *m-accretive* (see for example [10]) if $R(I+\lambda A) = X,\ \forall \lambda > 0$, where $R(I+\lambda A)$ is the range of $(I+\lambda A)$. The class of nonexpansive maps is a proper subclass of the class of pseudocontractive maps and T is pseudocontractive if and only if $(I-T)$ is accretive. Other important subclasses of pseudocontractive maps are the class of *strictly pseudocontractive operators* and the class of *strongly pseudocontractive operators*. T is strictly pseudocontractive if $\forall x, y \in D(T)$ and for some $k \in [0,1)$, we have

$$\langle (I-T)x - (I-T)y, j(x-y)\rangle \geq k\|(I-T)x-(I-T)y\|^2, \tag{1.6}$$

and T is strongly pseudocontractive if

$$\langle (I-T)x - (I-T)y, j(x-y)\rangle \geq k\|x-y\|^2,\ \forall x, y \in D(T) \text{ and for some } k \in [0,1), \tag{1.7}$$

(see for example [10]). The classes of strictly pseudocontractions and strongly pseudocontractions are independent and every strictly pseudocontraction is Lipschitz (see for example [3, 10]).

In the iterative approximation of fixed points of nonexpansive maps, strictly pseudocontractive maps and strongly pseudocontractive maps, the iterative scheme of Mann [22]:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n,\ n \geq 1; \tag{1.8}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ is a suitable sequence in $[0,1]$ have played pivotal role. In [8] Chidume and Mutangadura showed with an example in the 2-dimensional Euclidean plane that the original Mann iteration may fail to converge to a fixed point of a Lipschitz pseudocontractive selfmap T defined on a nonempty compact subset. For the class of pseudocontractive maps T, the Ishikawa iteration [14]:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T[(1-\beta_n)x_n + \beta_n T x_n],\ n \geq 1, \tag{1.9}$$

and its modifications have played very important role in the iterative approximation of fixed points of T when $F(T) \neq \emptyset$. However, the Ishikawa iteration scheme for Lipschitz pseudocontractions yield only weak convergence usually obtained mostly from $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$; and require "compactness" assumption either on the operator or the domain of the operator or even both to yield strong convergence. Most of the related results are also confined to Hilbert spaces. Often, very strong conditions are imposed on the fixed-point set, $F(T)$ to obtain strong convergence using the usual Ishikawa iteration process (see for example [3, 10, 30, 36]). For instance in [30], the author required that $F(T)$ is finite where T is a continuous pseudocontractive-type self-mapping of a nonempty convex compact of a Hilbert space, and in [36] the authors required that the interior of $F(T)$ is nonempty where T is a Lipschitz pseudocontractive self-mapping of a nonempty closed convex subset of a Hilbert space. Thus many other schemes have been recently studied by several authors to achieve relatively fast strong convergence with mild assumptions on the operator, its domain, its set of fixed points and other necessary components (see for example [1, 2, 3, 7, 10, 11, 12, 13, 16, 17, 18, 19, 20, 21, 23, 26, 27, 29, 31, 34, 35]). The Krasnoselskii-Mann and certain modifications and other prominent algorithms like the Halpern-type algorithms had been used to obtain weak and strong convergence results. In [9] the authors studied the following algorithm in arbitrary real Banach spaces:

Algorithm CZ ([9]). Let X be a real Banach space, K a nonempty closed convex subset of X and $T : K \to K$ a given operator. Then for arbitrary $x_1 \in K$ the sequence $\{x_n\}$ is generated by

(1.10) $$x_{n+1} = (1-\lambda_n)x_n + \lambda_n Tx_n - \lambda_n\theta_n(x_n - x_1),\ n \geq 1,$$

where $\{\lambda_n\}$ and $\{\theta_n\}$ are real sequences in $(0,1)$ such that: (i) $\lambda_n(1+\theta_n) < 1$, (ii) $\lim\limits_{n\to\infty}\theta_n = 0$, (iii) $\sum_{n=1}^{\infty}\lambda_n\theta_n = \infty$, (iv) $\lim\limits_{n\to\infty}\frac{\lambda_n}{\theta_n} = 0$, (v) $\lim\limits_{n\to\infty}\frac{\theta_{n-1}-\theta_n}{\lambda_n\theta_n^2} = 0$. They proved that the sequence $\{x_n\}$ is an approximate fixed point sequence for Lipschhitz pseudocontractive maps, T, in arbitrary real Banach spaces (i.e., $\lim\limits_{n\to\infty}\|x_n - Tx_n\| = 0$). They further obtained strong convergence of the sequence to $p \in F(T)$ when $F(T) \neq \emptyset$ in certain real Banach spaces much more general than Hilbert spaces. The algorithm has been studied extensively in recent years by various authors; it appears simpler than the Ishikawa iteration scheme and the results are proved in general Banach spaces. However, the implementation of the algorithm shows it as being very slow and hence there is a need to accelerate the scheme to achieve relatively fast convergence rate.
It is our purpose in this paper to introduce appropriate inertial term in Algorithm CZ to accelerate the convergence. We consider the following:
Algorithm 1.1. For arbitrary $x_0, x_1 \in X$, the sequence $\{x_n\}$ is generated by

(1.11) $$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}),\ n \geq 1 \\ x_{n+1} = (1-\lambda_n)w_n + \lambda_n Tw_n - \lambda_n\sigma_n(w_n - x_0),\ n \geq 1, \end{cases}$$

where $\{\lambda_n\}$ and $\{\sigma_n\}$ are suitable sequences in $(0,1)$; for some suitable $\theta \in (0,1)$, $0 \leq \theta_n \leq \bar{\theta_n}$, where

$$\bar{\theta_n} = \begin{cases} \min\{\theta, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\}, x_n \neq x_{n-1} \\ \theta,\ \text{otherwise}, \end{cases}$$

and $\{\epsilon_n\} \subseteq \mathbb{R}^+$ is a suitable sequence such that $\sum_{n=1}^{\infty} e_n < \infty$. We prove that under suitable conditions on $\{\lambda_n\}, \{\sigma_n\}, \{\theta_n\}, \{\epsilon_n\}$ and appropriate θ, the perturbed inertial Mann algorithm (1.11) is an approximate fixed point sequence for Lipschitz pseudocontractive maps in arbitrary real Banach spaces.

Interesting fast strong convergence results which generalize and improve several related recent results are proved. Implementation of the algorithm is demonstrated in finite and infinite dimensional Banach spaces with comparison with Algorithm CZ, the scheme without inertial term.

2. Preliminaries

We shall need the following results:

Lemma 2.1. ([15]) *Let X be an arbitrary Banach space, X^* the dual of X and let $x, y \in X$ be arbitrary. Then $\|x\| \leq \|x + \lambda y\|,\ \forall\lambda > 0$ if and only if there exists $j(x) \in J(x)$ such that* $\mathrm{Re}\langle y, j(x) \geq 0$.

Lemma 2.2. ([24]) *Let $\{a_n\}_{n=1}^{\infty}$, $\{\alpha_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of nonnegative real numbers and let $\psi : [0,\infty) \to [0,\infty)$ be a strictly increasing continuous function with $\psi(0) = 0$, $\psi(t) > 0,\ t \in (0,\infty)$. Let*
(i) $\lim\limits_{n\to\infty}\alpha_n = 0$,
(ii) $\sum_{n=1}^{\infty}\alpha_n = \infty$,
(iii) $\lim\limits_{n\to\infty}\frac{b_n}{\alpha_n} = 0$,
(iv) $a_{n+1}^2 \leq a_n^2 - \alpha_n\psi(a_{n+1}) + b_n$.
Then $\lim\limits_{n\to\infty} a_n = 0$.

Lemma 2.3. ([25]) *Let X be a Banach space, K a nonempty closed convex subset of X and $T : K \to X$ a continuous pseudocontractive mapping satisfying the weakly inward condition. Then for each $x_0 \in K$, there exists a unique path*

$$t \to y_t \in K; t \in [0,1), \text{ satisfying } y_t = tTy_t + (1-t)x_0. \tag{2.12}$$

Remark 2.1. If in Lemma 2.3 $F(T) \neq \emptyset$, then $\{y_t\}$ is bounded. Furthermore, if either
(i) X has a uniformly Gâteaux differentialble norm and is such that every closed convex bounded subset of K has the fixed point property (FPP) for nonexpansive self-mappings, or
(ii) X is a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm, then as $t \to 1$, the path converges strongly to a fixed point of T (see [25, 9]).

3. Main results

We now prove the following results:

Theorem 3.1. *Let X be an arbitrary real Banach space and let $T : X \to X$ be an L-Lipschitzian pseudocontractive mapping with $F(T) \neq \emptyset$. Let $\{\lambda_n\}$ and $\{\sigma_n\}$ be real sequence in $(0,1)$ such that (i) $\lambda_n(1+\sigma_n) < 1$, (ii) $\lim\limits_{n\to\infty} \sigma_n = 0$, (iii) $\sum_{n=1}^{\infty} \lambda_n\sigma_n = \infty$, (iv) $\sum_{n=1}^{\infty} \lambda_n^2 < \infty$; (v) $\lim\limits_{n\to\infty} \frac{|\sigma_n - \sigma_{n-1}|}{\lambda_n\sigma_n^2} = 0.$, (vi) $\lim\limits_{n\to\infty} \frac{\lambda_n}{\sigma_n} = 0$. Let $\{\epsilon_n\} \subseteq \mathbb{R}^+$ be such that $\sum_{n=1}^{\infty} \epsilon_n < \infty$, $\lim\limits_{n\to\infty} \frac{\epsilon_n}{\lambda_n\sigma_n} = 0$, and let $\theta \in [0,1)$. For arbitrary $x_0, x_1 \in X$, let $\{x_n\}$ be the sequence generated by* (1.11)*:*

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \ n \geq 1 \\ x_{n+1} = (1-\lambda_n)w_n + \lambda_n Tw_n - \lambda_n\sigma_n(w_n - x_0), \ n \geq 1, \end{cases}$$

where $\theta \in [0,1)$, $0 \leq \theta_n \leq \bar{\theta_n}$ and

$$\bar{\theta_n} = \begin{cases} \min\{\theta, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\}, x_n \neq x_{n-1} \\ \theta, \text{ otherwise.} \end{cases} \tag{3.13}$$

Then $\lim\limits_{n\to\infty} \|x_n - Tx_n\| = 0.$

Proof. We divide the proof as follows:
Claim 1: The sequences $\{x_n\}_{n=1}^{\infty}$, $\{w_n\}_{n=1}^{\infty}$, $\{Tx_n\}_{n=1}^{\infty}$ and $\{Tw_n\}_{n=1}^{\infty}$ are bounded.
Let $p \in F(T)$ be arbitrary and define $G_n : X \to X$ by
$G_n x := Tx - \sigma_n x + \sigma_n x_0, \ n \geq 1$. Then $G_n p = (1-\sigma_n)p + \sigma_n x_0$ and

$$\langle (I - G_n)x - (I-G_n)y, j(x-y)\rangle \geq \sigma_n \|x-y\|^2, \ \forall x, y \in X, \ n \geq 1. \tag{3.14}$$

From (3.14) we obtain

$$\langle (I - G_n - \sigma_n I)x - (I - G_n - \sigma_n I)y, j(x-y)\rangle \geq 0, \ \forall x, y \in X, \ n \geq 1, \tag{3.15}$$

and it follows from Lemma 2.1 that $\forall x, y \in X, \ n \geq 1$ we have

$$\|x - y\| \leq \|x - y + \lambda[(I - G_n - \sigma_n I)x - (I - G_n - \sigma_n I)y]\|. \tag{3.16}$$

From (1.11) we obtain

$$\begin{aligned} w_n &= (1 + \lambda_n + \lambda_n\sigma_n)x_{n+1} + \lambda_n(I - G_n - \sigma_n I)x_{n+1} - \lambda_n w_n \\ &\quad + \lambda_n(G_n x_{n+1} - G_n w_n) + 2\lambda_n^2(w_n - G_n w_n) \end{aligned} \tag{3.17}$$

Furthermore,

$$p = (1 + \lambda_n + \lambda_n\sigma_n)p + \lambda_n(I - G_n - \sigma_n I)p - \lambda_n p - \lambda_n\sigma_n(p - x_0). \tag{3.18}$$

Thus from (3.17) and (3.18) we obtain

$$\begin{aligned}\|w_n - p\| &= \|(1+\lambda_n+\lambda_n\sigma_n)(x_{n+1}-p) + \lambda_n[(I-G_n-\sigma_n I)x_{n+1}\\ &\quad -(I-G_n-\sigma_n I)p] - \lambda_n(w_n-p) + \lambda_n(Gx_{n+1}-G_nw_n)\\ &\quad +2\lambda_n^2(w_n-G_nw_n) + \lambda_n\sigma_n(p-x_0)\|\\ &\geq (1+\lambda_n+\lambda_n\sigma_n)\|x_{n+1}-p\| - \lambda_n\|w_n-p\| - \lambda_n\|Gx_{n+1}-G_nw_n\|\\ &\quad -2\lambda_n^2\|w_n-G_nw_n\| - \lambda_n\sigma_n\|p-x_0\|.\end{aligned}$$

Hence

$$\begin{aligned}(3.19)\ \|x_{n+1}-p\| &\leq \frac{(1+\lambda_n)}{(1+\lambda_n+\lambda_n\sigma_n)}\|w_n-p\| + \frac{1}{(1+\lambda_n+\lambda_n\sigma_n)}\big[\lambda_n\|G_nx_{n+1}-G_nw_n\|\\ &\quad +2\lambda_n^2\|w_n-G_nw_n\| + \lambda_n\sigma_n\|p-x_0\|\big]\\ &\leq (1+\lambda_n)[1-\lambda_n(1+\sigma_n)+\lambda_n^2(1+\sigma_n)^2]\|x_n-p\|\\ &\quad +(1+\lambda_n)\theta_n\|x_n-x_{n-1}\|\\ &\quad +\frac{1}{(1+\lambda_n+\lambda_n\sigma_n)}\big[\lambda_n^2(2+L+\sigma_n)\|w_n-G_nw_n\|\\ &\quad +\lambda_n\sigma_n\|p-x_0\|\big]\\ &\leq (1+\lambda_n)[1-\lambda_n(1+\sigma_n)+\lambda_n^2(1+\sigma_n)^2]\|x_n-p\|\\ &\quad +(1+\lambda_n)\theta_n\|x_n-x_{n-1}\|\\ &\quad +\frac{1}{(1+\lambda_n+\lambda_n\sigma_n)}\big[\lambda_n^2(2+L+\sigma_n)[(1+L+\sigma_n)(\|x_n-p\|\\ &\quad +\theta_n\|x_n-x_{n-1}\|)+\sigma_n\|p-x_0\|] + \lambda_n\sigma_n\|p-x_0\|\big]\\ &\leq \Big[1-\lambda_n\sigma_n+\lambda_n^2[(1+\sigma_n)(\sigma_n+\lambda_n(1+\sigma_n))\\ &\quad +(2+L+\sigma_n)(1+L+\sigma_n)]\Big]\|x_n-p\|\\ &\quad +\big[1+\lambda_n+\lambda_n^2(2+L+\sigma_n)\big]\theta_n\|x_n-x_{n-1}\|\\ &\quad +\lambda_n^2\sigma_n(2+L+\sigma_n)\|p-x_0\| + \lambda_n\sigma_n\|p-x_0\|\\ &= [1-\lambda_n\sigma_n+a_n]\|x_n-p\| + b_n + \lambda_n\sigma_n\|p-x_0\|,\end{aligned}$$

where

$$a_n = \lambda_n^2[(1+\sigma_n)(\sigma_n+\lambda_n(1+\sigma_n)) + (2+L+\sigma_n)(1+L+\sigma_n)],$$

and

$$b_n = \big[1+\lambda_n+\lambda_n^2(2+L+\sigma_n)\big]\theta_n\|x_n-x_{n-1}\| + \lambda_n^2\sigma_n(2+L+\sigma_n)\|p-x_0\|.$$

From (3.19) we obtain

$$\begin{aligned}\|x_{n+1}-p\| &\leq [1-\lambda_n\sigma_n+a_n]\max\{\|x_n-p\|,\|p-x_0\|\} + b_n\\ &\quad +\lambda_n\sigma_n\max\{\|x_n-p\|,\|p-x_0\|\}\\ &\leq [1+a_n]\max\{\|x_n-p\|,\|p-x_0\|\} + b_n\\ &\ \vdots\\ &\leq \prod_{j=1}^{n}(1+a_j)\|p-x_0\| + \sum_{j=1}^{n} b_j < \infty.\end{aligned}$$

Thus $\{x_n\}_{n=1}^\infty$, $\{w_n\}_{n=1}^\infty$, $\{Tx_n\}_{n=1}^\infty$ and $\{Tw_n\}_{n=1}^\infty$ are bounded.

Claim 2: The sequences $\{y_n\}_{n=1}^\infty$ and $\{Ty_n\}_{n=1}^\infty$, where $t_n := \frac{1}{1+\sigma_n}$, and $y_n \equiv y_{t_n} = t_nTy_n + (1-t_n)x_0$ are bounded.

Observe that $\lim_{n\to\infty} t_n = 1$ and since $F(T) \neq \emptyset$, it follows from Proposition 2 of [25] that $\{y_n\}$ is bounded. Since T is Lipschitz pseudocontractive, we have also that $\{Ty_n\}$ is bounded.

Claim 3: $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

$$\begin{aligned}
(3.20)\ \|x_{n+1}-y_n\|^2 &\leq \|x_n-y_n+\theta_n(x_n - x_{n-1})-\lambda_n(w_n - Tw_n)- \lambda_n\sigma_n(w_n - x_0)\|^2 \\
&\leq \|x_n - y_n\|^2 + 2\langle\theta_n(x_n - x_{n-1}) - \lambda_n(w_n - Tw_n) \\
&\quad -\lambda_n\sigma_n(w_n - x_0), j(x_{n+1} - y_n)\rangle \\
&\leq \|x_n - y_n\|^2 - 2\langle\lambda_n(w_n - Tw_n) + \lambda_n\sigma_n(w_n - x_0), j(x_{n+1} - y_n)\rangle \\
&\quad +2\theta_n\|x_n - x_{n-1}\|\|x_{n+1} - y_n\| \\
&= \|x_n - y_n\|^2 - 2\lambda_n\sigma_n\|x_{n+1} - y_n\|^2 \\
&\quad +2\lambda_n\langle\sigma_n(x_{n+1} - y_n) - (w_n - Tw_n) - \sigma_n(w_n - x_0), j(x_{n+1} - y_n)\rangle \\
&\quad +2\theta_n\|x_n - x_{n-1}\|\|x_{n+1} - y_n\| \\
&= \|x_n - y_n\|^2 - 2\lambda_n\sigma_n\|x_{n+1} - y_n\|^2 \\
&\quad +2\lambda_n\langle\sigma_n(x_{n+1} - w_n) + \sigma_n(x_0 - y_n) - (w_n - Tw_n), j(x_{n+1} - y_n)\rangle \\
&\quad +2\theta_n\|x_n - x_{n-1}\|\|x_{n+1} - y_n\| \\
&= \|x_n - y_n\|^2 - 2\lambda_n\sigma_n\|x_{n+1} - y_n\|^2 \\
&\quad +2\lambda_n\langle\sigma_n(x_{n+1} - w_n) + \sigma_n(x_0 - y_n) - (y_n - Ty_n) \\
&\quad -(x_{n+1} - Tx_{n+1} - (y_n - Ty_n)) + x_{n+1} - Tx_{n+1} \\
&\quad -(w_n - Tw_n), j(x_{n+1} - y_n)\rangle + 2\theta_n\|x_n - x_{n-1}\|\|x_{n+1} - y_n\|
\end{aligned}$$

Observe that $\sigma_n(x_0 - y_n) - (y_n - Ty_n) = 0$, and the pseudocontractive property of T implies that $\langle x_{n+1} - Tx_{n+1} - (y_n - Ty_n), j(x_{n+1} - y_n)\rangle \geq 0$. Thus it follows from (3.20) that

$$\begin{aligned}
(3.21)\ \|x_{n+1} - y_n\|^2 &\leq \|x_n-y_n\|^2 - 2\lambda_n\sigma_n\|x_{n+1} - y_n\|^2 \\
&\quad +2\lambda_n\sigma_n\|x_{n+1} - w_n\|\|x_{n+1} - y_n\| \\
&\quad +2\lambda_n\|x_{n+1} - w_n - (Tx_{n+1} - Tw_n)\|\|x_{n+1} - y_n\| \\
&\quad +2\theta_n\|x_n - x_{n-1}\|\|x_{n+1} - y_n\| \\
&\leq \|x_n - y_n\|^2 - 2\lambda_n\sigma_n\|x_{n+1} - y_n\|^2 \\
&\quad +[2\lambda_n\sigma_n + 2\lambda_n(1 + L)]\|x_{n+1} - w_n\|\|x_{n+1} - y_n\| \\
&\quad +2\theta_n\|x_n - x_{n-1}\|\|x_{n+1} - y_n\| \\
&\leq \|x_n - y_{n-1}\|^2 + 2\|y_{n-1} - y_n\|\|x_n - y_n\| - 2\lambda_n\sigma_n\|x_{n+1} - y_n\|^2 \\
&\quad +\lambda_n[2\lambda_n(\sigma_n + 1 + L)\|w_n - G_nw_n\| \\
&\quad +2\frac{\theta_n}{\lambda_n}\|x_n - x_{n-1}\|]\|x_{n+1} - y_n\|
\end{aligned}$$

Since T is pseudocontractive, then $\langle (I - T)x - (I - T)y, j(x - y)\rangle \geq 0$ and hence

$$\|x - y\| \leq \|x - y + \lambda((I - T)x - (I - T)y)\|,\ \forall x, y \in X,\ \lambda > 0.$$

Thus

$$
\begin{aligned}
(3.22)\qquad \|y_n - y_{n-1}\| &\le \|y_n - y_{n-1} + \frac{1}{\sigma_n}[(I-T)y_n - (I-T)y_{n-1}]\| \\
&= \frac{1}{\sigma_n}\|\sigma_n(y_n - y_{n-1}) + y_n - Ty_n - y_{n-1} + Ty_{n-1}\| \\
&= \frac{1}{\sigma_n}\|(1+\sigma_n)y_n - (1+\sigma_n)y_{n-1} - Ty_n + Ty_{n-1}\| \\
&= \frac{1}{\sigma_n}\|Ty_n + \frac{(1-t_n)}{t_n}x_0 - (1+\sigma_{n-1})y_{n-1} \\
&\quad +(1+\sigma_{n-1})y_{n-1} - (1+\sigma_n)y_{n-1} - Ty_n + Ty_{n-1}\| \\
&= \frac{1}{\sigma_n}\|\sigma_n x_0 - \sigma_{n-1}x_0 - (\sigma_n - \sigma_{n-1})y_{n-1}\| \\
&= \frac{1}{\sigma_n}|\sigma_n - \sigma_{n-1}|\|x_0 - y_{n-1}\| \\
&\le \frac{1}{\sigma_n}|\sigma_n - \sigma_{n-1}|[\|x_0\| + \|y_{n-1}\|].
\end{aligned}
$$

Using (3.22) in (3.21) yields

$$
\begin{aligned}
(3.23)\qquad \|x_{n+1} - y_n\|^2 &\le \|x_n - y_{n-1}\|^2 - 2\lambda_n\sigma_n\|x_{n+1} - y_n\|^2 \\
&\quad +2\lambda_n\sigma_n\big[\frac{1}{\lambda_n\sigma_n^2}|\sigma_n - \sigma_{n-1}|(\|x_0\| + \|y_{n-1}\|)\|x_n - y_n\| \\
&\quad +[\frac{\lambda_n}{\sigma_n}(\sigma_n + 1 + L)\|w_n - G_n w_n\| \\
&\quad +\frac{\theta_n}{\lambda_n\sigma_n}\|x_n - x_{n-1}\|]\|x_{n+1} - y_n\|\big]
\end{aligned}
$$

Since $\{x_n\}, \{y_n\}, \{w_n\}, \{Tw_n\}, \{G_n w_n\}$ and $\{\sigma_n\}$ are bounded, it follows from the conditions on $\{\lambda_n\}, \{\sigma_n\}, \{\theta_n\}$ and $\{\epsilon_n\}$ and Lemma 2.2 that $\lim_{n\to\infty}\|x_n - y_{n-1}\| = 0$. Thus $\lim_{n\to\infty}\|x_n - y_n\| = 0$. Finally we proof

Claim 4: $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$.

Since $0 \le \|y_n - Ty_n\| = (1-t_n)\|x_0 - Ty_n\| \to 0$ as $n \to \infty$, we obtain that

$$0 \le \|x_n - Tx_n\| \le (1+L)\|x_n - y_n\| + \|y_n - Ty_n\| \to 0 \text{ as } n \to \infty.$$

□

Prototype for our iteration parameters are: $\lambda_n = \frac{1}{(n+1)^a}$; $\sigma_n = \frac{1}{(n+1)^b}$; where $0 < b < a < 1; a+b < 1; 2a > 1$ (in particular $\lambda_n = \frac{1}{2(n+1)^{\frac{3}{5}}}, \sigma_n = \frac{1}{(n+1)^{\frac{1}{5}}}$); $\epsilon_n = \frac{1}{(n+1)^2}$, and $\theta = \frac{1}{2}$.

Theorem 3.2. *Let X be a real Banach space with a uniformly Gâteaux differentiable norm, and let $T : X \to X$ be an L-Lipschitzian pseudocontractive map with $F(T) \neq \emptyset$. Let every nonempty closed convex and bounded subset C of X have the fixed point property (FPP) for nonexpansive selfmaps and let $\{\lambda_n\}$, $\{\sigma_n\}, \{\epsilon_n\}, \theta, \{\theta_n\}$, and $\{\bar{\theta_n}\}$ be as in Theorem 3.1. Then the sequence $\{x_n\}$ generated from arbitrary $x_0, x_1 \in X$ by (1.11) converges strongly to a fixed point of T.*

Proof. It follows from Theorem 1 of [25] that $\{y_n\}$ converges to a fixed point $p \in F(T)$ and since $\lim_{n\to\infty}\|x_n - y_n\| = 0$, we obtain that $\{x_n\}$ converges strongly to $p \in F(T)$. □

Corollary 3.1. *Let X be a real uniformly smooth Banach space and let $T : X \to X$ be an L-Lipschitzian pseudocontractive map with $F(T) \neq \emptyset$. Let $\{\lambda_n\}$, $\{\sigma_n\}, \{\epsilon_n\}, \theta, \{\theta_n\}$, and $\{\bar{\theta_n}\}$*

be as in Theorem 3.1. Then the sequence $\{x_n\}$ generated from arbitrary $x_0, x_1 \in X$ by (1.11) converges strongly to a fixed point of T.

Proof. Every uniformly smooth Banach space has a Gâteaux differentiable norm and every nonempty closed convex bounded subset C of X has the fixed point property for nonexpansive selfmap (see for example [33]). □

Theorem 3.3. *Let X be a real reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm, and let $T : X \to X$ be a nonexpansive map with $F(T) \neq \emptyset$. Let $\{\lambda_n\}$, $\{\sigma_n\}, \{\epsilon_n\}, \theta, \{\theta_n\}$, and $\{\bar{\theta_n}\}$ be as in Theorem 3.1. Then the sequence $\{x_n\}$ generated from arbitrary $x_0, x_1 \in X$ by (1.11) converges strongly to a fixed point of T.*

Proof. Follows as in the proof Theorem 3.2 using Theorem 2 of [25]. □

Corollary 3.2. *Let X be a real Banach space with a uniformly Gâteaux differentiable norm and such that every nonempty closed convex and bounded subset C of X have the fixed point property (FPP) for nonexpansive selfmaps. Let $A : X \to X$ be an accretive operator with $A^{-1}0 = N(A) = \{x \in X : Ax = 0\} \neq \emptyset$ and let $\{\lambda_n\}$, $\{\sigma_n\}, \{\epsilon_n\}, \theta, \{\theta_n\}$, and $\{\bar{\theta_n}\}$ be as in Theorem 3.1. For arbitrary $x_0, x_1 \in X$, let $\{x_n\}$ be the sequence generated by*

$$(3.24) \qquad \begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \ n \geq 1 \\ x_{n+1} = w_n - \lambda_n A w_n - \lambda_n \sigma_n (w_n - x_0), \ n \geq 1. \end{cases}$$

Then $\{x_n\}$ converges strongly to a point $x* \in A^{-1}0$.

Corollary 3.3. *Let X be a real uniformly smooth Banach space Let $A : X \to X$ be an accretive operator with $A^{-1}0 = N(A) = \{x \in X : Ax = 0\} \neq \emptyset$ and let $\{\lambda_n\}$, $\{\sigma_n\}, \{\epsilon_n\}, \theta, \{\theta_n\}$, and $\{\bar{\theta_n}\}$ be as in Theorem 3.1. Then the sequence $\{x_n\}$ generated from arbitrary $x_0, x_1 \in X$ by (3.12) converges strongly to a point $x* \in A^{-1}0$.*

Corollary 3.4. *Let X be a real reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm. Let $A : X \to X$ be an accretive operator with $A^{-1}0 = N(A) = \{x \in X : Ax = 0\} \neq \emptyset$. and let $\{\lambda_n\}$, $\{\sigma_n\}, \{\epsilon_n\}, \theta, \{\theta_n\}$, and $\{\bar{\theta_n}\}$ be as in Theorem 3.1. Then the sequence $\{x_n\}$ generated from arbitrary $x_0, x_1 \in X$ by (3.12) converges strongly to a point $x* \in A^{-1}0$.*

4. NUMERICAL EXAMPLES

In this section, numerical illustration of the convergence of the iterative scheme 1.11 and iterative scheme CZ discussed in this paper are presented. The setting for the numerical example is that of a real Hilbert space. Using different examples, we show graphically and with a table of numerical values the convergence results discussed in this paper.

All codes are written in MATLAB, and implemented using an HP Elitebook 6930p computer with Pentium(R) DUAL-CORE CPU T4200 with 2.00Hz and 2GB RAM.

Example 4.1. Let $\mathbb{R}^2$ denote the 2-dimensional Euclidean plane. Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$Tx = T((x_1, x_2)) = (x_1, x_2) + (x_2, -x_1) = (x_1 + x_2, x_2 - x_1),$$

for each $x = (x_1, x_2) \in \mathbb{R}^2$. Then T is 2-Lipschitz pseudocontractive.
Let $x_0 = (-0.01, -0.01); x_1 = (1, 1)$; $\lambda_n = \frac{1}{(n+3)^{0.501}}, \sigma_n = \frac{1}{(n+3)^{0.2}}; \epsilon_n = \frac{1}{(n+3)^{1.01}}$, and $\theta = 0.5$. Then algorithm (1.11) converges strongly to $p = (0, 0)$ which is the only fixed point of T. It is shown in [8] that the original Mann algorithm (1.8) [22] fails to converge to the fixed point of T. Observe that algorithm (1.11) converges much faster than Algorithm CZ (see Table 1 and Figure 1 below).

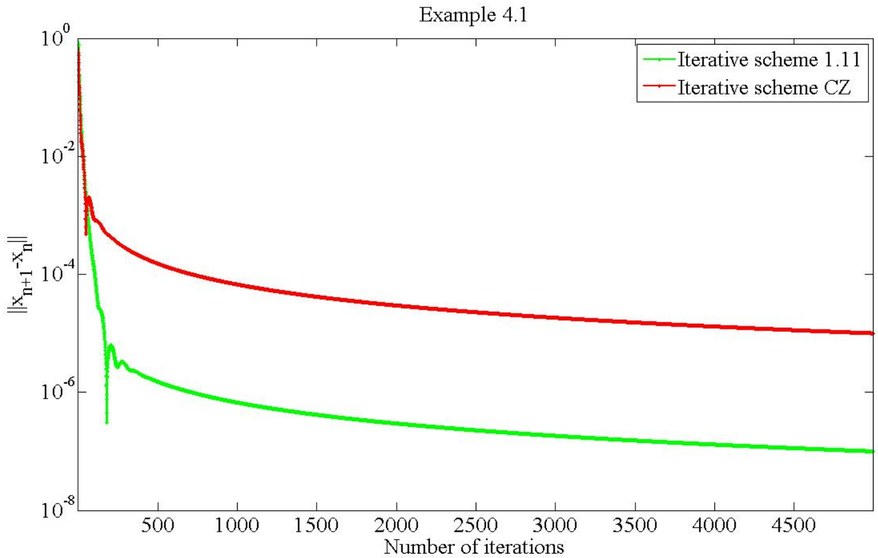

FIGURE 1. Graph showing the convergence of iterative schemes

No. of iter.		Iterative scheme 1.11		Iterative scheme CZ	
n	Time (Secs.)	Sequence $x_n = (x_n^1, x_n^2)$	$\|x_{n+1} - x_n\|$	Sequence $x_n = (x_n^1, x_n^2)$	$\|x_{n+1} - x_n\|$
1	0.181851	(1.214830, 0.129164)	0.896943	(1.499307, 0.500693)	0.706127
2	0.202965	(0.849417,-0.533430)	0.756676	(1.561283, -0.007158)	0.511619
592	1.300140	(-0.003305, 0.001887)	0.000001	(0.330551, -0.189437)	0.000124
593	1.304350	(-0.003304, 0.001886)	0.000001	(0.330432, -0.189404)	0.000124
2537	4.359200	(-0.002412, 0.001589)	0.000000	(0.241209, -0.159220)	0.000022
2538	4.361230	(-0.002412, 0.001589)	0.000000	(0.241188, -0.159211)	0.000022
4992	8.265880	(-0.002082, 0.001446)	0.000000	(0.208156, -0.144830)	0.000010
4993	8.267110	(-0.002082, 0.001446)	0.000000	(0.208147, -0.144826)	0.000010

TABLE 1. Table showing some terms of the sequence generated by iterative scheme 1.11 and CZ, values of $\|x_{n+1} - x_n\|$ and CPU time for the indicated values of n

Example 4.2. Let $X = \ell_2(\mathbb{R}) = \{x = \{x_i\}_{i=1}^{\infty} : x_i \in \mathbb{R} \text{ and } \sum_{i=1}^{\infty} |x_i|^2 < \infty\}$. Define $T : X \to X$ by $Tx = (0, -3x_2, -3x_3, \ldots)$. Then T is $\frac{1}{2}-$ strictly pseudocontractive and hence Lipschitz pseudocontractive. Let $x_0 = (-0.01, -0.01, -0.01, 0, 0, 0, \ldots); x_1 = (1, 1, 1, 0, 0, 0, \ldots)$,
$\lambda_n = \frac{1}{(n+3)^{0.51}}, \sigma_n = \frac{1}{(n+3)^{0.2}}; \epsilon_n = \frac{1}{(n+3)^{1.01}}$, and $\theta = 0.1$. Then algorithm (3.13) converges strongly to $p = (0, 0, 0 \ldots)$ which is the only fixed point of T. Observe that algorithm (1.11) converges much faster than Algorithm CZ (see Table 2 and Figure 2 below).

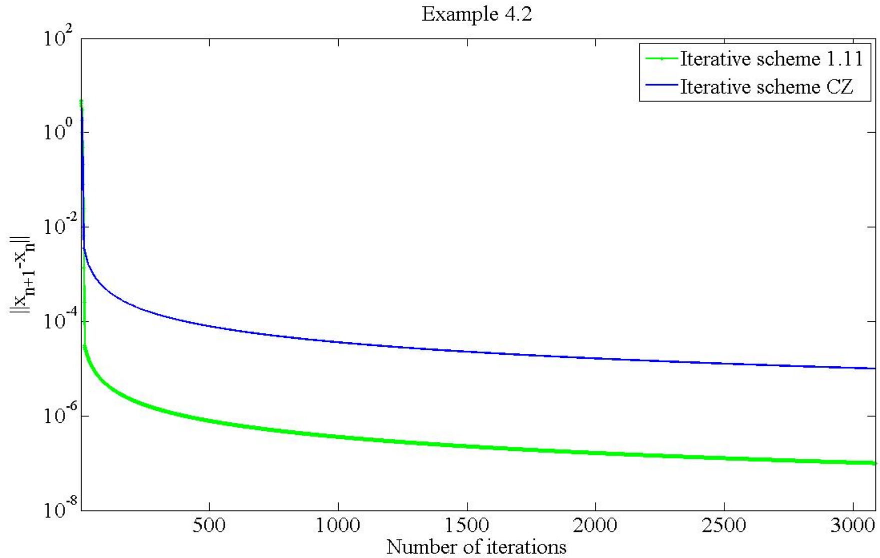

FIGURE 2. Graph showing the convergence of iterative schemes

No. of iter.		Iterative scheme 1.11	Iterative scheme CZ
n	Time (Secs.)	$\|x_{n+1} - x_n\|$	$\|x_{n+1} - x_n\|$
1	0.006275	3.618540	2.832740
2	0.008063	4.572730	3.311260
548	0.679860	0.000001	0.000071
549	0.680848	0.000001	0.000071
1999	2.489850	0.000000	0.000016
2000	2.490580	0.000000	0.000016
3083	3.872630	0.000000	0.000010
3084	3.873390	0.000000	0.000010

TABLE 2. Table showing some terms of the sequence generated by iterative scheme 1.11 and CZ, values of $\|x_{n+1} - x_n\|$ and CPU time for the indicated values of n

In the two examples given above the stopping criteria is $\|x_{n+1} - x_n\| \leq 10^{-7}$. This implies that the error of approximating the fixed point of the given maps is negligible. From Figure 1 and Table 1, it is clearly seen that as consecutive terms of the sequence get close enough (as close as a difference of 10^{-7}), the sequence generated is seen to approach the fixed point of the map which is 0. A lower stopping will make no much difference while a higher stopping criteria will truncate the computation too early which might leads to higher computational error. Clearly, the error = $\|x_n - 0\|$ tends to 0 as n tends to ∞ which shows that the convergence of the sequence generated by the algorithms converges. The same holds for Figure 2 and Table 2.

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