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In memoriam Professor Charles E. Chidume (1947- 2021)

On Fixed Points of Enriched Contractions and Enriched Nonexpansive Mappings

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ABSTRACT. We apply the concept of quasilinearization to introduce some enriched classes of Banach contraction mappings and analyse the fixed points of such mappings in the setting of Hadamard spaces. We establish existence and uniqueness of the fixed point of such mappings. To approximate the fixed points, we use an appropriate Krasnoselskij-type scheme for which we establish Δ and strong convergence theorems. Furthermore, we discuss the fixed points of local enriched contractions and Maia-type enriched contractions in Hadamard spaces setting. In addition, we establish demiclosedness-type property of enriched nonexpansive mappings. Finally, we present some special cases and corresponding fixed point theorems.

1. INTRODUCTION

Let $\mathcal{H}$ be a nonempty set and $T : \mathcal{H} \to \mathcal{H}$ be a mapping. A point $p \in \mathcal{H}$ is said to be a *fixed point* of T if $Tp = p$. We denote the set of fixed points of T by $F(T)$, that is, $F(T) = \{z \in \mathcal{H} : z = Tz\}$. Fixed points of certain mappings offer substantial role in solving many real-life problems. As a result, many theorems concerning existence, uniqueness or approximation of fixed points have been established and have proven to have vast applications in sciences and engineering. For details, we refer the interesting reader to see the monographs [33, 14, 9, 26] and the references therein.

One of the most applicable fixed point theorems in metric spaces is the remarkable Banach contraction mapping theorem - which can be traced to Banach [3]. This theorem affirms that a contraction mapping on a complete metric space onto itself always has a unique fixed point and that the Picard iteration approximates the fixed point. However, for wider classes of mappings (for example, the class of nonexpansive mappings), the mapping may not have fixed point or may have more than one fixed point or even have exactly one fixed point yet the Picard iteration fail to converge to the point. This, among other reasons, made the study of generalised classes of contraction mappings active area of research. See, for example, [29, 12, 21] and the references therein.

Fixed point theory involving class of nonexpansive mappings in Hadamard spaces were first studied and analysed by Kirk [27, 24]. The author proved, among other things, the following theorem.

Theorem 1.1. *Suppose $\mathcal{H}$ is a nonempty closed bounded convex subset of a Hadamard space and suppose $T : \mathcal{H} \to \mathcal{H}$ is nonexpansive. Then $F(T)$ is nonempty, closed and convex.*

Since then, the fixed point theory involving certain generalised classes of mappings in setting of Hadamard spaces has received much attention (see, for example, [15, 17, 1,

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31, 23, 30] and the references therein). In [4], Berg and Nikolaev consider a Hadamard space $(\mathcal{H}, d)$, denoted $(u, w) \in \mathcal{H} \times \mathcal{H}$ by $\overrightarrow{uw}$ and defined a quasilinearization map $\langle \cdot, \cdot \rangle : (\mathcal{H} \times \mathcal{H}) \times (\mathcal{H} \times \mathcal{H}) \to \mathbb{R}$ by

$$\langle \overrightarrow{uw}, \overrightarrow{vz} \rangle = \frac{1}{2}\left(d^2(u,z) + d^2(w,v) - d^2(u,v) - d^2(w,z)\right), \tag{1.1}$$

for every points $u, v, w, z \in \mathcal{H}$.

On the other hands, Berinde introduced and studied the class of enriched contraction mappings in [8] as a proper superclass of the class of Banach contraction mappings, and class of enriched nonexpansive in [6] as a superclass of the class of nonexpansive mappings. The authors definitions can be seen as follows: Let $(\mathcal{H}, \|\cdot\|)$ be a normed linear space and a mapping $T : \mathcal{H} \to \mathcal{H}$ is said to be an *enriched contraction* (or (α, β)-enriched contraction) if there exist $\alpha \in [0, +\infty)$ and $\beta \in [0, \alpha + 1)$ such that

$$\|\alpha(u - w) + Tu - Tw\| \leq \beta\|u - w\|, \ \forall u, w \in \mathcal{H}. \tag{1.2}$$

Moreover, if the inequality (1.2) holds for $\beta = \alpha + 1$, then T is said to be an *enriched nonexpansive mapping*. We state in the following the main result of Berinde [8].

Theorem 1.2. *Let $(\mathcal{H}, \|\cdot\|)$ be a Banach space and suppose $T : \mathcal{H} \to \mathcal{H}$ is an (α, β)-enriched contraction. Then*

(i) $F(T) = \{p\}$;

(ii) there exists $\sigma \in (0, 1]$ such that the iterative method $\{u_n\}$, given by

$$u_{n+1} = (1 - \sigma)u_n + \sigma T u_n, \ n \geq 1, \tag{1.3}$$

converges to p, for any $u_1 \in \mathcal{H}$;

(iii) the following estimate holds

$$\|u_{n+i-1} - p\| \leq \frac{c^i}{1 - c}\|u_n - u_{n-1}\|, n = 1, 2, \cdots; \ i = 1, 2, \cdots, \tag{1.4}$$

where $c = \dfrac{\beta}{\alpha + 1}$.

Although the celebrated Banach contraction mapping theorem was established in the setting of complete metric spaces which need not have linear structure, many results complementing the Banach contraction and its generalizations have been established in setting of linear spaces (see, for example, [8, 6, 32, 12, 11, 9, 7, 5]). These results are highly substantial in nonlinear convex analysis and optimizations. However, it was recently observed that many non-convex problems in the linear settings can be viewed as convex problems in Hadamard space (see, for example, [19, Example 5.2]). This, among other reasons, make Hadamard spaces appropriate setting for the study on nonlinear convex analysis and optimizations. Moreover, many results in this setting have applications in various fields, for example, the minimizers of the energy functionals have been used in geometry, the gradient flow investigates the asymptotic behavior of the Calabi flow in Kahler geometry (see, for example, [22, 2]). Our aim is to extend and unify the recent results in [8, 6] to the setting of Hadamard spaces. We establish our result using the concept of quasilinearization - which was introduced in [4] and our results generalise many existing results including the result of Kirk[27] in Theorem 1.1.

To this end, we need some basic concepts and known results related to Hadamard spaces which will be used in obtaining our main results.

2. PRELIMINARIES

Let $(\mathcal{H}, d)$ be a metric space and u, w be two points in $\mathcal{H}$. A geodesic from u to w is a mapping $\tau : \big[0, d(u,w)\big] \subset \mathbb{R} \to \mathcal{H}$ with the properties that $\tau(0) = u$, $\tau\big(d(u,w)\big) = w$ and $d\left(\tau(a), \tau(b)\right) = |a-b|$ for every $a, b \in \big[0, d(u,w)\big]$. The image $\tau\left(\big[0, d(u,w)\big]\right)$ of τ is called a *geodesic segment* connecting u and w. when $\tau\left(\big[0, d(u,w)\big]\right)$ is unique, it is denoted by $[u, w]$. A geodesic space is a metric space $(\mathcal{H}, d)$ in which every two elements are joined by a geodesic segment. For $u, w \in \mathcal{H}$ having unique geodesic segment and for any $\sigma \in [0,1]$, there exists a unique point z on the segment connecting u and w, denoted by $(1-\sigma)u \oplus \sigma w$ with the following conditions:

$$d(u,z) = \sigma d(u,w) \quad \text{and} \quad d(z,w) = (1-\sigma)d(u,w). \tag{2.1}$$

A *geodesic triangle* $\triangle(u_1, u_2, u_3)$ in a geodesic space $(\mathcal{H}, d)$ consists of three points in $\mathcal{H}$ and three geodesic segments-each for a pair of the vertices. A *comparison triangle* for a geodesic triangle $\triangle(u_1, u_2, u_3)$ in $(\mathcal{H}, d)$ is a triangle $\overline{\triangle}(\bar{u}_1, \bar{u}_2, \bar{u}_3)$ such that $\|\bar{u}_i - \bar{u}_j\|_2 = d(u_i, u_j)$ for $i, j \in \{1,2,3\}$. A geodesic space $(\mathcal{H}, d)$ is called a $CAT(0)$ *space* if every geodesic triangle $\triangle$ in $(\mathcal{H}, d)$ having comparison triangle $\overline{\triangle}$, the inequality

$$d(x,y) \le \|\bar{x} - \bar{y}\|_2 \tag{2.2}$$

holds for all points x, y in $\triangle$ with corresponding comparison points $\bar{x}, \bar{y}$ in $\overline{\triangle}$ (where a point $\bar{w} \in [\bar{x}, \bar{y}]$ is called a *comparison point* of a point $w \in [x, y]$ if $\|\bar{x} - \bar{w}\|_2 = d(x, w)$). Following (2.2), it is immediate that a geodesic space $(\mathcal{H}, d)$ is said to be a CAT(0) space if and only if it satisfies the CN-inequality of Bruhat and Tits [13] stated as follows. Let $u, w \in \mathcal{H}$, then

$$d\left(\frac{1}{2}u \oplus \frac{1}{2}w, y\right)^2 \le \frac{1}{2}d(w,y)^2 + \frac{1}{2}d^2(w,y) - \frac{1}{4}d^2(u,w), \tag{2.3}$$

for every $y \in \mathcal{H}$. Furthermore, CAT(0) spaces include pre-Hilbert spaces, Hilbert balls, Euclidean buildings, $\mathbb{R}$-trees and Hadamard manifolds, and a complete $CAT(0)$ space is called a *Hadamard space*. For further details on general $CAT(\kappa)$ spaces, see, for example, [10, 26] and the references therein.

As direct consequences of (2.1) and (2.2), for $u_1, u_2, u_3 \in \mathcal{H}$ and $t \in [0,1]$, the following inequalities hold (see also [18]):

$$d((1-t)u_1 \oplus tu_2, u_3) \le (1-t)d(u_1, u_3) + td(u_2, u_3); \tag{2.4}$$

$$d((1-t)u_1 \oplus tu_2, u_3)^2 \le (1-t)d(u_1, u_3)^2 + td(u_2, u_3)^2 - t(1-t)d(u_1, u_2)^2. \tag{2.5}$$

Let $(\mathcal{H}, d)$ be a Hadamard space and $\{v_n\}$ be a bounded sequence in $\mathcal{H}$. Then the *asymptotic center* $A(\{v_n\})$ of $\{v_n\}$ is defined by

$$A(\{v_n\}) := \big\{u \in \mathcal{H} : \limsup_{n\to\infty} d(u, v_n) = \inf_{u \in \mathcal{H}} \limsup_{n\to\infty} d(u, v_n)\big\}.$$

It is important to note that, as in [17, Proposition 7], in a Hadamard space, $A(\{v_n\})$ has exactly one element. Also, the sequence $\{v_n\}$ is said to Δ-converge to a point v in $\mathcal{H}$ if $\{v\}$ is the unique asymptotic centre for every subsequence $\{v_{n_k}\}$ of $\{v_n\}$. We write $v_n \xrightarrow{\Delta} v$ to mean $\{v_n\}$ is Δ-convergent to v. When a sequence $\{v_n\}$ converges to v in the usual sense, that is, $d(v_n, v) \to 0$, we say it is strongly convergent to v, denoted $v_n \to v$. Moreover, a subset E of CAT(0) space $\mathcal{H}$ is *convex* if all geodesic segments connecting any two points of E are contained in E.

In the sequel, we take $(\mathcal{H}, d)$ to be a Hadamard space and $E \subseteq \mathcal{H}$ nonempty closed convex. We shall say that a mapping T has *demiclosedness-type property* if the conditions $\{w_n\}$ Δ-converges to w and $d(w_n, Tw_n) \to 0$, imply $w = Tw$.

We now state definitions and lemmas that will be used in obtaining our main results.

Definition 2.1. A mapping $T : E \to E$ is called:

(i) *k-contraction* if there exists $k \in [0, 1)$ such that

$$d(Tu, Tw) \leq kd(u, w),\ \forall\, u, w \in E;$$

(ii) *nonexpansive* if

$$d(Tu, Tw) \leq d(u, w),\ \forall\, u, w \in E.$$

It is clear from Definition 2.1 that every k-contraction mapping is nonexpansive.

Lemma 2.1. [25, Proposition 3.7] *Every nonexpansive mapping $T : E \to E$ has demiclosedness-type property.*

Lemma 2.2. [16] *The asymptotic centre of any bounded sequence in E is contained in E.*

Lemma 2.3. [25, Proposition 3.6] *Every bounded sequence $\{w_n\}$ in E has a Δ-convergent subsequence $\{w_{n_k}\}$ in E.*

Lemma 2.4. [18, Lemma 2.8] *Let $\{v_n\}$ be a sequence in $\mathcal{H}$ with $A(\{v_n\}) = \{v\}$. Suppose that $\{v_{n_k}\}$ is a subsequence of $\{v_n\}$ with $A(\{v_{n_k}\}) = \{w\}$ and the sequence $\{d(v_n, w)\}$ converges. Then $v = w$.*

3. Enriched contraction mappings

Definition 3.2. Let $(\mathcal{H}, d)$ be a Hadamard space. We called a mapping $T : \mathcal{H} \to \mathcal{H}$ an enriched contractions if there exist two real numbers $\alpha \in [0, +\infty)$ and $\beta \in [0, \alpha + 1)$ such that

$$d(Tu, Tw)^2 + \alpha^2 d(u, w)^2 + 2\alpha\langle \overrightarrow{uw}, \overrightarrow{TuTw}\rangle \leq \beta^2 d(u, w)^2,\ \forall u, w \in \mathcal{H}. \tag{3.1}$$

To specify the constant involved in (3.1), we henceforth called T an (α, β)-enriched contraction. This name is motivated by [8, Definition 2.1].

It is very important to note from Definition 3.2 that:

(a) If $T : \mathcal{H} \to \mathcal{H}$ is k-contraction, then T is (α, β)-enriched contraction with $\alpha = 0$ and $\beta = k$. However, Example 1 of [8] furnishes a counterexample for the converse.

(b) If $\mathcal{H}$ is Hilbert space, then using the following known property of Hilbert spaces:

$$\|u + w\|^2 = \|u\|^2 + \|w\|^2 + 2\langle u, w\rangle$$

and the definition of quasilinearization, we can easily deduce that Definition 3.2 reduces to the definition of an (α, β)-enriched contraction as stated in (1.2).

Lemma 3.5. *Let $(\mathcal{H}, d)$ be a Hadamard space and let $T : \mathcal{H} \to \mathcal{H}$ be a mapping. For $\sigma \in (0, 1]$, let T_σ be defined by*

$$T_\sigma u := (1 - \sigma)u \oplus \sigma Tu,\ \forall u \in \mathcal{H}. \tag{3.2}$$

Then

(i) $F(T) = F(T_\sigma)$;

(ii) $d\big(T_\sigma u, T_\sigma w\big)^2 \leq (1 - \sigma)^2 d\big(u, w\big)^2 + \sigma^2 d\big(Tu, Tw\big)^2 + 2\sigma(1 - \sigma)\langle \overrightarrow{uw}, \overrightarrow{TuTw}\rangle$;

(iii) If T is an (α, β)-enriched contraction, then there exists $\sigma_o \in (0, 1]$ such that T_{σ_o} is k-contraction on $\mathcal{H}$.

Proof. For $\sigma = 1$, $T_\sigma = T$ and (i) follows trivially. Let $\sigma \in (0,1)$ and let $p \in F\big(T_\sigma\big)$. By (2.1), we have

$$d\big(p, Tp\big) = \frac{1}{(1-\sigma)} d\big(p, T_\sigma p\big) = 0, \tag{3.3}$$

which implies that $p \in F\big(T\big)$.

For the converse, suppose $p \in F\big(T\big)$. Then by (2.4), we have

$$d\big(p, T_\sigma p\big) \leq d\big(p, Tp\big) = 0, \tag{3.4}$$

which implies that $p \in F\big(T_\sigma\big)$. Hence, (i) is proven.

The inequality (ii) follows from from (3.2) and (2.5) as follows:

$$\begin{aligned} d\big(T_\sigma u, T_\sigma w\big)^2 &\leq (1-\sigma) d\big(u, T_\sigma w\big)^2 + \sigma d\big(Tu, T_\sigma w\big)^2 - \sigma(1-\sigma) d\big(u, Tu\big)^2 \\ &\leq (1-\sigma)\Big[(1-\sigma) d\big(u,w\big)^2 + \sigma d\big(u, Tw\big)^2 \\ &\quad - \sigma(1-\sigma) d\big(w, Tw\big)\Big] + \sigma\Big[(1-\sigma) d\big(Tu, w\big)^2 \\ &\quad + \sigma d\big(Tu, Tw\big)^2 - \sigma(1-\sigma) d\big(w, Tw\big)^2\Big] - \sigma(1-\sigma) d\big(u, Tu\big)^2 \\ &= (1-\sigma)^2 d\big(u,w\big)^2 + \sigma^2 d\big(Tu, Tw\big)^2 + \sigma(1-\sigma)\Big[d\big(u, Tw\big)^2 \\ &\quad + d\big(w, Tu\big)^2 - d\big(u, Tu\big)^2 - d\big(w, Tw\big)^2\Big] \\ &= (1-\sigma)^2 d\big(u,w\big)^2 + \sigma^2 d\big(Tu, Tw\big)^2 + 2\sigma(1-\sigma)\langle \overrightarrow{uw}, \overrightarrow{TuTw}\rangle. \end{aligned}$$

Next, suppose T is an (α,β)-enriched contraction. For $\alpha = 0$, take $\sigma_o = 1$. Then we obtain from (ii) that

$$d\big(T_{\sigma_o} u, T_{\sigma_o} w\big)^2 \leq d\big(Tu, Tw\big)^2 \leq \beta^2 d(u,w)^2,$$

for some $\beta \in [0,1)$. Consequently, we have

$$d\big(T_{\sigma_o} u, T_{\sigma_o} w\big) \leq \beta d(u,w)$$

which implies that T_{σ_o} is k-contraction with $k = \beta$.

Now for $\alpha > 0$, take $\sigma_o = \dfrac{1}{1+\alpha}$. Then we have from (ii) that

$$\begin{aligned} d\big(T_{\sigma_o} u, T_{\sigma_o} w\big)^2 &\leq \frac{\alpha^2}{(1+\alpha)^2} d\big(u,w\big)^2 + \frac{1}{(1+\alpha)^2} d\big(Tu, Tw\big)^2 \\ &\quad + 2\frac{\alpha}{(1+\alpha)^2}\langle \overrightarrow{uw}, \overrightarrow{TuTw}\rangle \\ &\leq \frac{1}{(1+\alpha)^2}\Big[d\big(Tu, Tw\big)^2 + \alpha^2 d\big(u,w\big)^2 + 2\alpha\langle \overrightarrow{uw}, \overrightarrow{TuTw}\rangle\Big] \\ &\leq \frac{\beta^2}{(1+\alpha)^2} d(u,w)^2. \end{aligned}$$

Consequently, we have

$$d\big(T_{\sigma_o} u, T_{\sigma_o} w\big) \leq \frac{\beta}{(1+\alpha)} d(u,w)$$

which implies that T_{σ_o} is k-contraction with $k = \dfrac{\beta}{(1+\alpha)} \in [0,1)$. Thus, the proof is complete. □

Theorem 3.3. *Let $(\mathcal{H}, d)$ be a Hadamard space and let $T : \mathcal{H} \to \mathcal{H}$ be an (α, β)-enriched contraction. Then the following statements hold.*

(i) T has a unique fixed point.

(ii) There exists $\sigma \in (0, 1]$ such that the sequence $\{u_n\}$ defined iteratively by

$$\begin{cases} u_1 \in \mathcal{H}, \\ u_{n+1} = (1-\sigma)u_n \oplus \sigma T u_n, \quad n \geq 1, \end{cases}$$

converges strongly to the fixed point of T.

(iii) There exists $k \in [0, 1)$ such that

$$d(u_{n+j-1}, p) \leq \frac{k^j}{1-k} d(u_n, u_{n-1}), \; n \geq 1, j \geq 1. \tag{3.5}$$

Proof. By Lemma 3.5, there exists $\sigma \in (0, 1]$ such that T_σ is k-contraction, that is, there exists $k \in [0, 1)$ such that

$$d\big(T_\sigma u, T_\sigma w\big) \leq kd(u, w), \quad \forall u, w \in \mathcal{H}. \tag{3.6}$$

We note that by definition of T_σ, the iterative scheme associated to T can be rewritten as follows:

$$\begin{cases} u_1 \in \mathcal{H}; \\ u_{n+1} = T_\sigma u_n, \quad n \geq 1. \end{cases} \tag{3.7}$$

Let $n \geq 2$. Replacing u by u_n and w by u_{n-1} in (3.6) and using (3.7), we get

$$d\big(u_{n+1}, u_n\big) \leq kd\big(u_n, u_{n-1}\big). \tag{3.8}$$

Inductively, we obtain

$$d\big(u_{n+1}, u_n\big) \leq k^{n-1} d\big(u_2, u_1\big), \quad n \geq 1. \tag{3.9}$$

Thus, for any $m, n \geq 1$, we have

$$\begin{aligned} d\big(u_{n+m}, u_n\big) &\leq d\big(u_n, u_{n+1}\big) + d\big(u_{n+1}, u_{n+2}\big) + \cdots + d\big(u_{n+m-1}, u_{n+m}\big) \\ &\leq k^{n-1} d\big(u_2, u_1\big) + k^n d\big(u_2, u_1\big) + \cdots + k^{n+m-2} d\big(u_2, u_1\big) \\ &= k^{n-1}\big[1 + k + \cdots + k^{m-1}\big] d\big(u_2, u_1\big) \\ &\leq k^{n-1} d\big(u_2, u_1\big) \sum_{j=1}^{+\infty} k^j \\ &\leq \frac{k^{n-1}}{1-k} d\big(u_2, u_1\big). \end{aligned}$$

Hence,

$$d\big(u_{n+m}, u_n\big) \leq \frac{k^{n-1}}{1-k} d\big(u_2, u_1\big), \quad n, m \geq 1. \tag{3.10}$$

This implies that $\{u_n\}$ is a Cauchy sequence and hence it converges in the Hadamard space $(\mathcal{H}, d)$. Let the limit of $\{u_n\}$ be u^*, that is,

$$\lim_{n \to \infty} u_n = u^*. \tag{3.11}$$

By letting $n \to \infty$ in (3.9), we have

$$d\big(T_\sigma u^*, u^*\big) \leq 0.$$

Hence, $u^* \in F\big(T_\sigma\big)$. Next, we prove that u^* is the unique fixed point of T_σ. Suppose that there exists $u^o \in F\big(T_\sigma\big)$ different from u^*. Then By (3.6), we get

$$0 < d\big(u^*, u^o\big) = d\big(T_\sigma u^*, T_\sigma u^o\big) \le kd\big(u^*, u^o\big) < d\big(u^*, u^o\big),$$

which is a contradiction. Hence, $F\big(T_\sigma\big) = \{u^*\}$. By Lemma 3.5(i), we have that $F\big(T\big) = \{u^*\}$. Thus, we have (i), and (ii) follows from (3.11).

To prove (iii), we estimate using (3.8) as follows:

$$\begin{aligned}
d\big(u_{n+m}, u_n\big) &\le d\big(u_n, u_{n+1}\big) + d\big(u_{n+1}, u_{n+2}\big) + \cdots + d\big(u_{n+m-1}, u_{n+m}\big)\\
&\le kd\big(u_n, u_{n-1}\big) + k^2 d\big(u_n, u_{n-1}\big) + \cdots + k^m d\big(u_n, u_{n-1}\big)\\
&= \left(\sum_{j=1}^{m} k^j\right) d\big(u_n, u_{n-1}\big)\\
&\le k\frac{1-k^m}{1-k} d\big(u_n, u_{n-1}\big)\\
&\le \frac{k}{1-k} d\big(u_n, u_{n-1}\big), \quad n, m \ge 1.
\end{aligned}$$

Hence, we have

$$d\big(u_{n+m}, u_n\big) \le \frac{k}{1-k} d\big(u_n, u_{n-1}\big).$$

Moreover, letting $m \to \infty$, we get

$$d\big(u^*, u_n\big) \le \frac{k}{1-k} d\big(u_n, u_{n-1}\big). \tag{3.12}$$

This and (3.8) imply

$$\begin{aligned}
d\big(u_{n+j-1}, u^*\big) &\le \frac{k}{1-k} d\big(u_{n+j-1}, u_{n+j-2}\big)\\
&\le \frac{k^2}{1-k} d\big(u_{n+j-2}, u_{n+j-3}\big)\\
&\;\;\vdots\\
&\le \frac{k^j}{1-k} d\big(u_n, u_{n-1}\big), \quad n, j \ge 1,
\end{aligned}$$

as desired.

We immediately have the following corollary.

□

Corollary 3.1. *Suppose that $\big(\mathcal{H}, \|\cdot\|\big)$ is a Hilbert space and $T : \mathcal{H} \to \mathcal{H}$ be an (α, β)-enriched contraction, that is, there exist $\alpha \in [0, +\infty)$ and $\beta \in [0, \alpha + 1)$ such that*

$$\|\alpha(u - w) + Tu - Tw\| \le \beta\|u - w\|, \ \forall u, w \in \mathcal{H}.$$

Then, the following statements hold.

(i) T has a unique fixed point.

(ii) There exists $\sigma \in (0, 1]$ such that the sequence $\{u_n\}$ defined iteratively by

$$\begin{cases} u_1 \in \mathcal{H},\\ u_{n+1} = (1-\sigma)u_n + \sigma T u_n, \quad n \ge 1, \end{cases}$$

converges to the fixed point of T;

(iii) There exists $k \in [0,1)$

$$\big\|u_{n+j-1} - p\big\| \le \frac{k^j}{1-k}\big\|u_n - u_{n-1}\big\|, \;\; n \ge 1, \; j \ge 1. \tag{3.13}$$

Due to applicability of Banach contraction mapping result, many different version of such result are published thereafter. Among which, in 2003, Granas and Dugundji [20] established a local version of contraction mapping theorem that involves an open ball in a complete matrix space $(\mathcal{H}, d)$ and a contraction map from the open ball to $\mathcal{H}$. The type of mapping considered does not displace the center of the ball very far and has shown to have many applications. We now present, in the following theorem, the analog of the mentioned result in the case of enriched contraction in Hadamard spaces.

Theorem 3.4. *Let $(\mathcal{H}, d)$ be a Hadamard space and let $T : B \to \mathcal{H}$ be a (α, β)-enriched contraction, where $B = B_r(u_o) := \{u \in \mathcal{H} : d(u, u_o) < r\}$ for some $r > 0$. If*

$$d\big(Tu_o, u_o\big) < (\alpha - \beta + 1)r,$$

then T has a fixed point.

Proof. Let ξ be choosing in $[0, r)$ such that

$$d\big(Tu_o, u_o\big) \le (\alpha - \beta + 1)\xi < (\alpha - \beta + 1)r. \tag{3.14}$$

Since T is an (α, β)-enriched contraction, it follows from Lemma 3.5 (iii) that there exists σ such that T_σ is k-contraction, that is,

$$d\big(T_\sigma u, T_\sigma w\big) \le kd(u, w), \;\; \forall u, w \in B, \tag{3.15}$$

with $k = \dfrac{\beta}{1+\alpha}$ as obtained in the proof. By (2.1), we have

$$d\big(T_\sigma u_o, u_o\big) = \sigma d\big(Tu_o, u_o\big). \tag{3.16}$$

It follows from (3.14) and (3.16) that

$$d\big(T_\sigma u_o, u_o\big) \le d\big(Tu_o, u_o\big) \le (\alpha - \beta + 1)\xi. \tag{3.17}$$

Consequently, we obtain that

$$d\big(T_\sigma u_o, u_o\big) \le \left(1 - \frac{\beta}{1+\alpha}\right)\xi. \tag{3.18}$$

We claim that the closed ball

$$\bar{B}_\xi(u_o) := \big\{w \in \mathcal{H} : d\big(w, u_o\big) \le \xi\big\}$$

is invarient under T_σ. To see this, let $u \in \bar{B}_\xi(u_o)$. Then from (3.15) and (3.18), we get

$$\begin{aligned} d\big(T_\sigma u, u_o\big) &\le d\big(T_\sigma u, T_\sigma u_o\big) + d\big(T_\sigma u_o, u_o\big) \\ &\le kd\big(u, u_o\big) + \left(1 - \frac{\beta}{1+\alpha}\right)\xi \\ &= \frac{\beta}{1+\alpha}\xi + \left(1 - \frac{\beta}{1+\alpha}\right)\xi = \xi. \end{aligned}$$

This implies that the range of T_σ over $\bar{B}_\xi(u_o)$ is contained in $\bar{B}_\xi(u_o)$. Since $\bar{B}_\xi(u_o)$ is complete, the conclusion follows by Theorem 3.3. □

We also have the following corrollary.

Corollary 3.2. *Let $(\mathcal{H}, \|\cdot\|)$ be a Hilbert space and let $T : B \to \mathcal{H}$ be a (α, β)-enriched contraction, where $B = B_r(u_o) := \{u \in \mathcal{H} : \|u - u_o\| < r\}$ for some $r > 0$. If*

$$\|u_o - Tu_o\| < (\alpha - \beta + 1)r,$$

then T has a fixed point.

It was observed in [8, Example 2] that there are mappings that do not satisfy (3.1) but have certain iterate that do satisfy. In such cases, we cannot apply Theorem 3.3 directly. However, the following result could be applicable.

Theorem 3.5. *Let $(\mathcal{H}, d)$ be a Hadamard space and let $G : \mathcal{H} \to \mathcal{H}$ be a mapping with the property that there exists $m \in \mathbb{N}$ such that G^m is an (α, β)-enriched contraction. Then*

(i) G has a unique fixed point;
(ii) there exists $\sigma \in (0, 1]$ such that the sequence $\{u_n\}$ defined iteratively by

$$\begin{cases} u_1 \in \mathcal{H}, \\ u_{n+1} = (1 - \sigma)u_n \oplus \sigma G^m u_n, \quad n \geq 1, \end{cases}$$

converges to the fixed point of G^m.

Proof. We apply Theorem 3.3(i) with $T = G^m$ and obtain that G^m has a unique fixed point. Moreover, if u^* is the fixed point of G^m, then

$$G^m\big(G(u^*)\big) = G^{m+1}(u^*) = G\big(G^m(u^*)\big) = G(u^*).$$

This implies that $G(u^*)$ is also a fixed point of G^m. Since, the fixed point of G^m is unique, we conclude that $G(u^*) = u^*$. Therefore, $u^* \in F(G)$. The remaining part of the proof follows similar lines to that of Theorem 3.3. □

Among the generalization of Banach contraction Mapping theorem is the Maia fixed point theorem discussed in the setting of metric space by Maia [28]. In the case of (α, β)-contraction, Berinde [8] gave the analog of Maia fixed point theorem in the setting of linear spaces as follows.

Theorem 3.6. [8, Theorem 3.6] *Let $\mathcal{H}$ be a linear space. Suppose $\|\cdot\|_d$ and $\|\cdot\|_\rho$ are two norms on $\mathcal{H}$, and $T : \mathcal{H} \to \mathcal{H}$ is a mapping. Suppose in addition, the following properties hold:*

(i) $\|u - w\|_d \leq \|u - w\|_\rho$, for each $u, w \in \mathcal{H}$;
(ii) $(\mathcal{H}, \|\cdot\|_d)$ is a Banach space;
(iii) $T : \mathcal{H} \to \mathcal{H}$ is continuous with respect to the norm $\|\cdot\|_d$;
(iv) T is an (α, β)-enriched contraction mapping with respect to the norm $\|\cdot\|_\rho$.

Then

(a) T has a unique fixed point;
(b) there exists $\sigma \in (0, 1]$ such that the sequence $\{u_n\}$ defined iteratively by

$$\begin{cases} u_1 \in \mathcal{H}, \\ u_{n+1} = (1 - \sigma)u_n + \sigma T u_n, \quad n \geq 1, \end{cases}$$

converges to the fixed point of T.

The next theorem concerns Maia-type fixed point theorem, which is an extension of Theorem 3.6 to setting of Hadamard spaces.

Theorem 3.7. *Let $\mathcal{H}$ be a geodesic space. Suppose d and ρ are two metrics on $\mathcal{H}$, and $T : \mathcal{H} \to \mathcal{H}$ is a mapping. Suppose in addition, the following properties hold:*

(i) $d(u,w) \le \rho(u,w)$, *for each* $u, w \in \mathcal{H}$*;*
(ii) $\big(\mathcal{H}, d\big)$ *is a Hadamard space;*
(iii) $T : \mathcal{H} \to \mathcal{H}$ *is continuous with respect to the metric* d*;*
(iv) T *is an* (α, β)*-enriched contraction mapping with respect to the metric* ρ*.*

Then

(a) T *has a unique fixed point;*
(b) there exists $\sigma \in (0,1]$ *such that the sequence* $\{u_n\}$ *defined iteratively by*

$$\begin{cases} u_1 \in \mathcal{H}, \\ u_{n+1} = (1-\sigma)u_n \oplus \sigma T u_n, \quad n \ge 1, \end{cases}$$

converges to the fixed point of T*.*

Proof. By hypothesis (iv) and Lemma 3.5, there exists $\sigma \in (0,1]$ such that for some $k \in [0,1)$,

$$\rho\big(T_\sigma u, T_\sigma w\big) \le k\rho(u,w), \quad \forall u, w \in \mathcal{H}. \tag{3.19}$$

Replacing u by u_n and w by u_{n-1} in (3.19) and using (3.7), we get

$$\rho\big(u_{n+1}, u_n\big) \le k\rho\big(u_n, u_{n-1}\big). \tag{3.20}$$

Inductively, we obtain

$$\rho\big(u_{n+1}, u_n\big) \le k^{n-1}\rho\big(u_2, u_1\big), \quad n \ge 1. \tag{3.21}$$

Thus, for any $m, n \ge 1$, we have

$$\begin{aligned} \rho\big(u_{n+m}, u_n\big) &\le \rho\big(u_n, u_{n+1}\big) + \rho\big(u_{n+1}, u_{n+2}\big) + \cdots + \rho\big(u_{n+m-1}, u_{n+m}\big) \\ &\le k^{n-1}\big[1 + k + \cdots + k^{m-1}\big]\rho\big(u_2, u_1\big) \\ &\le \frac{k^{n-1}}{1-k}\rho\big(u_2, u_1\big). \end{aligned}$$

Hence,

$$\rho\big(u_{n+m}, u_n\big) \le \frac{k^{n-1}}{1-k}\rho\big(u_2, u_1\big), \quad n, m \ge 1. \tag{3.22}$$

Therefore, $\{u_n\}$ is a Cauchy sequence in $(\mathcal{H}, \rho)$. By (i), we obtain that $\{u_n\}$ is also a Cauchy sequence in $(\mathcal{H}, d)$ and by (ii) we have that $\{u_n\}$ converges.

Now, let

$$u^* = \lim_{n\to\infty} u_n.$$

By (iii), we obtain that $u^* \in F(T_\sigma)$ for some $\sigma \in (0,1]$. By Theorem 3.3 (i) we have that u^* is unique. The conclusion follows from Lemma 3.5. □

4. Enriched Nonexpansive Mappings

Definition 4.3. Let $(\mathcal{H}, d)$ be a Hadamard space. A mapping $T : \mathcal{H} \to \mathcal{H}$ is called an enriched nonexpansive (or α-enriched nonexpansive) if there exists a real number $\alpha \in [0, +\infty)$ such that

$$d(Tu, Tw)^2 + \alpha^2 d(u,w)^2 + 2\alpha\langle \overrightarrow{uw}, \overrightarrow{TuTw}\rangle \le (\alpha+1)^2 d(u,w)^2, \tag{4.1}$$

for every points $u, w \in \mathcal{H}$.

It is important to note from Definition 4.3 that any nonexpansive mapping on $\mathcal{H}$ is also an α-enriched nonexpansive mapping with $\alpha = 0$. However, if we let $\mathcal{H} = \left[\frac{1}{2}, 2\right]$ be endowed with the usual metric and $T : \mathcal{H} \to \mathcal{H}$ be defined by $Tx = \frac{1}{x}$, then according to Example 1 of [6], T is $3/2$-enriched nonexpansive but not nonexpansive mapping.

Definition 4.4. Let $(\mathcal{H}, d)$ be a Hadamard space and E be a nonempty closed convex subset of $\mathcal{H}$. A mapping $T : E \to E$ is said to be asymptotically regular if, for each $u \in E$,

$$d\big(T^{n+1}u, T^n u\big) \to 0, \ as\ n \to \infty. \tag{4.2}$$

Lemma 4.6. *Let $(\mathcal{H}, d)$ be a Hadamard space and E be a nonempty closed convex subset of $\mathcal{H}$. Suppose $G : E \to E$ is nonexpansive and $F\big(G\big) \neq \emptyset$. Then, for any fixed $\lambda \in (0, 1)$, the mapping G_λ defined by $G_\lambda u = (1 - \lambda)u \oplus \lambda Gu$ is asymptotically regular.*

Proof. Let $u \in E$ and let $p \in F\big(G\big)$. Then for any natural number n, we have

$$\begin{aligned}
d\big(G_\lambda^{n+1}u, p\big)^2 &\leq d\big(G(G_\lambda^n u), p\big)^2 \\
&= d\big((1-\lambda)G_\lambda^n u \oplus \lambda G\big(G_\lambda^n u), p\big)^2 \\
&\leq (1-\lambda)d\big(G_\lambda^n u, p\big)^2 + \lambda d\big(G(G_\lambda^n u), p\big)^2 \\
&\quad - \lambda(1-\lambda)d\big(G(G_\lambda^n u), G_\lambda^n u\big)^2 \\
&= (1-\lambda)d\big(G_\lambda^n u, p\big)^2 + \lambda d\big(G(G_\lambda^n u), G(p)\big)^2 \\
&\quad - \lambda(1-\lambda)d\big(G(G_\lambda^n u), G_\lambda^n u\big)^2 \\
&\leq d\big(G_\lambda^n u, p\big)^2 - \lambda(1-\lambda)d\big(G(G_\lambda^n u), G_\lambda^n u\big)^2 &(4.3)\\
&\leq d\big(G_\lambda^n u, p\big)^2. &(4.4)
\end{aligned}$$

From (4.4), we have that $d\big(G_\lambda^{n+1}u, p\big) \leq d\big(G_\lambda^n u, p\big)$. Consequently the sequence $\left\{d\big(G_\lambda^n u, p\big)\right\}$ converges. Moreover, we obtain from (4.3) that

$$d\big(G(G_\lambda^n u), G_\lambda^n u\big)^2 \leq \frac{1}{\lambda(1-\lambda)}\left[d\big(G_\lambda^n u, p\big)^2 - d\big(G_\lambda^{n+1}u, p\big)^2\right]. \tag{4.5}$$

Letting $n \to \infty$ in (4.5), we have

$$d\big(G(G_\lambda^n u), G_\lambda^n u\big) \to 0, \ as\ n \to \infty. \tag{4.6}$$

Moreover, $G_\lambda^{n+1}u = (1-\lambda)G_\lambda^n u \oplus \lambda G\big(G_\lambda^n u)$, which by (2.1) implies,

$$d\big(G_\lambda^{n+1}u, G_\lambda^n u\big) \leq d\big(G(G_\lambda^n u), G_\lambda^n u\big) \to 0, \ as\ n \to \infty. \tag{4.7}$$

We have the lemma. □

Kirk and Panyanak [25] established that nonexpansive mapping has demiclosedness-type property as stated in Lemma 2.1. The next Lemma extends their result to the case of enriched nonexpansive mappings.

Lemma 4.7. *Let $T : \mathcal{H} \to \mathcal{H}$ be an enriched nonexpansive mapping. Then T has the demiclosedness-type property. That is, for a sequnce $\{x_n\}$ in $\mathcal{H}$ such that $x_n \xrightarrow{\Delta} x^*$ and $d(x_n, Tx_n) \to 0$, then $Tx^* = x^*$.*

Proof. Let $\sigma \in (0, 1]$. We obtain from Lemma 3.5 that

(i) $F(T) = F(T_\sigma)$;
(ii) $d\big(T_\sigma u, T_\sigma w\big)^2 \leq (1-\sigma)^2 d\big(u, w\big)^2 + \sigma^2 d\big(Tu, Tw\big)^2 + 2\sigma(1-\sigma)\langle\overrightarrow{uw}, \overrightarrow{TuTw}\rangle$.

From (ii), we have

$$\frac{1}{\sigma^2}d\big(T_\sigma u, T_\sigma w\big)^2 \le \left(\frac{1}{\sigma}-1\right)^2 d\big(u,w\big)^2 + d\big(Tu,Tw\big)^2 + 2\left(\frac{1}{\sigma}-1\right)\langle\overrightarrow{uw},\overrightarrow{TuTw}\rangle. \tag{4.8}$$

Since T is α-enriched nonexpansive mapping, then for $\sigma = \dfrac{1}{\alpha+1} \in (0,1]$, we obtain from (4.8) that

$$\begin{aligned}(\alpha+1)^2 d\big(T_\sigma u, T_\sigma w\big)^2 &\le \alpha^2 d\big(u,w\big)^2 + d\big(Tu,Tw\big)^2 + 2\alpha\langle\overrightarrow{uw},\overrightarrow{TuTw}\rangle\\ &\le (\alpha+1)^2 d(u,w)^2. \end{aligned}\tag{4.9}$$

This implies that

$$d\big(T_\sigma u, T_\sigma w\big) \le d(u,w). \tag{4.10}$$

Consequently, we have

$$\begin{aligned} d\big(x_n, T_\sigma x^*\big) &\le d\big(x_n, T_\sigma x_n\big) + d\big(T_\sigma x_n, T_\sigma x^*\big)\\ &\le d\big(x_n, T_\sigma x_n\big) + d\big(x_n, x^*\big)\\ &\le d\big(x_n, Tx_n\big) + d\big(x_n, x^*\big). \end{aligned}\tag{4.11}$$

Thus, by the hypothesis, we have

$$\limsup_{n\to\infty} d\big(x_n, T_\sigma x^*\big) \le \limsup_{n\to\infty} d\big(x_n, x^*\big). \tag{4.12}$$

This and the fact that $A(\{x_n\})$ has exactly one element imply $x^* = T_\sigma x^*$. Hence, $x^* \in F\big(T_\sigma\big) = F\big(T\big)$, by (i). Therefore $x^* = Tx^*$. □

We need the following definition for our main results of this section.

Definition 4.5. Let $(\mathcal{H}, d)$ be a Hadamard space and E be a nonempty subset of $\mathcal{H}$. A mapping $T : E \to \mathcal{H}$ is called demicompact if it has the property that whenever $\{w_n\}$ is bounded sequence and $d\big(Tw_n, w_n\big)$ converges strongly, then there exists a subsequence $\big\{w_{n_k}\big\}$ of $\{w_n\}$ that converges strongly.

Now we state and prove the main result of this section.

Theorem 4.8. *Let E be a nonempty bounded closed convex subset of $\mathcal{H}$ and $T : E \to E$ be an α-enriched nonexpansive. Then*

(i) the set $F\big(T\big)$ of fixed points of T is a nonempty closed and convex set;

(ii) there exists $\sigma_o \in (0,1]$ such that the sequence $\{u_n\}$ defined iteratively by

$$\begin{cases} u_1 \in E, \\ u_{n+1} = (1-\sigma_o)u_n \oplus \sigma_o Tu_n, \quad n \ge 1, \end{cases}$$

Δ-converges to a fixed point of T;

(iii) if in addition T is demicompact, then the convergence in (ii) *is strong.*

Proof. As in similar arguments to (4.10), we have that for $\sigma = \dfrac{1}{\alpha+1}$, T_σ is nonexpansive mapping. Thus, by Theorem 1.1, we have that $F\big(T_\sigma\big)$ is nonempty, closed and convex. Therefore, by Lemma 3.5 (i), $F(T)$ is nonempty, closed and convex. So, we have (i).

In order to prove part (ii) of the theorem, consider the sequence $\{u_n\}$ given by

$$u_{n+1} = (1-\mu)u_n \oplus \mu T_\sigma u_n, \; u_1 \in E, \; n \ge 1. \tag{4.13}$$

It is obvious that $\{u_n\}$ lies in E and hence it is bounded.

For $\sigma = \dfrac{1}{\alpha+1}$, T_σ is nonexpansive mapping by (4.10). By Lemma 4.6, the mapping G_μ defined by $G_\mu w = (1-\mu)w \oplus \mu T_\sigma w$ is asymptotically regular, that is,

$$d\big(u_n, G_\mu u_n\big) \to 0,\ as\ n \to \infty. \tag{4.14}$$

This is the same as

$$d\big(u_{n+1}, u_n\big) \to 0,\ as\ n \to \infty. \tag{4.15}$$

Thus, by (2.1), we get

$$d\big(u_n, T_\sigma u_n\big) \le \frac{1}{\mu} d\big(u_{n+1}, u_n\big) \to 0,\ as\ n \to \infty. \tag{4.16}$$

Let $p = T_\sigma p$. Then by (4.10), we have

$$\begin{aligned} d\big(u_{n+1}, p\big) &= d\big((1-\sigma)u_n \oplus \sigma T u_n, p\big) \\ &= d\big(T_\sigma u_n, T_\sigma p\big) \\ &\le d(u_n, p). \end{aligned} \tag{4.17}$$

This implies that $\big\{d(u_n,p)\big\}$ converges in $\mathbb{R}$.

Now, let $w \in \bigcup A(\{x_n\})$, where the union is taken over subsequences $\{x_n\}$ of $\{u_n\}$. Then there exists a subsequence $\{w_n\}$ of $\{u_n\}$ such that $A(\{w_n\}) = \{w\}$. By Lemma 2.3 there exists a subsequence $\{v_n\}$ of $\{w_n\}$ such that $v_n \xrightarrow{\Delta} v$ and by Lemma 2.2 we have that $v \in E$. Since T_σ is nonexpansive, using (4.16) and Lemma 4.7, we have $v \in F\big(T_\sigma\big)$ and hence, by (4.17), $\{d(u_n, v)\}$ converges. Moreover, Lemma 2.4 implies that $w = v \in F\big(T_\sigma\big)$. Thus $\bigcup A(\{x_n\}) \subseteq F\big(T_\sigma\big)$.

We now claim that the set $\bigcup A(\{x_n\})$ is a singleton set. Indeed, let $A(\{u_n\}) = \{u\}$ and let $\{x_n\}$ be arbitrary subsequence of $\{u_n\}$ with $A(\{x_n\}) = \{x\}$. We have $u \in F\big(T_\sigma\big)$ and by (4.17), $\{d(u_n, x)\}$ converges. Lemma 2.4 implies that $x = u$. Therefore, since $\bigcup A(\{x_n\})$ is a singleton set that is contained in $F\big(T_\sigma\big)$, we conclude that $\{u_n\}$ Δ-converges to an element of $F\big(T_\sigma\big) = F\big(T\big)$.

On the other hands, let $n \ge 1$. Then from (2.1), we have that $T_\sigma u_n \in \big[u_n, Tu_n\big]$ and $u_{n+1} \in \big[u_n, T_\sigma u_n\big]$. Hence, $u_{n+1} \in \big[u_n, Tu_n\big]$. Moreover,

$$d\big(u_{n+1}, u_n\big) = \mu d\big(u_n, T_\sigma u_n\big) = \mu\sigma d\big(u_n, Tu_n\big). \tag{4.18}$$

Thus, by (2.1), we have

$$d\big(u_{n+1}, Tu_n\big) = d\big(u_n, Tu_n\big) - d\big(u_{n+1}, u_n\big) = (1-\mu\sigma) d\big(u_n, Tu_n\big). \tag{4.19}$$

Therefore,

$$u_{n+1} = (1-\mu\sigma)u_n \oplus \mu\sigma T u_n,\ u_1 \in E,\ n \ge 1.$$

Hence, the proof of (ii) is complete.

We now prove the last part of the theorem. Since T is demicompact, then it is immediate from (2.1) that T_σ is demicompact. Hence, there exists some subsequence $\big\{u_{n_k}\big\}$ of $\{u_n\}$ which converges strongly in E. Denote

$$\lim_{k\to\infty} u_{n_k} = q. \tag{4.20}$$

Thus, $u_{n_k} \xrightarrow{\Delta} q$ and by the proof of part (ii), we have that $\bigcup A(\{x_n\}) \subseteq F\big(T_\sigma\big) = F\big(T\big)$. Therefore, $q \in F\big(T\big)$ and by the argument in (4.17), the proof is completed. □

This theorem immediately implies the following corollaries.

Corollary 4.3. *Let E be a nonempty bounded closed convex subset of $\mathcal{H}$ and $T : E \to E$ be a nonexpansive mapping. Then*

(i) the set $F(T)$ of fixed points of T is a nonempty closed and convex set;
(ii) there exists $\sigma_o \in (0,1]$ such that the sequence $\{u_n\}$ defined iteratively by

$$\begin{cases} u_1 \in E, \\ u_{n+1} = (1-\sigma_o)u_n \oplus \sigma_o T u_n, \quad n \geq 1, \end{cases}$$

Δ-converges to a fixed point of T;
(iii) if in addition T is demicompact, then the convergence in (ii) *is strong.*

In case of $\alpha = 0$, we have:

Corollary 4.4. *Let E be a nonempty bounded closed convex subset of a Hilbert space $\mathcal{H}$ and $T : E \to E$ be an α-enriched nonexpansive,that is, there exists $\alpha \in [0, +\infty)$ such that*

$$\|\alpha(u-w) + Tu - Tw\| \leq (\alpha+1)\|u-w\|, \ \forall u, w \in \mathcal{H}. \tag{4.21}$$

Then

(i) the set $F(T)$ of fixed points of T is a nonempty closed and convex set;
(ii) there exists $\sigma_o \in (0,1]$ such that the sequence $\{u_n\}$ defined iteratively by

$$\begin{cases} u_1 \in E, \\ u_{n+1} = (1-\sigma_o)u_n + \sigma_o T u_n, \quad n \geq 1, \end{cases}$$

converges weakly to a fixed point of T;
(iii) if in addition T is demicompact, then the convergence in (ii) *is strong.*

Proof. The proof follows trivially from Theorem 4.8 using the fact that Hilbert spaces are contained in Hadamard spaces and Δ-convergence coincides with weak convergence in that setting. □

Corollary 4.5. *Let E be a nonempty bounded closed convex subset of a Hilbert space $\mathcal{H}$ and $T : E \to E$ be a nonexpansive mapping. Then*

(i) the set $F(T)$ of fixed points of T is a nonempty closed and convex set;
(ii) there exists $\sigma_o \in (0,1]$ such that the sequence $\{u_n\}$ defined iteratively by

$$\begin{cases} u_1 \in E, \\ u_{n+1} = (1-\sigma_o)u_n + \sigma_o T u_n, \quad n \geq 1, \end{cases}$$

converges weakly to a fixed point of T;
(iii) if in addition T is demicompact, then the convergence in (ii) *is strong.*

Example 4.1. *Let $\mathcal{H} = \mathbb{R}^2$ be endowed with the metric $d_{\mathcal{H}} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by*

$$d_{\mathcal{H}}\big((w_1, w_2), (z_1, z_2)\big) = \sqrt{(w_1 - z_1)^2 + (w_1^2 - w_2 - z_1^2 + z_2)^2}.$$

Then $(\mathcal{H}, d_{\mathcal{H}})$ is a nonlinear Hadamard space (see, e.g., [19, Example 5.2]*). Moreover the operation, $(1-t)(w_1, w_2) \oplus t(v_1, v_2)$, is given by*

$$\left((1-t)w_1 + tv_1, \big((1-t)w_1 + tv_1\big)^2 - \big((1-t)(w_1^2 - w_2) + t(v_1^2 - v_2)\big)\right).$$

Let η_1, η_2, η_3 be fixed real numbers such that $0 < \eta_1 \leq \dfrac{1}{\eta_2}, \eta_2 > 1$, and consider the bounded closed convex set $E = \{\eta_3\} \times [\eta_1, \eta_2]$. Let $T : E \to E$ such that $(x_1, x_2) \mapsto \left(\eta_3, \frac{1}{x_2}\right)$. Then

(i) T is enriched nonexpansive.

(ii) T *is not nonexpansive.*
(iii) $F(T)=\{(\eta_3,1)\}$.

Proof. The proof of (iii) is trivial. We now prove only (i) and (ii).
(i) Let $u=(u_1,u_2)\,,w=(w_1,w_2)\in E$. Then

$$\begin{aligned}
d_{\mathcal{H}}(Tu,Tw)^2+\alpha^2 d_{\mathcal{H}}(u,w)^2+2\alpha\langle\overrightarrow{uw},\overrightarrow{TuTw}\rangle &= \left(\frac{1}{u_2}-\frac{1}{w_2}\right)^2+\alpha^2\left(u_2-w_2\right)^2 \\
&\quad+\alpha\left[\left(u_2-\frac{1}{w_2}\right)^2+\left(w_2-\frac{1}{u_2}\right)^2-\left(u_2-\frac{1}{u_2}\right)^2-\left(w_2-\frac{1}{w_2}\right)^2\right] \\
&=\left(\frac{1}{(u_2w_2)^2}+\alpha^2\right)\left(u_2-w_2\right)^2-2\alpha\left[\frac{u_2}{w_2}+\frac{w_2}{u_2}-2\right] \\
&=\left(\frac{1}{u_2w_2}-\alpha\right)^2\left(u_2-w_2\right)^2 \\
&=\left(\frac{1}{u_2w_2}-\alpha\right)^2 d_{\mathcal{H}}\left(u,w\right)^2.
\end{aligned}$$

Thus, for $\alpha=\frac{\eta_2^2-1}{2}$, we have

$$d(Tu,Tw)^2+\alpha^2 d(u,w)^2+2\alpha\langle\overrightarrow{uw},\overrightarrow{TuTw}\rangle\leq(\alpha+1)^2 d(u,w)^2.$$

(ii) For $u=(\eta_3,1)$ and $w=(\eta_3,1/\eta_2)$, we have

$$d_{\mathcal{H}}(Tu,Tw)=\left|\frac{1}{u_2}-\frac{1}{w_2}\right|=|1-\eta_2|>\frac{|1-\eta_2|}{n_2}=\left|1-\frac{1}{\eta_2}\right|=d_{\mathcal{H}}(u,w).$$

□

Clearly, in Example 4.1, $F(T)$ is nonempty closed and convex set. Moreover, the sequence $\{u_n\}$ defined iteratively by

$$\text{(4.22)}\qquad \begin{cases} u_1\in E, \\ u_{n+1}=\frac{\eta_2^2-1}{\eta_2^2+1}u_n\oplus\frac{2}{\eta_2^2+1}Tu_n,\quad n\geq 1, \end{cases}$$

yields the results in TABLE 1.

TABLE 1. Few values of the sequence $\{u_n\}$ generated by (4.22)

n	$\eta_1 = 1/100,\ \eta_2 = 50$					
	$\eta_3 = 3$			$\eta_3 = -5$		
1	(3, 42)	(3, 1/80)	(3, 17)	(-5, 21)	(-5, 1/50)	(-5, 3)
2	(3, 38.7711)	(3, 6.1654)	(3, 15.6968)	(-5, 19.3883)	(-5, 3.8646)	(-5, 2.7949)
3	(3, 35.7907)	(3, 5.7036)	(3, 14.4943)	(-5, 17.9008)	(-5, 3.5872)	(-5, 2.6074)
4	(3, 33.0397)	(3, 5.2783)	(3, 13.3846)	(-5, 16.5281)	(-5, 3.3327)	(-5, 2.4363)
5	(3, 30.5005)	(3, 4.8869)	(3, 12.3608)	(-5, 15.2614)	(-5, 3.0995)	(-5, 2.2805)
⋮	⋮	⋮	⋮	⋮	⋮	⋮
45	(3, 1.5835)	(3, 1.0163)	(3, 1.1113)	(-5, 1.1679)	(-5, 1.0059)	(-5, 1.0028)
46	(3, 1.5103)	(3, 1.0138)	(3, 1.095)	(-5, 1.1439)	(-5, 1.005)	(-5, 1.0024)
47	(3, 1.4451)	(3, 1.0117)	(3, 1.081)	(-5, 1.1232)	(-5, 1.0043)	(-5, 1.002)
48	(3, 1.3871)	(3, 1.0099)	(3, 1.069)	(-5, 1.1053)	(-5, 1.0036)	(-5, 1.0017)
49	(3, 1.3359)	(3, 1.0084)	(3, 1.0588)	(-5, 1.0898)	(-5, 1.003)	(-5, 1.0015)
50	(3, 1.2907)	(3, 1.0071)	(3, 1.05)	(-5, 1.0766)	(-5, 1.0026)	(-5, 1.0012)

5. Conclusions

(1) In this work we studied and analysed, in the setting of Hadamard spaces, enriched classes of contractions (enriched contractions, Maia-type enriched contraction, local enriched contractions) and the class of enriched nonexpansive mappings. The studied classes are known to include the classical Banach contractions, some nonexpansive mappings and Lipchitzian mappings, among others.

(2) We established that any enriched contraction in a Hadamard space has a unique fixed point, and the fixed point can be approximated by a suitable Kransnoselskijj-type scheme (Theorem 3.3). In particular, from the established result herein, we obtained the classical Banach contraction mapping theorem in the setting of Hadamard spaces. These result extend the main result of Berinde and Păcurar [8] to nonlinear setting.

(3) We proved that an enriched nonexpansive mapping T defined on a bounded convex closed subset of Hadamard space has demiclosedness-type property (Lemma 4.7). We have also shown that the fixed point set $F(T)$ is nonempty closed and convex, and by an appropriate Krasnoselskij we established Δ and strong convergence theorems for approximating a member of the set $F(T)$ (Theorem 4.8).

(4) We analysed a local fixed point result (Theorem 3.4) and asymptotic fixed point result (Theorem 3.5) in the setting of Hadamard spaces. The established results complement the corresponding ones in [8] from linear space to nonlinear setting.

(5) We obtained Maia-type fixed point theorem for enriched contractions in the setting of Hadamard spaces (Theorem 3.7).

(6) Our work extends and complements the work of Berinde in [8, 6] and many other results to the setting of Hadamard space.

(7) Recall that the concept of optimization has better representation in Hadamard spaces. However, we could not find example of the studied mappings from the optimization perspective. It will be interesting future works if we can find examples of such mappings.

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