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In memoriam Professor Charles E. Chidume (1947- 2021)

On normalized distance Laplacian eigenvalues of graphs and applications to graphs defined on groups and rings

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ABSTRACT. The normalized distance Laplacian matrix of a connected graph G, denoted by $D^{\mathcal{L}}(G)$, is defined by $D^{\mathcal{L}}(G) = Tr(G)^{-1/2}D^L(G)Tr(G)^{-1/2}$, where $D(G)$ is the distance matrix, the $D^L(G)$ is the distance Laplacian matrix and $Tr(G)$ is the diagonal matrix of vertex transmissions of G. The set of all eigenvalues of $D^{\mathcal{L}}(G)$ including their multiplicities is the normalized distance Laplacian spectrum or $D^{\mathcal{L}}$-spectrum of G. In this paper, we find the $D^{\mathcal{L}}$-spectrum of the joined union of regular graphs in terms of the adjacency spectrum and the spectrum of an auxiliary matrix. As applications, we determine the $D^{\mathcal{L}}$-spectrum of the graphs associated with algebraic structures. In particular, we find the $D^{\mathcal{L}}$-spectrum of the power graphs of groups, the $D^{\mathcal{L}}$-spectrum of the commuting graphs of non-abelian groups and the $D^{\mathcal{L}}$-spectrum of the zero-divisor graphs of commutative rings. Several open problems are given for further work.

1. INTRODUCTION

In this article, all graphs are connected, simple, undirected and finite. A graph is denoted by $G(V, E)$ (or simply G), where V and E are its vertex and edge set, respectively. The cardinality of V is the *order* n of G and the cardinality of E is the *size* m of G. The *degree* of a vertex $v \in V$ in G is the number of edges incident with v and is denoted by $d_G(v)$ (or simply d_v). The *neighborhood* of a vertex v, denoted by $N(v)$, is the set of vertices of G adjacent to v, so that $d_v = |N(v)|$. A graph is called r-*regular* if the degree of every vertex is r. For other undefined notations, see [18].

The *adjacency matrix* $A(G)$ of graph G is a $(0, 1)$-square matrix of order n with rows and columns indexed by vertices, where (i, j)-entry is 1, if v_i is adjacent to v_j and 0, otherwise, $1 \leq i, j \leq n$. Let $Deg(G)$ be the diagonal matrix of vertex degrees d_i. The real symmetric matrices $L(G) = Deg(G) - A(G)$ and $Q(G) = Deg(G) + A(G)$ are called the *Laplacian matrix* and the *signless Laplacian matrix* of G, respectively. More about these matrices can be seen in [18].

The *normalized Laplacian matrix* of a connected graph G is a square matrix of order n, defined by

$$\mathcal{L}(G) = (l_{ij})_{n\times n} = \begin{cases} 1 & i = j, \\ \frac{-1}{\sqrt{d_{v_i} d_{v_j}}} & i \neq j, \end{cases}$$

where d_{v_i} is the degree of vertex v_i and $1 \leq i, j \leq n$. The relation of the adjacency matrix and the Laplacian matrix of G with the normalized Laplacian matrix is given below

$$\mathcal{L}(G) = Deg(G)^{-1/2}L(G)Deg(G)^{-1/2} = I - Deg(G)^{-1/2}A(G)Deg(G)^{-1/2},$$

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where $Deg(G)^{-1/2}$ is the diagonal matrix of square root of vertex degrees of G. For some recent work on the normalized Laplacian, see [19, 20, 50, 26] and for the normalized Laplacian Estrada index, see [47, 48].

For two vertices u and v in a connected graph G, the *distance* between u and v, denoted by $d(u,v)$, is the length of a shortest path between them. The *distance matrix* $D(G)$ of a connected G, is defined as $D(G) = (d(u,v))_{u,v\in V(G)}$, if $u \neq v$ and is taken as zero when u coincides with v. The *transmission* (or transmission degree) $Tr_G(v)$(if the graph is clear from the context then we write Tr_i instead of $Tr_G(i)$) of the vertex v is defined to be the sum of the distances from v to all other vertices of G, that is, $Tr_G(v) = \sum_{u\in V(G)} d(u,v)$. We note that $Tr_G(v)$ is same as the v-th row sum of the matrix $D(G)$. Let $Tr(G)$ be the diagonal matrix of row sums of G. The *Weiner index* $W(G)$ of G is the sum of distances between all unordered pairs of vertices. The distance Laplacian matrix is denoted by $D^L(G)$ and is defined as $D^L(G) = Tr(G) - D(G)$. It immediately follows that $D^L(G)$ is a real symmetric and positive semi-definite matrix. Similarly the matrix $D^Q(G) = Tr(G) + D(G)$ is called the distance signless Laplacian matrix of G. More about these matrices can be seen in [8, 9, 22, 23, 39].

Recently, Reinhart [46] put forward the concept of the normalized distance Laplacian matrix of a connected graph G and defined as

$$D^{\mathcal{L}}(G) = (d_{ij})_{n\times n} = \begin{cases} 1 & \text{if } i = j, \\ \frac{-d(v_i,v_j)}{\sqrt{Tr(v_i)Tr(v_j)}} & \text{if } i \neq j, \end{cases}$$

where $1 \leq i, j \leq n$. Just as the normalized Laplacian matrix can be represented in terms of Laplacian and adjacency matrix of a graph, the normalized distance Laplacian matrix is equivalently defined in terms of distance matrix and distance Laplacian matrix as follows

$$D^{\mathcal{L}}(G) = Tr(G)^{-1/2}D^L(G)Tr(G)^{-1/2} = I - Tr(G)^{-1/2}D(G)Tr(G)^{-1/2}.$$

The normalized distance Laplacian matrix is real symmetric positive semi-definite and its eigenvalues are ordered as $\rho_1^{\mathcal{L}} \geq \rho_2^{\mathcal{L}} \geq \cdots \geq \rho_{n-1}^{\mathcal{L}} \geq \rho_n^{\mathcal{L}} = 0$, where $\rho_1^{\mathcal{L}}$ is known as the distance normalized Laplacian spectral radius of G. The set of all eigenvalues of the matrix $D^{\mathcal{L}}(G)$ together with their multiplicities is called the normalized distance Laplacian spectrum or $D^{\mathcal{L}}$-spectrum of G. For some spectral properties of the normalized distance Laplacian matrix, we refer to [46, 30, 24].

There are various types of graphs defined on groups, rings, vector spaces, like Cayley graphs [12], intersection graphs, commuting graphs [10], primes graphs, power graphs [32], zero divisor graphs [14] and several other types of graphs defined on algebraic structures. In fact algebra and discrete mathematics are two vital disciplines of mathematics, they interact in several ways by extending tools from each other. These types of graphs have numerous applications and are well studied, see [33, 7]. Researchers have studied various spectral properties of algebraic graphs for different types of matrices and interesting results were obtained, [40, 38, 15, 16, 29, 28, 35].

The rest of the paper is organised as: in Section 2, we find the $D^{\mathcal{L}}$-spectrum of joined union of regular graphs in terms of the adjacency spectrum of the respective components. As applications to the results in Section 2, we determine the $D^{\mathcal{L}}$-eigenvalues of power graphs of groups, the commuting graphs of non-abelain groups and the zero-divisors of the commutative ring $\mathbb{Z}_n$ in Section 3, Section 4 and Section 5, respectively.

2. Normalized Laplacian eigenvalues of the joined union of graphs

In this section, we find the $D^{\mathcal{L}}$-spectrum of the joined union of regular graphs in terms of the adjacency spectrum of its components and an auxiliary matrix determined by the structure of the graph. We also obtain the $D^{\mathcal{L}}$-spectrum of the join of regular graphs. Further, the $D^{\mathcal{L}}$-spectrum of some well known families of graphs are determined.

Consider the matrix

$$M = \begin{pmatrix} m_{1,1} & m_{1,2} & \cdots & m_{1,s} \\ m_{2,1} & m_{2,2} & \cdots & m_{2,s} \\ \vdots & \vdots & \ddots & \vdots \\ m_{s,1} & m_{s,2} & \cdots & m_{s,s} \end{pmatrix}_{n\times n},$$

whose rows and columns are partitioned according to a partition $S = \{S_1, S_2, \ldots, S_m\}$ of the index set $X = \{1, 2, \ldots, s\}$. The quotient matrix $\mathcal{Q}$ of M is the $s \times s$ matrix whose (i, j)-th entry is the average row sum of the block $a_{i,j}$. The partition S is said to be *regular* if each block $a_{i,j}$ of M has constant row (and column) sum and in this case the matrix $\mathcal{Q}$ is called a *regular quotient matrix*. In general, the eigenvalues of $\mathcal{Q}$ are interlaced by the eigenvalues of M, while for regular partitions [18], each eigenvalue of $\mathcal{Q}$ is an eigenvalue of M.

Let $G(V, E)$ be a graph of order n and $G_i(V_i, E_i)$ be graphs of order n_i, where $i = 1, \ldots, n$. The *joined union* $G[G_1, \ldots, G_n]$ of graphs $G_1, G_2, \ldots, G_n$ with respect to graph G is the graph $H(W, F)$ with vertex set W and edge set F defined as

$$W = \bigcup_{i=1}^{n} V_i \text{ and } F = \bigcup_{i=1}^{n} E_i \bigcup \Big(\bigcup_{\{v_i,v_j\}\in E} V_i \times V_j \Big).$$

Equivalently, the joined union $G[G_1, \ldots, G_n]$ of graphs $G_1, G_2, \ldots, G_n$ with respect to graph G is obtained by replacing each vertex v_i of G by the graph G_i and joining edges from each vertex of G_i to every vertex of G_j whenever v_i and v_j are adjacent in G. It is clear that order of the graph $G[G_1, \ldots, G_n]$ is $\sum_{i=1}^{n} n_i$ and its size is $\sum_{i=1}^{n} m_i + \sum_{v_i\sim v_j} n_i n_j$, where $v_i \sim v_j$ means that v_i adjacent to v_j. It is clear from the definition that the usual join $G_1 \nabla G_2$ is a particular case of the joined union and can be written as $K_2[G_1, G_2]$. The following example illustrates the joined union of three graphs.

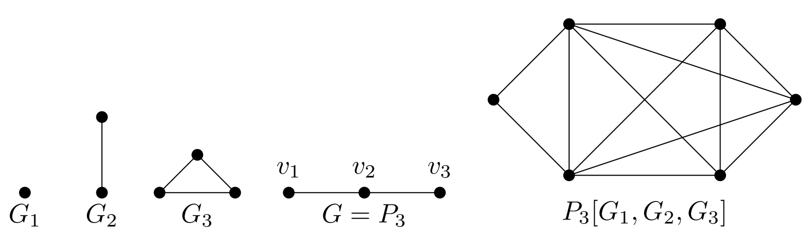

FIGURE 1. Joined union of three graphs

The following result gives the $D^{\mathcal{L}}$-spectrum of $G[G_1, \ldots, G_n]$, when each G_i is r_i-regular graph of order n_i.

Theorem 2.1. *Let G be a connected graph of order $n \geq 2$ and size m and let G_i be r_i-regular connected graphs of order n_i having adjacency eigenvalues $\lambda_{i1} = r_i \geq \lambda_{i2} \geq \ldots \geq \lambda_{in_i}$, where*

$i = 1, 2, \dots, n$. Then the normalized distance Laplacian eigenvalues of $G[G_1, \dots, G_n]$ are

$$1 + \frac{1}{T_i}\Big(2 + \lambda_{ik}(G_i)\Big), \text{ for } i = 1, \dots, n \text{ and } k = 2, 3, \dots, n_i,$$

where $T_i = 2n_i - 2 - r_i + n_i^{'}$, and $n_i^{'} = \sum\limits_{k=1, k\neq i}^{n} n_k d_G(v_i, v_k)$. The other n normalized distance Laplacian eigenvalues are given by the eigenvalues of the following regular quotient matrix

$$M = \begin{pmatrix} d_1^{'} & \frac{-n_2 d_G(v_1,v_2)}{\sqrt{T_1T_2}} & \dots & \frac{-n_n d_G(v_1,v_n)}{\sqrt{T_1T_n}} \\ \frac{-n_1 d_G(v_2,v_1)}{\sqrt{T_1T_2}} & d_2^{'} & \dots & \frac{-n_n d_G(v_2,v_n)}{\sqrt{T_2T_n}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{-n_1 d_G(v_n,v_1)}{\sqrt{T_1T_n}} & \frac{-n_2 d_G(v_n,v_2)}{\sqrt{T_2T_n}} & \dots & d_n^{'} \end{pmatrix}, \tag{2.1}$$

where $d_i^{'} = 1 - \frac{1}{T_i}\Big(2n_i - r_i - 2\Big)$.

Proof. Let $V(G) = \{v_1, \dots, v_n\}$ be the vertex set of G and let $V(G_i) = \{v_{i1}, \dots, v_{in_i}\}$ be the vertex set of G_i, for $i = 1, 2, \dots, n$. Let $\mathcal{H} = G[G_1, \dots, G_n]$ be the joined union of r_i-regular graphs G_i, where $i = 1, 2, \dots, n$. Clearly order of $\mathcal{H}$ is $\mathcal{N} = \sum\limits_{i=1}^{n} n_i$. As G_i is r_i-regular graph, the transmission degree of every vertex $v_{ij} \in V(\mathcal{H})$, for $1 \leq i \leq n$ and $1 \leq j \leq n_i$, is equal to the distance of j-th vertex in G_i to all other vertices of G, which is further equal to the distance within vertices of G_i and the distance of vertices outside of G_i. Thus,

$$\begin{aligned} Tr(v_{ij}) &= d_{G_i}(v_{ij}) + 2(n_i - 1 - d_{G_i}(v_{ij})) + n_1 d(v_i, v_1) + n_2 d(v_i, v_2) + \dots + n_n d(v_i, v_n) \\ &= 2n_i - 2 - d(v_{ij}) + n_i^{'} = 2n_i + n_i^{'} - r_i - 2. \end{aligned}$$

Further, we note that $Tr(v_{i1}) = Tr(v_{i2}) = \dots = Tr(v_{in_i}) = T_i$ (say), for $i = 1, 2, \dots, n$. Without loss of generality let us label the vertices of $\mathcal{H}$ in a suitable way so that the normalized distance Laplacian matrix $D^{\mathcal{L}}(\mathcal{H})$ can be put as

$$D^{\mathcal{L}}(H) = \begin{pmatrix} D_1 & \frac{-d_G(v_1,v_2)}{\sqrt{T_1T_2}} J_{n_1\times n_2} & \dots & \frac{-d_G(v_1,v_n)}{\sqrt{T_1T_n}} J_{n_1\times n_n} \\ \frac{-d_G(v_1,v_2)}{\sqrt{T_1T_2}} J_{n_2\times n_1} & D_2 & \dots & \frac{-d_G(v_2,v_n)}{\sqrt{T_2T_n}} J_{n_2\times n_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{-d_G(v_1,v_n)}{\sqrt{T_1T_n}} J_{n_n\times n_1} & \frac{-d_G(v_2,v_n)}{\sqrt{T_2T_n}} J_{n_n\times n_{n-1}} & \dots & D_n \end{pmatrix},$$

where, for $i = 1, 2, \dots, n$,

$$D_i = I_{n_i} - \frac{1}{T_i}\Big(2(J_{n_i} - I_{n_i}) - A(G_i)\Big),$$

$A(G_i)$ is the adjacency matrix of G_i, J_{n_i} is the matrix with all entries equal to 1 and I_{n_i} is the identity matrix of order n_i.

Since each of G_i is r_i-regular graph, it is well known that r_i is the adjacency eigenvalue [see, [18]] of $A(G_i)$ with the corresponding eigenvector $e_{n_i} = (\underbrace{1, 1, \dots, 1}_{n_i})^T$ and any other eigenvector of $A(G_i)$ is orthogonal to it. Let λ_{ik}, where $\lambda_{ik} \neq r_i$, $2 \leq k \leq n_i$, be an arbitrary eigenvalue of $A(G_i)$ with its associated eigenvector $Y = (y_{i1}, y_{i2}, \dots, y_{in_i})^T$ satisfying $e_{n_i}^T Y = 0$. Clearly, the column vector Y can be considered as a function defined on $V(G_i)$ associating the vertex v_{ij} to y_{ij}, that is, $Y(v_{ij}) = y_{ij}$, with $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n_i$. Now, take the vector $X = (x_1, x_2, \ldots, x_n)^T$, where

$$x_j = \begin{cases} y_{ij} & \text{if } v_{ij} \in V(G_i) \\ 0 & \text{otherwise.} \end{cases}$$

Recall that $e_{n_i}^T Y = 0$ and coordinates of X corresponding to vertices in $\bigcup_{j\neq i} V_j$ of $\mathcal{H}$ are zeros, so we obtain

$$D^{\mathcal{L}}(\mathcal{H})X = \begin{pmatrix} \frac{-d_G(v_1,v_i)}{\sqrt{T_1T_i}} J_{n_1\times n_i}Y \\ \vdots \\ \frac{-d_G(v_{i-1},v_i)}{\sqrt{T_1T_{i-1}}} J_{n_{i-1}\times n_i}Y \\ D_iY \\ \frac{-d_G(v_{i+1},v_i)}{\sqrt{T_{i+1}T_i}} J_{n_{i+1}\times n_i}Y \\ \vdots \\ \frac{-d_G(v_1,v_2)}{\sqrt{T_1T_2}} J_{n_n\times n_i}Y \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \Big(I_{n_i} - \frac{1}{T_i}(2(J_{n_i} - I_{n_i}) - A(G_i)\Big)Y \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$
$$= \left(I_{n_i} - \frac{1}{T_i}\Big(2(J_{n_i} - I_{n_i}) - A(G_i)\Big)\right)X = 1 + \frac{1}{T_i}\Big(2 + \lambda_{ik}A(G_i)\Big).$$

This shows that X is the eigenvector of the normalized distance Laplacian matrix $D^{\mathcal{L}}(\mathcal{H})$ associated to the eigenvalue $1 + \frac{1}{T_i}\Big(2 + \lambda_{ik}(G_i)\Big)$, for each eigenvalue λ_{ik} of $A(G_i)$ other than r_i, $2 \leq k \leq n_i$. This implies that $1 + \frac{1}{T_i}\Big(2 + \lambda_{ik}(G_i)\Big)$, with $1 \leq i \leq n$ and $2 \leq k \leq n_i$, are the eigenvalues of $D^{\mathcal{L}}(\mathcal{H})$. In this way, we obtain $\sum_{i=1}^{n} n_i - n$ normalized distance Laplacian eigenvalues of $D^{\mathcal{L}}(\mathcal{H})$. The other n normalized distance Laplacian eigenvalues of $D^{\mathcal{L}}(\mathcal{H})$ can be obtained by using the concept of regular quotient matrix and are given by the matrix (2.1). This proves the result. □

The *lexicographic product* $G\big[H\big]$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ and $(a, x)(b, y) \in E(G\big[H\big])$ whenever $ab \in E(G)$, or $a = b$ and $xy \in E(H)$. It is easy to see that the lexicographic product $G\big[H\big]$ of the graphs G and H is the joined union $G[H, H, \ldots, H]$. That is, $G\big[H\big] = G[H, H, \ldots, H]$. Taking, $G_i = H$ for all i in Theorem 2.1, we obtain the following result which gives the normalized distance Laplacian spectrum of the lexicographic product $G\big[H\big]$.

Corollary 2.1. *Let G be a connected graph of order $n \geq 2$ and let H be a connected r-regular graph of order n_1. Let $\lambda_1(H) \geq \lambda_2(H) \geq \ldots \geq \lambda_{n_1}$ be the adjacency eigenvalues of H. The normalized distance Laplacian spectrum of the lexicographic product $G\Big[H\Big] = G[H, \ldots, H]$ consists of the eigenvalues $1 + \frac{1}{T_i}\Big(2 + \lambda_k(G_i)\Big)$, for $i = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, n_1 - 1$, where $T_i = 2n_1 - 2 - r + n_1 \sum_{k=1,k\neq i} nd_G(v_i, v_k)$. The remaining n eigenvalues are given by the eigenvalues of the matrix 2.1 with $n_i = n_1$ and $T_i = 2n_1 - 2 - r + n_1 \sum_{k=1,k\neq i} nd_G(v_i, v_k)$, $1 \leq i \leq n$.*

Next result is a consequence of Theorem 2.1 and gives the normalized distance Laplacian eigenvalues of the join of two regular graphs.

Corollary 2.2. *Let G_1 and G_2 be r_1-regular and r_2-regular connected graphs of order n_1 and n_2, respectively. The normalized distance Laplacian spectrum of $G_1 \nabla G_2$ consists of the eigenvalues*

$1+\frac{1}{T_1}\Big(2+\lambda_{1k}(G_1)\Big),\ k=2,\ldots,n_1$, *the eigenvalues* $1+\frac{1}{T_2}\Big(2+\lambda_{2k}(G_1)\Big),\ k=2,\ldots,n_2$ *and the other two normalized distance Laplacian eigenvalues of* $G_1\nabla G_2$ *are the eigenvalues of the matrix given below*

$$\begin{pmatrix} 1-\frac{1}{T_1}\Big(2n_1-r_1-2\Big) & \frac{-n_2}{\sqrt{T_1T_2}} \\ \frac{-n_1}{\sqrt{T_1T_2}} & 1-\frac{1}{T_2}\Big(2n_2-r_2-2\Big) \end{pmatrix}, \tag{2.2}$$

where $T_1=2n_1+n_2-r_1-2$ *and* $T_2=2n_2+n_1-r_2-2$.

Proof.

$$\begin{pmatrix} I_1-\frac{1}{T_1}\Big(2(J_{n_1}-I_{n_1})-A(G_1)\Big) & \frac{-1}{\sqrt{T_1T_2}}J_{n_1\times n_2} \\ \frac{-1}{\sqrt{T_1T_2}}J_{n_2\times n_1} & I_1-\frac{1}{T_2}\Big(2(J_{n_2}-I_{n_2})-A(G_2)\Big) \end{pmatrix}.$$

Now, result follows by Theorem 2.1. □

Consider the *complete split graph*, denoted by $CS_{n,\omega}$, with clique number ω and independence number $n-\omega$. Clearly, $CS_{n,\omega}=K_\omega\nabla\overline{K}_{n-\omega}$. Taking $n_1=\omega, n_2=n-\omega, T_1=n-1, T_2=2n-\omega-2$ in Corollary 2.2 and using the fact adjacency spectrum of K_ω is $\{\omega-1,-1^{[\omega-1]}\}$, it follows that the spectrum of $CS_{n,\omega}$ consists of the eigenvalue $1+\frac{1}{n-1}=\frac{n}{n-1}$ with multiplicity $\omega-1$, the eigenvalue $1+\frac{2}{2n-\omega-2}=\frac{2n-\omega}{2n-\omega-2}$ with multiplicity $n-\omega-1$. The remaining two normalized distance Laplacian eigenvalues of $CS_{n,\omega}$ are given by the eigenvalues of the matrix 2.2 and are 0, $\frac{n-\omega}{n-1}+\frac{\omega}{2n-\omega-2}$.

Let W_{n+1} be the *wheel graph* of order $n+1$. It is well known that $W_{n+1}=C_n\nabla K_1$. Taking $n_1=n, n_2=1, r_1=2, r_2=0, T_1=2n-3, T_2=n$ in Corollary 2.2 and using the fact that the adjacency spectrum of K_ω is $\{\omega-1,-1^{[\omega-1]}\}$, we get the normalized distance Laplacian spectrum of W_{n+1}, which consists of the eigenvalues $1+\frac{1}{2n-3}\Big(1+2\cos\big(\frac{2\pi(k-1)}{n}\big)\Big),\ k=2,\ldots,n$ and the eigenvalues $\frac{2n-2}{2n-3}, 0$.

3. Normalized distance Laplacian eigenvalues of power graphs of groups

In this section, we consider the power graphs of finite groups. As applications of Theorem 2.1 and its consequences obtained in Section 2, we determine their normalized distance Laplacian eigenvalues.

If $\mathcal{G}$ is a finite group of order n with identity e. The power graph of $\mathcal{G}$, denoted by $\mathcal{P}(\mathcal{G})$, is the simple graph with vertex set as the elements of group $\mathcal{G}$ and two distinct vertices $x,y\in\mathcal{P}(\mathcal{G})$ are adjacent if and only if one is the positive power of the other, that is, $x^i=y$ or $y^j=x$, for positive integers i,j with $2\le i,j\le n$. Such graphs were introduced in [32], see also [17], and have valuable applications both in algebra and combinatorics. They are related to automata theory [33] besides being useful in characterizing finite groups. Let $\mathbb{Z}_n$ be the additive integer modulo n group and we let $U_n^o=\{a\in\mathbb{Z}_n:(a,n)=1\}\cup\{0\}$, where (a,n) denotes their greatest common divisor. Our other group theory notations are standard and can be taken from [37]. More work on power graphs can be seen in [11, 17, 34, 1] and the references therein.

The adjacency spectrum, the Laplacian, the normalized Laplacian and the signless Laplacian spectrum of power graphs of finite cyclic and dihedral groups have been investigated in [13, 16, 28, 29, 43, 35, 44].

The divisor d of positive integer n (written as $d|n$) is the proper divisor of n, if $1<d<n$. Let $\mathbb{D}_n$ be a simple graph with vertex set as the proper divisor set $\{d_i:1,n\neq d_i|n,\ 1\le i\le t\}$ and edge set $\{d_id_j:d_i|d_j,\ 1\le i<j\le t\}$, for $1\le i<j\le t$. If the *canonical decomposition* of n is $n=p_1^{n_1}p_2^{n_2}\ldots p_r^{n_r}$, where $r,n_1,n_2,\ldots,n_r$ are non-negative integers

and $p_1, p_2, \ldots, p_r$ are distinct prime numbers, then the order of graph $\mathbb{D}_n$ is $|V(\mathbb{G}_n)| = \prod_{i=1}^{r}(n_i + 1) - 2$.

Suppose $d_1, d_2, \ldots, d_t$ be the proper divisors of n and x, y are vertices in $\mathcal{P}(\mathbb{Z}_n)$. Clearly, if x and y are adjacent then $o(x)|o(y)$ or $o(y)|o(x)$, where $o(x)$ denotes the order of x in $\mathbb{Z}_n$. Thus, $\mathcal{P}(\mathbb{Z}_n)$ has complete subgroups of orders $\phi(d_i), 1 \leq i \leq t$. If for some i and j, $di|dj$, then since $\mathbb{Z}_n$ is a cyclic group, all vertices of degree d_i and d_j are adjacent. Also, all the generators of $\mathbb{Z}_n$ together with identity element constitute a complete subgraph of order $\phi(n) + 1$ and all its vertices are adjacent to every other vertex of $\mathcal{P}(\mathbb{Z}_n)$. This fact is made precise in the following result.

Theorem 3.2. [34] *If $\mathbb{Z}_n$ is a finite cyclic group of order n, then the power graph $\mathcal{P}(\mathbb{Z}_n)$ is given by*

$$\mathcal{P}(\mathbb{Z}_n) = K_{\phi(n)+1} \nabla \mathbb{D}_n[K_{\phi(d_1)}, K_{\phi(d_2)}, \ldots, K_{\phi(d_t)}],$$

where $\phi(n)$ is the Euler's totient function.

The following result gives the information about some of the distance normalized Laplacian eigenvalues of a graph when the graph contains a clique in which each vertex have the same neighbour set outside the clique.

Lemma 3.1. [24] *Let G be a connected graph of order n and $K = \{v_1, v_2, \ldots, v_\omega\}$ induces a clique in G satisfying $N(v_i) - K = N(v_j) - K$, $1 \leq i, j \leq \omega$. Then G has $\frac{T+1}{T}$ as the normalized distance Laplacian eigenvalue with multiplicity at least $\omega - 1$, where $T = Tr(v_i) = Tr(v_j)$ for $1 \leq i, j \leq \omega$.*

The next result implies that $\frac{n}{n-1}$ is always the $D^{\mathcal{L}}$-eigenvalue of $\mathcal{P}(\mathbb{Z}_n)$.

Proposition 3.1. *If $\mathcal{P}(\mathbb{Z}_n)$ is the power graph of the cyclic group $\mathbb{Z}_n$, then $\frac{n}{n-1}$ is normalized distance Laplacian eigenvalue of $\mathcal{P}(\mathbb{Z}_n)$ with multiplicity at least $\phi(n)$.*

Proof. Let $\mathcal{P}(\mathbb{Z}_n)$ be the power graph of the integer modulo group $\mathbb{Z}_n$. Then we know that there are exactly $\phi(n)$ number of elements in $\mathbb{Z}_n$ which generate every element of $\mathbb{Z}_n$. So, by the definition of power graphs, there are $\phi(n) + 1$ numbers of elements adjacent to every other vertex of $\mathcal{P}(\mathbb{Z}_n)$. Therefore, power graph of $\mathbb{Z}_n$ can be written as

$$\mathcal{P}(\mathbb{Z}_n) = K_{\phi(n)+1} \nabla \mathcal{P}(\mathbb{Z}_n \setminus U_n^o).$$

Thus, by using Lemma 3.1, we see that $1 + \frac{1}{n-1} = \frac{n}{n-1}$ is the normalized distance Laplacian eigenvalue with multiplicity $\phi(n)$. Also we note that $\frac{n}{n-1}$ may be the eigenvalue of the equitable quotient matrix of $K_{\phi(n)+1} \nabla \mathcal{P}(\mathbb{Z}_n \setminus U_n^o)$. □

Next, problem is immediate from Proposition 3.1.

Problem 3.1. *Characterize the power graphs of $\mathbb{Z}_n$ which attain equality in Proposition 3.1.*

Now, we will find the normalized distance Laplacian eigenvalues of $\mathcal{P}(\mathbb{Z}_n)$ using Theorems 2.1 and 3.2.

Theorem 3.3. *The normalized distance Laplacian spectrum of $\mathcal{P}(\mathbb{Z}_n)$ consists of the eigenvalue $\frac{n}{n-1}$ with multiplicity $\phi(n)$, the eigenvalues $\frac{\phi(d_i)+n'_{i+1}}{\phi(d_i)+n'_{i+1}-1}$, with multiplicities $\phi(d_i) - 1$, for $i =$*

$1, 2, \ldots, t$. *The remaining* $t+1$ *eigenvalues are the eigenvalues of matrix* (3.3) *given below*

$$M = \begin{pmatrix} 1-\frac{\phi(n)}{n-1} & \frac{-\phi(d_1)}{\sqrt{(n-1)T_2}} & \cdots & \frac{-\phi(d_t)}{\sqrt{(n-1)T_{t+1}}} \\ \frac{-(\phi(n)+1)}{\sqrt{(n-1)T_2}} & d_2^{'} & \cdots & \frac{-\phi(d_t)d(v_2,v_{t+1})}{\sqrt{T_2T_{t+1}}} \\ \frac{-(\phi(n)+1)}{\sqrt{(n-1)T_3}} & \frac{-\phi(d_1)d(v_2,v_3)}{\sqrt{T_2T_3}} & \cdots & \frac{-\phi(d_t)d(v_3,v_{t+1})}{\sqrt{T_3T_{t+1}}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{-(\phi(n)+1)}{\sqrt{(n-1)T_{t+1}}} & \frac{-(\phi(d_1))d(v_{t+1},v_2)}{\sqrt{T_2T_{t+1}}} & \cdots & d_{t+1}^{'} \end{pmatrix}, \tag{3.3}$$

where $T_{i+1} = \phi(d_i) + n_{i+1}^{'} - 1$ *and* $d_{i+1}^{'} = 1 - \frac{\phi(d_i)-1}{\phi(d_i)+n_i^{'}-1}$ *for* $i = 1, \ldots, t$, *where* $n_i^{'}$ *is defined in Theorem 2.1.*

Proof.

$$\mathcal{P}(\mathbb{Z}_n) = H[K_{\phi(n)+1}, K_{\phi(d_1)}, K_{\phi(d_2)}, \ldots, K_{\phi(d_t)}],$$

where $H = K_1 \nabla \mathbb{D}_n$ is the graph with vertex set $\{v_1, \ldots, v_{t+1}\}$. By using Proposition 3.1, it follows that $\frac{n}{n-1}$ is the normalized distance Laplacian eigenvalue of $\mathcal{P}(\mathbb{Z}_n)$ with multiplicity $\phi(n)$. For $i = 1, \ldots, t$, $G_{i+1} = K_{\phi(d_i)}$ and $T_{i+1} = 2n_{i+1} - 2 - r_{i+1} + n_{i+1}^{'} = 2n_{i+1} - 2 - n_{i+1} + 1 + n_{i+1}^{'} = n_{i+1} + n_{i+1}^{'} - 1 = \phi(d_i) + n_i^{'} - 1$. Now, by applying Theorem 2.1, we see that

$$1 + \frac{1}{T_{i+1}} = 1 + \frac{1}{\phi(d_i) + n_{i+1}^{'} - 1} = \frac{\phi(d_i) + n_{i+1}^{'}}{\phi(d_i) + n_{i+1}^{'} - 1}$$

are the normalized distance Laplacian eigenvalues of $\mathcal{P}(\mathbb{Z}_n)$ with multiplicity $\phi(d_i) - 1$, for $i = 1, 2, \ldots, n$. The other $t+1$ distance normalized Laplacian eigenvalues of $\mathcal{P}(\mathbb{Z}_n)$ are the eigenvalues of matrix (3.3). □

If $n = p^z$, where p is prime and z is positive integer, then $\mathcal{P}(\mathbb{Z}_n)$ [17] is the complete graph K_n and its $D^{\mathcal{L}}$-spectrum is $\left\{0, \left(\frac{n}{n-1}\right)^{[n-1]}\right\}$.

Let $n = pq$, where p and q, $p < q$, are primes. Then the power graph $\mathcal{P}(\mathbb{Z}_n)$ can be written as

$$P_3[K_{p-1}, K_{\phi(pq)+1}, K_{q-1}].$$

Here, $H = P_3, n_1 = p-1, n_2 = \phi(pq)+1, n_3 = q-1, n_1^{'} = \phi(pq)+1+2q-2 = pq+q-p$, $n_2^{'} = p+q-2$, $n_3^{'} = pq+p-q$, $T_1 = n_1 + n_1^{'} - 1 = pq+q-2$, $T_2 = pq-1$ and $T_3 = pq+p-2$. Thus, by Proposition 3.1, $\frac{n}{n-1}$ is the normalized distance Laplacian eigenvalue of $\mathcal{P}(\mathbb{Z}_n)$ with multiplicity $\phi(n)$ and by Theorem 3.3, $\frac{\phi(p)+n_2^{'}}{\phi(p)+n_2^{'}-1} = \frac{pq+q-1}{pq+q-2}$ and $\frac{\phi(q)+n_3^{'}}{\phi(q)+n_3^{'}-1} = \frac{pq+p-1}{pq+p-2}$ are the normalized distance Laplacian eigenvalues of $\mathcal{P}(\mathbb{Z}_n)$ with multiplicities $p-2$ and $q-2$, respectively. The other three $D^{\mathcal{L}}$-eigenvalues of $\mathcal{P}(\mathbb{Z}_n)$ are the eigenvalues of the following matrix

$$\begin{pmatrix} d_1^{'} & \frac{-n_2}{\sqrt{T_1T_2}} & \frac{-n_3}{\sqrt{T_1T_3}} \\ \frac{-n_1}{\sqrt{T_1T_2}} & d_2^{'} & \frac{-n_3}{\sqrt{T_2T_3}} \\ \frac{-n_1}{\sqrt{T_1T_3}} & \frac{-n_2}{\sqrt{T_2T_3}} & d_3^{'} \end{pmatrix},$$

where $d_i^{'} = 1 - \frac{1}{T_i}(n_i - 1)$, for $i = 1, 2, 3$. □

We recall that the proper power graph $\mathcal{P}(\mathcal{G}^*)$ of a group $\mathcal{G}$ is the power graph of group upon identity deletion, that is, $\mathcal{P}(\mathcal{G} \setminus \{e\})$. The proper power graph $\mathcal{P}(\mathbb{Z}_n^*)$ is a connected graph of order $n-1$, since there are generators of cyclic group $\mathbb{Z}_n$ which are adjacent to every other vertices of $\mathcal{P}(\mathbb{Z}_n^*)$. Thus the normalized distance Laplacian matrix makes sense on $\mathcal{P}(\mathbb{Z}_n^*)$. An analogues of Theorem 3.3 can be proved on the proper power graph $\mathcal{P}(\mathbb{Z}_n^*)$.

The dihedral group of order $2n$ and dicyclic groups of order $4n$ are denoted and presented as follows

$$D_{2n} =< a, b \mid a^n = b^2 = e, bab = a^{-1} >,$$
$$Q_n =< a, b \mid a^{2n} = e, b^2 = a^n, ab = ba^{-1} > .$$

If n is a power of 2, then Q_n is called the *generalized quaternion group* of order $4n$.

If $n = p_1^m$, where m_1 is the positive integer, then

$$\mathcal{P}(D_{2n}) = P_3[K_{n-1}, K_1, \overline{K}_n],$$

that is, $\mathcal{P}(D_{2p^{m_1}})$ is the *pineapple graph*, the graph obtained from K_n by appending vertices of degree 1 at some vertex of K_n. The normalized distance Laplacian spectrum of $\mathcal{P}(D_{2p^{m_1}}) = P_3[K_{n-1}, K_1, \overline{K}_n]$ can be easily discussed as above in $\mathcal{P}(\mathbb{Z}_{pq})$.

Also, if n is a power of 2, then it is clear that a^n and e are adjacent to all other vertices of $\mathcal{P}(Q_n)$, so the power graph $\mathcal{P}(Q_n)$ can be written as joined union of cliques

$$\mathcal{P}(Q_n) = S[K_2, K_{2n-2}, \underbrace{K_2, K_2, \ldots, K_2}_{n}],$$

where $S = K_{1,n+1}$. Therefore, by using Theorems 2.1 and 3.3, the normalized distance Laplacian spectrum of $\mathcal{P}(Q_{2^n})$ can be easily determined. When $n \neq p_1^m$ in case of the group D_{2n} and n not a power of 2 in case of the group Q_n, it will be an interesting problem to determine the normalized distance Laplacian spectrum of the power graphs. Therefore, we leave the following problems for the future research.

Problem 3.2. *Find normalized distance Laplacian spectrum of $\mathcal{P}(D_{2n})$ for $n \in \{pqr, p^2q, (pq)^2\}$ and generalize for any n?*

Problem 3.3. *Find normalized distance Laplacian spectrum of $\mathcal{P}(Q_n)$ for $n \in \{pq, pqr, p^2q, (pq)^2\}$ and generalize for any n?*

4. Normalized distance Laplacian eigenvalues of commuting graphs of groups

Let $\mathcal{G}$ be a finite group and X be a non empty subset of $\mathcal{G}$. The *commuting graph*, denoted by $\mathcal{C}(\mathcal{G}, X)$, is defined with X as vertex set and two vertices x and y are adjacent if and only if x and y commute in X. commuting graphs of matrix rings and semirings over finite fields were studied in [2, 21]. Metric dimension, resolving polynomial, clique number and chromatic number of commuting graphs on dihedral groups were discussed in [4]. Recent results on the commuting graph of generalized dihedral groups can be found in [31] and the references therein. The connectivity and spectral radius of adjacency matrix of commuting graphs were studied in [5], Laplacian and signless Laplacian spectrum of commuting graphs on dihedral groups were investigated in [3], common neighborhood energy of commuting graphs [36]. For other spectral properties of commuting graphs, we refer to [27] and the references therein.

The presentation of semi dihedral SD_{8n} of order $8n$ is given by

$$SD_{8n} = \langle a, b : a^{4n} = e = b^2, ab = ba^{2n-2} \rangle,$$

The center of $\mathcal{G}$, denoted by $Z(\mathcal{G})$, is defined by

$$Z(\mathcal{G}) = \{z \in \mathcal{G} : za = az \text{ for each } a \in \mathcal{G}.\}$$

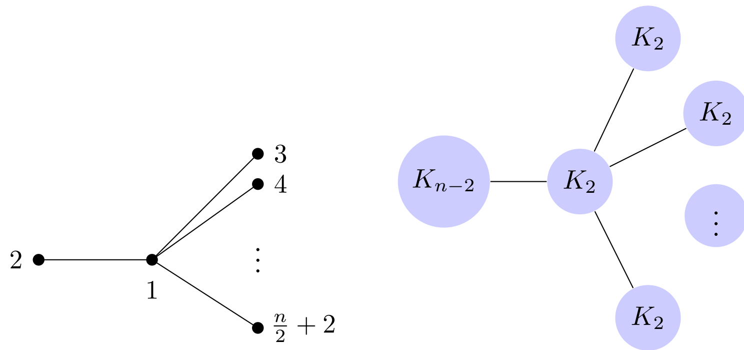

FIGURE 2. Graph S and the commuting graph $\mathcal{C}(D_{2n}, D_{2n})$ with even n.

Clearly, the commuting graph $G = \mathcal{C}(\mathbb{Z}_n, \mathbb{Z}_n)$ is the complete graph K_n, as every element of $\mathbb{Z}_n$ commutes with every other element. The normalized distance Laplacian spectrum of $\mathcal{C}(\mathbb{Z}_n, \mathbb{Z}_n) \cong K_n$ is already discussed. It is well known that $Z(D_{2n}) = \{e\}$, for odd n and $Z(D_{2n}) = \left\{e, a^{\frac{n}{2}}\right\}$, for even n. Also, $Z(Q_{4n}) = \{e, a^n\}$ is the center of dicyclic group. For the commuting graph $G = \mathcal{C}(D_{2n}, Z(D_{2n}))$, we have $G = K_1$, for odd n and $G = K_2$, for even n. So, the commuting graphs $\mathcal{C}(\mathcal{G}, Z(\mathcal{G}))$ have simple structures. However non trivial structures are obtained when we consider commuting graphs of D_{2n}, with $X = D_{2n}$ itself.

Lemma 4.2. [4] *For the commuting graph $G = \mathcal{C}(D_{2n}, D_{2n})$ of the dihedral group D_{2n}, we have*

$$G = \begin{cases} K_1 \triangledown \left(K_{n-1} \cup \overline{K}_n\right) & \text{if } n \text{ is odd,} \\ K_2 \triangledown \left(K_{n-2} \cup \frac{n}{2} K_2\right) & \text{if } n \text{ is even.} \end{cases}$$

In the following result, we find the normalized distance Laplacian eigenvalues of the commuting graphs of dihedral group D_{2n}.

Theorem 4.4. *For the commuting graph $\mathcal{C}(D_{2n}, D_{2n})$ of the dihedral group D_{2n}, the following hold.*

- **(i)** *If n is odd, then the normalized distance Laplacian spectrum of $\mathcal{C}(D_{2n}, D_{2n})$ consists of the eigenvalue $\frac{4n-1}{4n-2}$ with multiplicity $n-1$, the eigenvalue $\frac{3n}{3n-1}$ with multiplicity $n-2$ and three eigenvalues of matrix* (4.4).
- **(ii)** *If n is even, then the normalized distance Laplacian spectrum of $\mathcal{C}(D_{2n}, D_{2n})$ consists of the simple eigenvalue $\frac{2n}{2n-1}$, the eigenvalue $\frac{3n-1}{3n-2}$ with multiplicity $n-3$, the eigenvalue $\frac{4n-4}{4n-5}$ with multiplicity $\frac{n}{2}$ and other eigenvalues are the eigenvalue of matrix* (4.5).

Proof.

$$K_1 \triangledown \left(K_{n-1} \cup \overline{K}_n\right) = P_3[\overline{K}_n, K_1, K_{n-1}],$$

that is, $\mathcal{C}(D_{2n}, D_{2n})$ is the *pineapple graph* (a graph obtained by appending pendent edges to a vertex of complete graph). For $i = 1, 2, 3$, we have

$$n_1^{'} = 1 + 2(n-1) = 2n-1, n_2^{'} = n + n - 1 = 2n - 1 \text{ and } n_3^{'} = 2n+1.$$

Also, $T_1 = 2n_1 - 2 - r_1 + n_1^{'} = 4n-3$, $T_2 = 2n-1$ and $T_3 = 3n-1$. Thus, from Theorem 2.1, we see that $1 + \frac{2}{T_1} = \frac{4n-1}{4n-2}$ and $\frac{3n}{3n-1}$ are the normalized distance Laplacian eigenvalues of $\mathcal{C}(D_{2n}, D_{2n})$ with multiplicities $n-1$ and $n-2$, respectively. The other three normalized distance Laplacian eigenvalues of $\mathcal{C}(D_{2n}, D_{2n})$ are the eigenvalues of the following matrix

$$\begin{pmatrix} 1-\frac{2n-2}{T_1} & \frac{-1}{\sqrt{T_1T_2}} & \frac{-2(n-1)}{\sqrt{T_1T_3}} \\ \frac{-n}{T_1T_2} & 1-\frac{2}{T_2} & \frac{-(n-1)}{T_2T_3} \\ \frac{-2n}{\sqrt{T_1T_3}} & \frac{-1}{\sqrt{T_2T_3}} & 1-\frac{n-2}{T_3} \end{pmatrix}. \tag{4.4}$$

(ii). By Lemma 4.2, the commuting graph $\mathcal{C}(D_{2n}, D_{2n})$ of D_{2n} for even n is given below

$$\mathcal{C}(D_{2n}, D_{2n}) = S[K_2, K_{n-2}, \underbrace{K_2, K_2, \dots, K_2}_{\frac{n}{2}}],$$

where $S = K_{\frac{n}{2}+1,1}$, see figure (1). Further, we have

$$n_1^{'} = n-2+2\frac{n}{2} = 2n-2, n_2^{'} = 2n+2, n_3^{'} = \cdots = n_{\frac{n}{2}+2}^{'} = 4n-6$$

and

$$T_1 = 2n_1 - 2 - r_1 + n_1^{'} = 2n-1,\ T_2 = 3n-2,\ T_3 = \cdots = T_{\frac{n}{2}+2} = 4n-5.$$

By applying Theorem 2.1, the normalized distance Laplacian spectrum of $\mathcal{C}(D_{2n}, D_{2n})$ consists of the eigenvalue $1+\frac{1}{T_1}$ with multiplicity one, the eigenvalue $1+\frac{1}{T_2}$ with multiplicity $n-3$ and the eigenvalue $1+\frac{1}{T_3}$ with multiplicity $\frac{n}{2}$. The other normalized distance Laplacian eigenvalues of $\mathcal{C}(D_{2n}, D_{2n})$ are the eigenvalues of matrix (4.5),

$$\begin{pmatrix} \frac{2n}{2n-1} & \frac{-(n-2)}{\sqrt{T_1T_2}} & \frac{-2}{\sqrt{T_1T_3}} & \cdots & \frac{-2}{\sqrt{T_1T_3}} \\ \frac{-2}{\sqrt{T_1T_2}} & \frac{3n}{3n-1} & \frac{-4}{\sqrt{T_2T_3}} & \cdots & \frac{-4}{\sqrt{T_2T_3}} \\ \frac{-2}{\sqrt{T_1T_3}} & \frac{-2(n-2)}{\sqrt{T_2T_3}} & \frac{4n-4}{4n-5} & \cdots & \frac{-4}{T_3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{-2}{\sqrt{T_1T_3}} & \frac{-2(n-2)}{\sqrt{T_2T_3}} & \frac{-4}{T_3} & \cdots & \frac{4n-4}{4n-5} \end{pmatrix}. \tag{4.5}$$

□

For the commuting graph $\mathcal{C}(SD_{8n}, SD_{8n})$ of the semi-dihedral group SD_{8n} [[51], Lemma 2.10], we have the following observations:

$$Z(SD_{8n}) = \begin{cases} a^n & \text{if } n \text{ is even} \\ a^{2n} & \text{if } n \text{ is odd} \end{cases}.$$

Let $H = \langle a \rangle$ and $G = C_{Q_{4n}}(a^{2n}) = C_{Q_{4n}}(a^{4n})$ be centralized subgroup of a^{2n} in Q_{4n}. Then following observations are clear from the definition of commuting graphs

$$H_1 = \begin{cases} \mathcal{C}(SD_{8n}, H - Z(SD_{8n})) = K_{4n-2} & \text{if } n \text{ is even}, \\ \mathcal{C}(SD_{8n}, H - Z(SD_{8n})) = K_{4n-4} & \text{if } n \text{ is odd}, \end{cases}$$

$$H_2 = \begin{cases} \mathcal{C}(SD_{8n}, Z(SD_{8n})) = K_2 & \text{if } n \text{ is even}, \\ \mathcal{C}(SD_{8n}, Z(SD_{8n})) = K_4 & \text{if } n \text{ is odd}, \end{cases}$$

$$H_2 = \begin{cases} \mathcal{C}(SD_{8n}, G - (Z(SD_{8n}) \cup H)) = 2nK_2 & \text{if } n \text{ is even}, \\ \mathcal{C}(SD_{8n}, G - (Z(SD_{8n}) \cup H)) = nK_4 & \text{if } n \text{ is odd}. \end{cases}$$

By above calculations, the commuting graphs of Q_{4n} is

$$\mathcal{C}(SD_{8n}, SD_{8n}) = P_3[H_1, H_2, H_3] = \begin{cases} K_4 \nabla \Big(K_{4n-4} \cup \big(\underbrace{K_4 \cup \cdots \cup K_4}_{n} \big) \Big) & \text{if } n \text{ is odd,} \\ K_2 \nabla \Big(K_{4n-4} \cup \big(\underbrace{K_2 \cup \cdots \cup K_2}_{2n} \big) \Big) & \text{if } n \text{ is even.} \end{cases}$$

The subgraph induced $\mathcal{C}(\mathbb{Z}_n, \mathbb{Z}_n)$ of a in $\mathcal{C}(Q_n, Q_n)$ corresponds to K_{2n}, the indiced subgraph $\mathcal{C}(\mathbb{Z}_4, \mathbb{Z}_4)$ of b corresponds to K_2 and they are repeated n times, and all these vertices shares the identity element e and a^n of $\mathcal{C}(Q_n, Q_n)$. Based on these observations, the commuting graph $\mathcal{C}(Q_n, Q_n)$ of Q_n [5] can be written:

$$\mathcal{C}(Q_n, Q_n) = K_2 \nabla \Big(K_{2n-2} \cup \underbrace{K_2 \cup K_2 \cup \cdots \cup K_2}_{n} \Big).$$

Now, proceeding as in Theorem 4.4, the normalized distance Laplacian spectrum of commuting graphs of semi-dihedral and dicyclic groups can be found.

5. Normalized distance Laplacian eigenvalues of zero-divisor graphs of commutative rings

Let $\mathcal{R}$ be a commutative ring with multiplicative identity different from zero. A non-zero element x of $\mathcal{R}$ is known as a zero-divisor of $\mathcal{R}$ if there exists a non-zero $y \in \mathcal{R}$ such that $xy = 0$. The zero-divisor graph of commutative ring $\mathcal{R}$, denoted by $\Gamma(\mathcal{R})$, is a simple, connected and undirected graphs with vertex set as the set of non-zero zero-divisors of $\mathcal{R}$, in which two vertices x and y are adjacent if and if $xy = 0$. The zero divisor graphs were initially used in colouring of graphs, but nowadays they are extensively studied in investigating both the algebraic and the combinatorial properties of rings, see [49]. More about these graphs can be found in [42, 14, 6, 52, 15, 41, 45].

For integers d and n with $1 < d < n$, if d divides n (written as $d|n$), we say d is a proper divisor of n. Let $d_1, d_2, \dots, d_t$ be the distinct proper divisors of n and let γ_n be the simple graph with vertex set $\{d_1, d_2, \dots, d_t\}$ and edge set $\{d_i d_j : n | d_i d_j\}$. The graph γ_n is connected (see [15]) and plays the role of underlying graph in the joined union graph of $\Gamma(\mathbb{Z}_n)$.

For $1 \le i \le t$, we consider the sets $A_{d_i} = \{x \in \mathbb{Z}_n : (x, n) = d_i\}$, where (x, n) is their greatest common divisor. We see that $A_{d_i} \cap A_{d_j} = \phi$, when $i \neq j$, implying that the sets $A_{d_1}, A_{d_2}, \dots, A_{d_t}$ are pairwise disjoint and partitions the vertex set of $\Gamma(\mathbb{Z}_n)$ as $V(\Gamma(\mathbb{Z}_n)) = A_{d_1} \cup A_{d_2} \cup \cdots \cup A_{d_t}$. From the definition of A_{d_i}, a vertex of A_{d_i} is adjacent (see [15]) to the vertex of A_{d_j} in $\Gamma(\mathbb{Z}_n)$ if and only if n divides $d_i d_j$, for $i, j \in \{1, 2, \dots, t\}$. Also, it is clear that cardinality [52] of $|A_{d_i}| = \phi\left(\frac{n}{d_i}\right)$, for $1 \le i \le t$.

The induced subgraphs $\Gamma(A_{d_i})$ of $\Gamma(\mathbb{Z}_n)$ are either cliques or totally disconnected and $\Gamma(\mathbb{Z}_n)$ can be written as their joined union. This observation is made precise in the following result which can be found in [15].

Lemma 5.3. [15] *Let n be a positive integer and d_i be a proper divisor of n. Then the following holds.*

- **(i)** *For $i \in \{1, 2, \dots, t\}$, the induced subgraph $\Gamma(A_{d_i})$ of $\Gamma(\mathbb{Z}_n)$ on the vertex set A_{d_i} is either the complete graph $K_{\phi(\frac{n}{d_i})}$ or its complement $\overline{K}_{\phi(\frac{n}{d_i})}$. Indeed, $\Gamma(A_{d_i})$ is $K_{\phi(\frac{n}{d_i})}$ if and only if n divides d_i^2.*
- **(ii)** *For $i, j \in \{1, 2, \dots, t\}$ with $i \neq j$, a vertex of A_{d_i} is adjacent to either all or none of the vertices of A_{d_j} in $\Gamma(\mathbb{Z}_n)$.*

(iii) *Let $\Gamma(\mathbb{Z}_n)$ be the zero-divisor graph of the commutative ring $\mathbb{Z}_n$. Then*

$$\Gamma(\mathbb{Z}_n) = \gamma_n[\Gamma(A_{d_1}), \Gamma(A_{d_2}), \ldots, \Gamma(A_{d_l})].$$

The next result describes the $D^{\mathcal{L}}$-spectrum of the zero-divisor graph $\Gamma(\mathbb{Z}_n)$ of the commutative ring $\mathbb{Z}_n$.

Theorem 5.5. *Let $d_1, d_2, \ldots, d_l$ be the proper divisors of n. Then the normalized distance Laplacian spectrum of $\Gamma(\mathbb{Z}_n)$ consists of the eigenvalues $1 + \frac{c_i}{T_i}$, with multiplicity $\phi\left(\frac{n}{d_i}\right) - 1$, where $c_i = \begin{cases} 1 & \text{if } n \text{ divides } d_i^2 \\ 2 & \text{if } n \text{ does not divides } d_i^2 \end{cases}$, for $i = 1, 2, \ldots, l$. The remaining normalized distance Laplacian eigenvalues of $\Gamma(\mathbb{Z}_n)$ are the eigenvalues of the following matrix*

$$M = \begin{pmatrix} d_1' & \frac{-\phi\left(\frac{n}{d_2}\right) d_{\gamma_n}(v_1,v_2)}{\sqrt{T_1T_2}} & \cdots & \frac{-\phi\left(\frac{n}{d_l}\right) d_{\gamma_n}(v_1,v_l)}{\sqrt{T_1T_l}} \\ \frac{-\phi\left(\frac{n}{d_1}\right) d_{\gamma_n}(v_2,v_1)}{\sqrt{T_1T_2}} & d_2' & \cdots & \frac{-\phi\left(\frac{n}{d_l}\right) d_G(v_{\gamma_n},v_l)}{\sqrt{T_2T_l}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{-\phi\left(\frac{n}{d_1}\right) d_{\gamma_n}(v_n,v_1)}{\sqrt{T_1T_l}} & \frac{-\phi\left(\frac{n}{d_2}\right) d_{\gamma_n}(v_n,v_2)}{\sqrt{T_2T_l}} & \cdots & d_l' \end{pmatrix},$$

where $d_i' = 1 - \frac{e_i}{T_i}$ with $e_i = \begin{cases} 2n_i - 2 & \text{if } n \text{ does not divide } d_i^2 \\ n_i - 1 & \text{if } n \text{ divides } d_i^2 \end{cases}$, for $i = 1, 2, \ldots, l$.

Proof. We note that $\Gamma(\mathbb{Z}_n)$ can be written as joined union of cliques and their complements, so by Lemma 5.3 and Theorem 2.1, result follows. □

If $n = p^2$, where p is prime, then p is the only proper divisor and so in this case the zero-divisor graph $\Gamma(\mathbb{Z}_n)$ is isomorphic to $K_{\phi(p)}$ and therefore its $D^{\mathcal{L}}$-spectrum is $\left\{0, \left(\frac{\phi(p)}{\phi(p)-1}\right)^{[p-2]}\right\}$.

If $n = pq$ $(p < q)$, p and q are primes, then p and q are the only proper divisors of n. Thus, $\gamma_n \cong K_2$ and neither p^2 nor q^2 divides pq. So, the zero-divisor graph of $\mathbb{Z}_{pq}$ is $K_2[\overline{K}_{\phi(p)}, \overline{K}_{\phi(q)}]$ and $T_1 = 2\phi(p) - 2 + \phi(q)$ and $T_2 = 2\phi(q) - 2 + \phi(p)$. Therefore, by Theorem 5.5, $1 + \frac{1}{T_1}$ and $1 + \frac{1}{T_2}$ are the $D^{\mathcal{L}}$-eigenvalues of $\Gamma(\mathbb{Z}_n)$ with multiplicities $\phi(p) - 1$ and $\phi(q) - 1$, respectively. The other two $D^{\mathcal{L}}$ of $\Gamma(\mathbb{Z}_n)$ are the eigenvalues of following matrix

$$\begin{pmatrix} \frac{\phi(q)}{T_1} & \frac{-\phi(q)}{\sqrt{T_1T_2}} \\ \frac{-\phi(p)}{\sqrt{T_1T_2}} & \frac{\phi(p)}{T_2} \end{pmatrix},$$

and its characteristic polynomial is $x\left(x - \frac{\phi(q)}{T_1} - \frac{\phi(p)}{T_2}\right)$.

The next consequences of Theorem 5.5 gives the $D^{\mathcal{L}}$-spectrum of the commutative graph $\Gamma(\mathbb{Z}_n)$ of $\mathbb{Z}_n$ when n is a prime power. □

Corollary 5.3. *Let $n = p^M$, where $M = 2m$, p is prime and $m \geq 3$ is positive integer. Then the $D^{\mathcal{L}}$-spectrum of $\Gamma(\mathbb{Z}_n)$ consists of the eigenvalue $1 + \frac{2}{T_i}$ with multiplicity $\phi(p^{2m-i}) - 1$, for $i = 1, 2, \ldots, m-1$, the eigenvalue $1 + \frac{1}{T_j}$ with multiplicity $\phi(p^{2m-j}) - 1$, for $j = m, m+1, \ldots, 2m-1$, and*

$$T_i = \begin{cases} 2(p^{2m-1} - 1) - p^i - 1, & \text{for } i = 1, 2, \ldots, m-1, \\ 2(p^{2m-1} - 1) + \phi(p^{2m-i}) - p^i, & \text{for } i = m, m+1, \ldots, 2m-1. \end{cases}$$

The other $D^{\mathcal{L}}$-eigenvalues of $\Gamma(\mathbb{Z}_n)$ are the eigenvalues of matrix (5.7).

Proof. For $n = p^{2m}$, where m is positive integer and p is prime, the proper divisors of p^{2m} are $p, p^2, p^3, \dots, p^{m-1}, p^m, p^{m+1}, \dots, p^{2m-2}, p^{2m-1}$ and the structure of $\Gamma(\mathbb{Z}_n)$ with the help of Lemma 5.3 was found by Rather et al. [42] and is given below

$$\Gamma(\mathbb{Z}_n) = \gamma_n\left[\overline{K}_{\phi(p^{2m-1})}, \overline{K}_{\phi(p^{2m-2})}, \dots, \overline{K}_{\phi(p^{m+1})}, K_{\phi(p^m)}, \dots, K_{\phi(p^2)}, K_{\phi(p)}\right]. \tag{5.6}$$

In order to proceed further, we need effective informative about the underlying graph γ_n.

By the definition of γ_n, the adjacency relation of vertices are

$$\begin{aligned}
p &\sim p^{2m-1}\\
p^2 &\sim p^{2m-1}, p^{2m-2}\\
p^3 &\sim p^{2m-1}, p^{2m-2}, p^{2m-3}\\
&\vdots\\
p^{m-1} &\sim p^{2m-1}, p^{2m-2}, \dots, p^{m+2}, p^{m+1}\\
p^m &\sim p^{2m-1}, p^{2m-2}, \dots, p^{m+1}, p^m\\
p^{m+1} &\sim p^{2m-1}, p^{2m-2}, \dots, p^m, p^{m-1}\\
&\vdots\\
p^{2m-2} &\sim p^{2m-1}, p^{2m-2}, \dots, p^{m+1}, p^m, p^{m-1}, \dots, p^3, p^2\\
p^{2m-1} &\sim p^{2m-1}, p^{2m-2}, \dots, p^{m+1}, p^m, p^{m-1}, \dots, p^3, p^2, p
\end{aligned}$$

Using the above adjacency relations and by (5.6), $n_i = \phi(p^{2m-i})$ for $i = 1, 2, \dots, m-1, m, m+1, \dots, 2m-2, 2m-1$, the value of $n_i^{'}$'s are

$$\begin{aligned}
n_1^{'} &= n_{2m-1} + 2n_2 + 2n_3 + \cdots + 2n_{2m-2} = 2\sum_{i=2}^{2m-1} n_i - n_{2m-1},\\
n_2^{'} &= n_{2m-1} + n_{2m-2} + 2(n_{2m-3} + \cdots + n_3 + n_1) = 2\sum_{i=1, i\neq 2}^{2m-1} n_i - (n_{2m-1} + n_{2m-2}),\\
&\vdots\\
n_{m-1}^{'} &= n_{2m-1} + \cdots + n_{m+1} + 2(n_m + n_{m-2} + \cdots + n_2 + n_1) = 2\sum_{i=1, i\neq m-1}^{2m-1} n_i - \sum_{i=1}^{m-1} n_{2m-i},
\end{aligned}$$

that is,

$$n_i^{'} = 2\sum_{j=1, j\neq i}^{2m-1} n_j - \sum_{j=1}^{i} n_{2m-j}, \text{ for } i = 1, 2, \dots, m-2, m-1.$$

Similarly,

$$n_m^{'} = n_{2m-1} + n_{2m-2} + \cdots + n_{m+2} + n_{m+1} + 2(n_{m-1} + n_{m-2} + \cdots + n_2 + n_1)$$

and in general

$$n_i^{'} = \sum_{j=1, j\neq i}^{2m-1} n_j + \sum_{j=1}^{2m-1-i} n_j, \text{ for } i = m, m+1, \dots, 2m-2, 2m-1.$$

Also, $T_1 = 2n_1 + n_1^{'} - 2 = \sum\limits_{i=1}^{2m-1} n_i - 2 - \phi(p) = 2N - 2 - \phi(p)$, where N is the order of $\Gamma(\mathbb{Z}_n)$ and equals $N = \sum\limits_{i=1}^{2m-1} \phi(p^i) = p^{2m-1} - 1$. Arguing as above, we have

$$T_2 = 2n_2 + n_2^{'} - 2 = 2\sum_{i=1}^{2m-1} n_i - 2 - (\phi(p^2) + \phi(p)),$$

$$\vdots$$

$$T_{m-1} = 2n_{m-1} + n_{m-1}^{'} - 2 = 2\sum_{i=1}^{2m-1} n_i - 2 - \sum_{i=1}^{m-1} \phi(p^i),$$

or

$$T_i = 2N - 1 - p^i, \text{ for } i = 1, 2, \ldots, m-1.$$

Also,

$$T_m = 2n_m + n_m^{'} - 2 = \sum_{i=1}^{2m-1} n_i - 2 + n_m + \sum_{i=1}^{m-1} n_i = N - 2 + \phi(p^m) + p^{2m-1} - p^m,$$

and for $i = m+1, m+2, \ldots, 2m-2, 2m-1$, we have

$$T_i = \sum_{j=1}^{2m-1} n_j - 2 + n_i + \sum_{j=1}^{i} n_j = N - 2 + \phi(p^{2m-i}) + p^{2m-1} - p^i.$$

Now, applying Theorem 5.5, the $D^{\mathcal{L}}$-spectrum of $\Gamma(\mathbb{Z}_n)$ consists of the eigenvalues $1 + \frac{2}{T_i}$ with multiplicity $\phi(p^{2m-i}) - 1$, for $i = 2, 3, \ldots, m-1$, and the eigenvalues $1 + \frac{1}{T_i}$ with multiplicities $\phi(p^{2m-i}) - 1$, for $i = m, m+1, \ldots, 2m-2, 2m-1$. While calculating the values of T_i's, we have used the fact that $\sum\limits_{i=1}^{r} \phi(p^r) = p^r - 1$. The other $D^{\mathcal{L}}$-eigenvalues of $\Gamma(\mathbb{Z}_n)$ are the eigenvalues of following matrix

$$\begin{pmatrix}
d_1^{'} & \frac{-2\phi(p^{M-2})}{\sqrt{T_1T_2}} & \cdots & \frac{-2\phi(p^{m+1})}{\sqrt{T_1T_{m-1}}} & \frac{-2\phi(p^{m})}{\sqrt{T_1T_m}} & \cdots & \frac{-2\phi(p^{2})}{\sqrt{T_1T_{M-2}}} & \frac{-\phi(p)}{\sqrt{T_1T_{M-1}}} \\
\frac{-2\phi(p^{M-1})}{\sqrt{T_2T_1}} & d_2^{'} & \cdots & \frac{-2\phi(p^{m+1})}{\sqrt{T_2T_{m-1}}} & \frac{-2\phi(p^{m})}{\sqrt{T_2T_m}} & \cdots & \frac{-\phi(p^{2})}{\sqrt{T_2T_{M-2}}} & \frac{-\phi(p)}{\sqrt{T_2T_{M-1}}} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots & \vdots \\
\frac{-2\phi(p^{M-1})}{\sqrt{T_{m-1}T_1}} & \frac{-2\phi(p^{M-2})}{\sqrt{T_{m-1}T_2}} & \cdots & d_{m-1}^{'} & \frac{-2\phi(p^{m})}{\sqrt{T_{m-1}T_m}} & \cdots & \frac{-\phi(p^{2})}{\sqrt{T_{m-1}T_{M-2}}} & \frac{-\phi(p)}{\sqrt{T_{m-1}T_{M-1}}} \\
\frac{-2\phi(p^{M-1})}{\sqrt{T_mT_1}} & \frac{-2\phi(p^{M-2})}{\sqrt{T_mT_2}} & \cdots & \frac{-2\phi(p^{m+1})}{\sqrt{T_mT_{m-1}}} & d_m^{'} & \cdots & \frac{-\phi(p^{2})}{\sqrt{T_mT_{M-2}}} & \frac{-\phi(p)}{\sqrt{T_mT_{M-1}}} \\
\frac{-2\phi(p^{M-1})}{\sqrt{T_{m+1}T_1}} & \frac{-2\phi(p^{M-2})}{\sqrt{T_{m+1}T_2}} & \cdots & \frac{-\phi(p^{m+1})}{\sqrt{T_{m+1}T_{m-1}}} & \frac{-\phi(p^{m})}{\sqrt{T_{m+1}T_m}} & \cdots & \frac{-\phi(p^{2})}{\sqrt{T_{m+1}T_{M-2}}} & \frac{-\phi(p)}{\sqrt{T_{m+1}T_{M-1}}} \\
\vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\frac{-2\phi(p^{M-1})}{\sqrt{T_{M-2}T_1}} & \frac{-2\phi(p^{M-2})}{\sqrt{T_{M-2}T_2}} & \cdots & \frac{-\phi(p^{m+1})}{\sqrt{T_{M-2}T_{m-1}}} & \frac{-\phi(p^{m})}{\sqrt{T_{M-2}T_m}} & \cdots & d_{M-2}^{'} & \frac{-\phi(p)}{\sqrt{T_{M-2}T_{M-1}}} \\
\frac{-2\phi(p^{M-1})}{\sqrt{T_{M-1}T_1}} & \frac{-\phi(p^{M-2})}{\sqrt{T_{M-1}T_2}} & \cdots & \frac{-\phi(p^{m+1})}{\sqrt{T_{M-1}T_{m-1}}} & \frac{-\phi(p^{m})}{\sqrt{T_{M-1}T_m}} & \cdots & \frac{-\phi(p^{2})}{\sqrt{T_{M-1}T_{M-2}}} & d_{M-1}^{'}
\end{pmatrix} \tag{5.7}$$

where $d_i^{'} = \begin{cases} 1 - \frac{2\phi(p^{2m-i})-2}{T_i}, & \text{for } i = 1, 2, \ldots, m-1, \\ 1 - \frac{\phi(p^{2m-i})-1}{T_i}, & \text{for } i = m, m+1, \ldots, 2m-1. \end{cases}$ □

Following the similar steps as in Corollary 5.3, the proof of odd case $n = p^{2m+1}$ can be worked out.

Let the *canonical decomposition* of n be $n = p_1^{n_1} p_2^{n_2} \ldots p_r^{n_r}$, where $r, n_1, n_2, \ldots, n_r$ are positive integers and $p_1, p_2, \ldots, p_r$ are distinct prime numbers. Then we have the following problem.

Problem 5.4. *Discuss the $D^{\mathcal{L}}$-eigenvalues of $\Gamma(\mathbb{Z}_n)$, where n is in canonical decomposition and relate the spectral properties with the algebraic properties of the ring $\mathbb{Z}_n$.*

6. Conclusion

The normalized distance Laplacian matrix $D^{\mathcal{L}}(G)$ is a newly introduced distance based analogue of the normalized Laplacian matrix. Basics properties of $D^{\mathcal{L}}$-matrix were given in [46]. The results were further elaborated and some spectral characterizations including energy of $D^{\mathcal{L}}$-matrix are given in [24]. In the present paper, the $D^{\mathcal{L}}$-eigenvalues of joined union of regular graphs were obtained along with their applications to graphs defined from algebraic structures, like the power graphs of integer modulo groups, the commuting graphs of non-abelian groups and the zero-divisor graphs of commutative rings.

Eigenvalues of a graph matrix are almost connected to each structural property of the graph. It is this connection which makes the spectral study of graphs with respect to a given graph matrix interesting in Spectral graph theory. Regarding the graphs which arise from algebraic structures, spectral analysis with respect to a given graph matrix can be helpful to answer many questions which are concerned to their algebraic structure. Further, from Matrix theory point of view the spectral study of graphs which arise from algebraic structures is important as these graphs form a small class of graphs and so many interesting problems like spectral determination, extremal graphs for spectral norm, extremal graphs for trace norm, integral spectrum, graphs with few eigenvalues, etc, which are important from Matrix theory point of view, can be easily discussed here. The work in this paper is an effort to lay down a foundation for the study the algebraic graphs with respect to a new graph matrix, namely the normalized distance Laplacian matrix. We conclude this paper with some interesting problems which can be considered in future.

Although we have discussed the $D^{\mathcal{L}}$-eigenvalues of the power graphs of integer modulo groups, the commuting graphs of non-abelian groups and the zero-divisor graphs of commutative rings. The following problem can be of interest.

Problem 6.5. *Discuss the $D^{\mathcal{L}}$-eigenvalues of the algebraic graphs like Cayley graphs, compressed zero divisor graphs of commutative rings, zero divisor graphs of modules.*

The authors in [24] have introduced the distance analogue of the general Randić topological index. Also, the relations between the distance Randić matrix and the $D^{\mathcal{L}}$-matrix are presented in [25]. The fruitful properties of the $D^{\mathcal{L}}$-matrix, the distance Randić matrix and the distance general Randić matrix are yet to be elaborated more. Therefore, we leave the following problem.

Problem 6.6. *Discuss the distance general Randić index as a topological index and explore its properties.*

The spectral graph invariants like, Estrada index is well studied for other graph matrices. Therefore, the following problem can be considered in future.

Problem 6.7. *Define the Estrada index for the matrix $D^{\mathcal{L}}(G)$ of graph G and explore its properties.*

Further, one can think about the study of the Ky Fan norm, the Schatten p-norm, extremal spectral graph spectral characterizations, the spectral spread and several other interesting problems about the matrix $D^{\mathcal{L}}(G)$ in future.

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