CARPATHIAN J. MATH.
Volume **39** (2023), No. 1,
Pages 255 - 264

Online version at https://semnul.com/carpathian/
Print Edition: ISSN 1584 - 2851; Online Edition: ISSN 1843 - 4401
DOI: https://doi.org/10.37193/CJM.2023.01.17

In memoriam Professor Charles E. Chidume (1947- 2021)

Diophantine triples with distinct binary recurrences

FLORIAN LUCA and LÁSZLÓ SZALAY

ABSTRACT. In this paper, we look at Diophantine triples with values in three different binary recurrence sequences. These are the Fibonacci and Pell sequences and the sequence of one more of powers of a given prime p. The novelty of the article is the appearance of three different sequences, as up to now the analogous problem had been investigated only for one sequence.

1. INTRODUCTION

A Diophantine m-tuple is a set of $\{a_1, \ldots, a_m\}$ of positive rational numbers, or integers, such that $a_ia_j + 1$ is a square for all $1 \le i < j \le m$. Several variations of this problem have been studied. The most common variation is to take $m = 3$ and replace the squares by some other set of numbers with interesting arithmetic properties. For example, replacing the squares with the $\mathcal{S}$-units, which are integers whose prime factors belong to a fixed finite set $\mathcal{S}$ of primes, one gets the problem of studying triples of positive integers $\{a_1, a_2, a_3\}$ such that all prime factors of $(a_1a_2 + 1)(a_2a_3 + 1)(a_3a_1 + 1)$ are in $\mathcal{S}$. In [8] it was conjectured that given $\mathcal{S}$ there are only finitely many such triples (a_1, a_2, a_3). This was confirmed to be so in [7] in a stronger form and in [3] in a quantitative form. See [9], [17], [21], [22], [23] for more results in this direction. A different popular variation is when the squares are replaced by terms of a given binary recurrence. In the paper [13], the authors characterized the non-degenerate binary recurrence sequences $(u_n)_{n\ge0}$ with positive discriminant for which there exist infinitely many 6-tuples of non-negative integers $(a, b, c; x, y, z)$ with $1 \le a < b < c$ such that

$$ab + 1 = u_x, \qquad ac + 1 = u_y \qquad \text{and} \qquad bc + 1 = u_z. \tag{1.1}$$

There are some papers which compute all the solutions corresponding to equation (1.1) when the recurrence is given. This was done for the Fibonacci sequence in [15], for the sequence of Lucas numbers in [16], and for the sequence of balancing numbers in [1]. The papers [10], [11], [12] investigate the same problem with members of higher order recurrence sequences.

In this paper, we look at the situation when the right-hand side of (1.1) consists of terms from three distinct binary recurrences. Up to our knowledge, this is the first attempt to handle such a composite problem. The recurrent sequences we consider are the sequence of Fibonacci numbers, the sequence of Pell numbers, and the sequence $(p^n + 1)_{n\ge0}$ where p is a given prime of general term denoted $\Phi_n = p^n + 1$. The method which worked in [15, 16, 1] cannot be applied here. On the other hand, in the present problem we can exploit that if $bc + 1 = p^n + 1$, then b and c are powers of p.

Received: 06.09.2020. In revised form: 31.03.2021. Accepted: 20.12.2021
2010 *Mathematics Subject Classification.* 11B39, 11D61.
Key words and phrases. *Diophantine triple, linear recurrence.*
Corresponding author: László Szalay; szalay.laszlo@uni-sopron.hu

Note that (1.2) has two obvious solutions, namely $(a,b,c,x,y,z) = (1,1,1,3,2,0)$, and $(a,b,c,x,y,z) = (4,1,1,5,3,0)$, independently of the prime p. These are called trivial.

Theorem 1.1. *Assume that $p < 20$ is a prime. Then the non-trivial solutions to the system*

$$ab + 1 = F_x, \qquad ac + 1 = P_y, \qquad bc + 1 = \Phi_z \tag{1.2}$$

in $x, y \in \mathbb{N}^+$ and $z \in \mathbb{N}$ are the following quadruples (p, x, y, z):

$$(2,5,3,2),(2,5,3,4),(2,3,3,2),(2,4,3,1),(2,4,3,3),(2,6,5,2),(2,4,2,1),(2,5,2,2),$$

$$(2,11,4,3),(3,7,3,1),(3,9,4,1),(5,8,3,1),(7,5,5,1),(7,8,3,1),(11,3,4,1),(13,12,4,1).$$

2. Preliminaries

We let $(F_n)_{n\ge 0}$ and $(P_n)_{n\ge 0}$ be the sequences of Fibonacci and Pell numbers, respectively, given by $F_0 = P_0 = 0,\ F_1 = P_1 = 1$, and by the recurrence relations

$$F_{n+2} = F_{n+1} + F_n \qquad \text{and} \qquad P_{n+2} = 2P_{n+1} + P_n \qquad \text{for all } n \ge 0,$$

respectively. Putting $\alpha := (1+\sqrt{5})/2$, $\beta := -\alpha^{-1}$, $\gamma := 1+\sqrt{2}$, and $\delta := -\gamma^{-1}$ the formulae of the general terms of these particular sequences are

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \qquad \text{and} \qquad P_n = \frac{\gamma^n - \delta^n}{\gamma - \delta} \tag{2.3}$$

for all $n \ge 0$, respectively. The terms of their associate sequences are denoted by L_n and Q_n. They can be expressed by

$$L_n = \alpha^n + \beta^n \qquad \text{and} \qquad Q_n = \gamma^n + \delta^n \tag{2.4}$$

for all $n \ge 0$. There are many identities and inequalities involving Fibonacci or Pell numbers and sometimes their associates. Some of them are well known, for instance

$$\alpha^{n-2} \le F_n \le \alpha^{n-1} \qquad \text{and} \qquad \gamma^{n-2} \le P_n \le \gamma^{n-1} \qquad \text{for all } n \ge 0, \tag{2.5}$$

so we refer to these only in the text of the paper. We remark that the same bounds can be applied for $F_n - 1$ and $P_n - 1$ whenever $n \ge 6$ and $n \ge 3$, respectively. A few other results are emphasized below. The following can be deduced from Theorem VII in [5].

Lemma 2.1. *The following divisibility relation holds:*

$$\gcd(F_u, L_v) = \begin{cases} L_{\gcd(u,v)}, & \text{if } \frac{u}{\gcd(u,v)} \not\equiv \frac{v}{\gcd(u,v)} \equiv 1 \pmod 2; \\ 1 \text{ or } 2, & \text{otherwise.} \end{cases}$$

Lemma 2.1 implies the following result.

Corollary 2.1. *Assume that $\varepsilon_1 \in \{\pm 1, \pm 2\}$, and $\varepsilon_2 \in \{\pm 1\}$. Then we have*

$$\gcd\left(F_{\frac{n-\varepsilon_1}{2}}, L_{\frac{n+\varepsilon_1}{2}}\right) \le 3, \qquad \gcd\left(P_{\frac{n-\varepsilon_2}{2}}, Q_{\frac{n+\varepsilon_2}{2}}\right) \le 2.$$

Lemma 2.2. *The following formulae hold, where in the second case we assume that n is odd:*

$$\begin{aligned} F_n - 1 &= F_{\frac{n-\varepsilon_1}{2}} L_{\frac{n+\varepsilon_1}{2}}, \text{ with } \varepsilon_1 \in \{\pm 1, \pm 2\}, \\ P_n - 1 &= P_{\frac{n-\varepsilon_2}{2}} Q_{\frac{n+\varepsilon_2}{2}}, \text{ with } \varepsilon_2 \in \{\pm 1\}. \end{aligned}$$

Let $\nu_p(n)$ denote the largest exponent ν such that $p^\nu \mid n$. The result below is already implicit in Lucas' seminal paper [18] (see the Theorem on page 210 in [18]). It also appears in Lengyel [14] for Fibonacci numbers, and Lucas numbers and Sanna [20] in general.

Lemma 2.3. *Let* $p \in \{2, 3, 5, 7, 11, 13, 17, 19\}$. *Then*

$$\nu_p(u_n) \leq \nu_p(n) + 2 \tag{2.6}$$

for $u_n = F_n, L_n, P_n,$ *and* Q_n.

In the proof of Theorem 1.1, the case when y is even causes the main difficulty because of the lack of an algebraic factorization for $P_y - 1$. We analyze this case separately.

Theorem 2.2. *If* $y \geq 2$ *is even, then*

(i) $\nu_p(P_y - 1) = 0$ *if* $p = 2, 7, 17, 19$;
(ii) $\nu_p(P_y - 1) \leq 1$ *if* $p = 3, 5, 13$.
(iii) *For* $p = 11$ *we have* $\nu_p(P_y - 1) \leq 614(\log(1.05(y+1)))^2$.

Proof. We deal with them one prime at a time.

Case $p = 2$. P_y is even so $P_y - 1$ is odd. Thus, $\nu_2(P_y - 1) = 0$ for y even.

Case $p = 3$. We check that $\{P_{2m}\}_{m \geq 0}$ is periodic modulo 9 with period 12. Listing $P_{2m} - 1$ modulo 9 for $m = 0, \ldots, 11$, we do not get a 0. Hence, $\nu_3(P_y - 1) \leq 1$.

Case $p = 5,\ 7,\ 13,\ 17,\ 19$. The treatment is similar to the previous two cases. Thus, we have dealt with (i) and (ii), now we turn our attention to (iii).

Case $p = 11$. Here, we get some non-trivial divisibilities. For example, $11^5 \mid P_{200194} - 1$. We illustrate a general procedure which works in order to bound $\nu_p(P_y - 1)$ for every even input y. Write

$$P_y - 1 = \frac{\gamma^y - \gamma^{-y}}{2\sqrt{2}} - 1 = \frac{\gamma^{-y}}{2\sqrt{2}}(\gamma^{2y} - 2\sqrt{2}\gamma^y - 1) = \frac{\gamma^{-y}}{2\sqrt{2}}(\gamma^y - \zeta_1)(\gamma^y - \zeta_2). \tag{2.7}$$

Here, $\zeta_{1,2} = \sqrt{2} \pm \sqrt{3}$ are the roots of $\zeta^2 - 2\sqrt{2}\zeta - 1 = 0$. Let $\mathbb{K} := \mathbb{Q}(\zeta_1) = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. Its discriminant is divisible only by 2 and 3. All other primes p have the property that $p = \prod_{i=1}^{k} \pi_i(p)$, where $\pi_i(p)$ are coprime distinct ideals and $k \in \{2, 4\}$. It is easy to decide the value of k. If both $2, 3$ are quadratic residues modulo p, then $k = 4$. Otherwise, $k = 2$.

Let π be any prime ideal dividing $p \geq 5$. Let ν_π be the normalised valuation of π. That is, if $p^f = |\mathcal{O}_{\mathbb{K}}/\pi|$, and η is any non-zero element of $\mathbb{K}$, then $\nu_\pi(\eta) = (f/4)\mathrm{ord}_\pi(\eta)$, where $\mathrm{ord}_\pi(\eta)$ is the exponent of the ideal π in the factorisation of the principal ideal $\eta\mathcal{O}_{\mathbb{K}}$. In particular,

$$\mathrm{ord}_p(P_y - 1) = \mathrm{ord}_\pi(P_y - 1) = (4/f)\nu_\pi(P_y - 1).$$

Thus, using (2.7), we get

$$\begin{aligned} \mathrm{ord}_p(P_y - 1) &= (4e/f)\left(\nu_\pi(\delta^y - \zeta_1) + \nu_\pi(\delta^y - \zeta_2) - \nu_\pi(2\sqrt{2})\right) \\ &= (4/f)\max\{\nu_\pi(\delta^y - \zeta_i) : i = 1, 2\} \\ &+ (4/f)\min\{\nu_\pi(\delta^y - \zeta_i) : i = 1, 2\} - (4/f)\nu_\pi(2\sqrt{2}) \\ &\leq (4/f)\max\{\nu_\pi(\delta^y - \zeta_i : i = 1, 2\}. \end{aligned} \tag{2.8}$$

The only step that needs justification is the last one. Well, if we put

$$t := \min\{\nu_\pi(\alpha^y - \zeta_i), i = 1, 2\},$$

it follows that $\pi^{4t/f}$ divides $\delta^y - \zeta_1$ and $\delta^y - \zeta_2$, so their difference which is $\zeta_1 - \zeta_2 = 2\sqrt{2}$. Thus, $\min\{\nu_\pi(\delta^y - \zeta_i : i = 1, 2\} \leq \nu_\pi(2\sqrt{2})$, which implies the last inequality (2.8).

For $\nu_\pi(\delta^y - \zeta_i)$, we use Théorème 3 of [2]. We take in the notation there

$$\alpha_1 := \gamma, \quad \alpha_2 := \zeta_i, \quad b_1 := y, \quad b_2 := 1.$$

We put $D := 4/f$, and g for a common order of α_1 and α_2 modulo π. That is, the smallest positive integer such that $\delta^g \equiv 1 \pmod{\pi}$ and $\zeta_i^g \equiv 1 \pmod{\pi}$. We put

$$\log A_i \geq \max\left\{h(\alpha_i), \frac{\log p}{D}\right\},$$

where $h(\eta)$ is the Weil height of the algebraic number η. This is given by

$$h(\eta) := \frac{1}{u}\left(\log a_0 + \sum_{j=1}^{u} \max\{0, \log|\eta^{(j)}|\}\right),$$

where the minimal polynomial of η is $f(X) = a_0(X - \eta^{(1)}) \cdots (X - \eta^{(u)}) \in \mathbb{Z}[X]$ with positive a_0. The properties

(i) $h(\mu + \nu) \leq h(\mu) + h(\nu) + \log 2$,
(ii) $h(\mu\nu^{\pm 1}) \leq h(\mu) + h(\nu)$,
(iii) $h(\mu^\ell) \leq |\ell| h(\mu)$

are valid for all algebraic numbers μ, ν, and integers ℓ.

For us, $h(\alpha_1) = (1/2)\log(1 + \sqrt{2}) = 0.4406\ldots$ and $h(\zeta_i) = 0.573108\ldots$. It follows since $p \geq 5$, that $f \leq 2$, so $\log A_i = \log p/D$. Further, $f = 1, 2$ according to whether both 2, 3 are quadratic residues modulo p or not. Then we put

$$b' := \frac{y}{D \log A_2} + \frac{1}{D \log A_1}.$$

Now Théorème 3 in [2] gives

$$\nu_\pi(\delta^y - \zeta_i) \leq \frac{24pg}{(p-1)\log p)^4} D^4 \max\left\{\log b' + \log\log p + 0.4, \frac{10 \log p}{D}, 10\right\}^2.$$

We need to check that α_1 and α_2 are multiplicatively independent. Well, if not there is some relation $\alpha_1^u \alpha_2^v = 1$, where u, v are integers not both 0. We may assume they are both even. Then $\alpha_1^u \in \mathbb{Q}(\sqrt{2})$ and

$$\alpha_2^{-v} = (\alpha_2^2)^{-v/2} = (5 \pm 2\sqrt{6})^{v/2} \in \mathbb{Q}(\sqrt{6}).$$

Since $\alpha_1^u = \alpha_2^{-v}$ and $\mathbb{Q}(\sqrt{2}) \cap \mathbb{Q}(\sqrt{6}) = \mathbb{Q}$, we get that $\alpha_1^u = \alpha_2^{-v} \in \mathbb{Q}$. Since they are also units (so algebraic integers whose reciprocal is an algebraic integer) it follows that $\alpha_1^u = \alpha_2^v = 1$, which leads to $u = v = 0$, a contradiction.

This is in general. Let us apply the above scheme for $p = 11$. Since 2 is not a quadratic residue modulo 11, it follows that $f = 2$. Let π be some prime ideal in $\mathcal{O}_{\mathbb{K}}$ dividing 11. Since $f = 2$, we have $D = 2$. Further, we can take $g = 24$. Indeed

$$\alpha_1^{11} = (1 + \sqrt{2})^{11} \equiv 1 + 2^{(11-1)/2}\sqrt{2} \equiv 1 - \sqrt{2} \pmod{\pi}.$$

Thus, $\alpha_1^{12} \equiv (1 + \sqrt{2})(1 - \sqrt{2}) \equiv -1 \pmod{\pi}$, and so $\alpha_1^{24} \equiv 1 \pmod{\pi}$. Further,

$$\zeta_1^{11} \equiv (\sqrt{2} + \sqrt{3})^{11} \equiv 2^{(11-1)/2}\sqrt{2} + 3^{(11-1)/2}\sqrt{3} \equiv -\sqrt{2} + \sqrt{3} \pmod{\pi},$$

so $\zeta_1^{12} \equiv (\sqrt{2} + \sqrt{3})(-\sqrt{2} + \sqrt{3}) \equiv 1 \pmod{\pi}$. The same argument works for ζ_2. Next,

$$b' = \frac{y}{\log 11} + \frac{1}{\log 11} < \frac{y+1}{\log 11}.$$

We thus get

$$\begin{aligned}\nu_\pi(\delta^y-\zeta_i) &\leq \frac{24\times 11\times 24}{(11-1)(\log 11)^4}\times 2^4\\ &\times \max\left\{\log\left(\frac{y+1}{\log 11}\right)+\log\log 11+0.04, \frac{10\log 3}{2}, 10\right\}^2.\end{aligned}$$

So, either the maximum is at 10, in which case $y+1<e^{10-0.04}<22,000$, or

$$\nu_\pi(\delta^y-\zeta_i)<307(\log(1.05(y+1)))^2. \tag{2.9}$$

Together with (2.8), we get

$$\mathrm{ord}_{11}(P_y-1)\leq 614(\log(1.05(y+1)))^2 \tag{2.10}$$

for even $y>22,000$. We checked that $\nu_{11}(P_y-1)\leq 4$ for all even $y\leq 22,000$. In particular, the inequality (2.10) holds for all $y\geq 2$ even. This finishes the proof of (iii). □

The above proof used lower bound for p-adic linear forms in two logarithms. We also need some results from the theory of lower bounds in non-zero linear forms in logarithms of algebraic numbers. We start by recalling Theorem 9.4 of [4], which is a modified version of a result of Matveev [19]. Let $\mathbb{L}$ be an algebraic number field of degree $d_{\mathbb{L}}$. Let $\eta_1,\eta_2,\ldots,\eta_l\in\mathbb{L}$ not 0 or 1 and $d_1,\ldots,d_l$ be non-zero integers. We put

$$\Gamma=\prod_{i=1}^{l}\eta_i^{d_i}-1,\qquad \text{and}\qquad D=\max\{|d_1|,\ldots,|d_l|,3\}.$$

Let $A_1,\ldots,A_l$ be positive integers such that

$$A_j\geq h'(\eta_j):=\max\{d_{\mathbb{L}}h(\eta_j),|\log\eta_j|,0.16\},\quad \text{for}\quad j=1,\ldots l.$$

The following consequence of Matveev's theorem is Theorem 9.4 in [4].

Theorem 2.3. *If* $\Gamma\neq 0$ *and* $\mathbb{L}\subseteq\mathbb{R}$, *then*

$$\log|\Gamma|>-1.4\cdot 30^{l+3}l^{4.5}d_{\mathbb{L}}^2(1+\log d_{\mathbb{L}})(1+\log D)A_1A_2\cdots A_l.$$

3. The proof of Theorem 1.1

In order to keep uniformity, we always take the worst case depending on the primes p.

Assume that a,b,c,x,y,z is a solution to (1.2). Suppose that x,y,z are large enough, say that $x\geq 10^3$ and $y\geq 9\cdot 10^5$. Under these conditions, we present Theorem 3.4. We notice that the third equation $bc+1=p^n+1$ of (1.2) implies that $b=p^{b_1}$, $c=p^{c_1}$, where the exponents are non-negative integers. Let $a=Ap^{a_1}$ such that $p\nmid A$. Thus, according to Lemma 2.2,

$$Ap^{a_1+b_1}=F_x-1=F_{\frac{x-\varepsilon_1}{2}}L_{\frac{x+\varepsilon_1}{2}},$$

with $\varepsilon_1\in\{\pm1,\pm2\}$. Hence, Corollary 2.1, together with Lemma 2.3 provide

$$\begin{aligned}\nu_p(F_x-1) &= \nu_p\left(F_{\frac{x-\varepsilon_1}{2}}\right)+\nu_p\left(L_{\frac{x+\varepsilon_1}{2}}\right)\leq\nu_p\left(\frac{x-\varepsilon_1}{2}\right)+2+1\\ &\leq \log_p\left(\frac{x+2}{2}\right)+3\leq 1.5\log\left(\frac{x}{2}\right)+3.\end{aligned}$$

In the last inequality we used that $p\geq 2$ and $x\geq 100$. Thus,

$$b_1\leq 1.5\log\left(\frac{x}{2}\right)+3. \tag{3.11}$$

This machinery also works for the Pell sequence if y is odd. We similarly write

$$Ap^{a_1+b_c} = P_y - 1 = P_{\frac{y-\varepsilon_2}{2}}\, Q_{\frac{y+\varepsilon_2}{2}},$$

where $\varepsilon_2 \in \{\pm 1\}$, and if $y \geq 100$, then

$$\nu_p(P_y - 1) \quad \leq \quad 1.5 \log\left(\frac{y}{2}\right) + 3.$$

When y is even we need an other treatment. The situation here is described in Theorem 2.2. Combining this and the above observation, we obtain

$$c_1 \leq \begin{cases} 1.5\log\left(\frac{y}{2}\right) + 3, & \text{for } y \text{ odd}, \\ 614(\log(1.05(y+1)))^2, & \text{for } y \text{ even}. \end{cases} \tag{3.12}$$

Now turn our attention to the first two equations of (1.2). Obviously,

$$\frac{F_x - 1}{p^{b_1}} = \frac{P_y - 1}{p^{c_1}}. \tag{3.13}$$

We distinguish two cases. First assume that $b_1 \geq c_1$. Then the previous equation leads to

$$F_x - 1 = p^{b_1 - c_1}(P_y - 1). \tag{3.14}$$

Using the explicit formulae (2.3) we rewrite it as

$$\frac{\alpha^x - \beta^x}{\sqrt{5}} - 1 = p^{b_1-c_1}\left(\frac{\gamma^y - \delta^y}{\sqrt{8}} - 1\right).$$

Thus,

$$\left|\frac{\alpha^x}{\sqrt{5}} - p^{b_1-c_1}\frac{\gamma^y}{\sqrt{8}}\right| = \left|\frac{\beta^x}{\sqrt{5}} + 1 - p^{b_1-c_1}\left(\frac{\delta^y}{\sqrt{8}} + 1\right)\right|.$$

Dividing both sides of it by $p^{b_1-c_1}\gamma^y/\sqrt{8}$, we obtain the expression

$$\left|\frac{\alpha^x}{\gamma^y}\frac{\sqrt{8}}{\sqrt{5}}\frac{1}{p^{b_1-c_1}} - 1\right|, \tag{3.15}$$

in the left-hand side, while for the right hand-side we get a bound of

$$\begin{aligned} \frac{\sqrt{8}}{p^{b_1-c_1}\gamma^y}\left|\frac{\beta^x}{\sqrt{5}} + 1 - p^{b_1-c_1}\left(\frac{\delta^y}{\sqrt{8}} + 1\right)\right| \quad &\leq \quad \frac{\sqrt{8}}{\gamma^y p^{b_1-c_1}}\left(\frac{|\beta|^x}{\sqrt{5}} + 1\right) + \frac{\sqrt{8}}{\gamma^y}\left(\frac{|\delta|^y}{\sqrt{8}} + 1\right) \\ &< \quad \frac{8}{\gamma^y}. \end{aligned} \tag{3.16}$$

Indeed, the above inequalities follow since $\max\{|\beta|^x, |\delta|^y, 1/p^{b_1-c_1}\} \leq 1$. Put

$$\Gamma_1 := \frac{\alpha^x}{\gamma^y}\frac{\sqrt{8}}{\sqrt{5}}\frac{1}{p^{b_1-c_1}} - 1.$$

This is non-zero, because otherwise $\alpha^x/\sqrt{5} = p^{b_1-c_1}\gamma^y/\sqrt{8}$ would hold. The left-hand side is in $\mathbb{Q}(\sqrt{5})$, while the right-hand side is in $\mathbb{Q}(\sqrt{2})$. Consequently, both are in $\mathbb{Q}$, a contradiction. Now we apply Theorem 2.3 for (3.15) with the conditions

$$l := 4, \quad \mathbb{L} := \mathbb{Q}(\sqrt{2}, \sqrt{5}), \quad d_{\mathbb{L}} = 4,$$

and furthermore $\eta_1 := \alpha$, $\eta_2 := \gamma$, $\eta_3 := p$, $\eta_4 := \sqrt{8/5}$. Clearly, $d_1 = x$, $d_2 = y$, $d_3 = b_1 - c_1$, $d_4 = 1$. Since

$$b_1 - c_1 \leq b_1 \leq 1.5\log\left(\frac{y}{2}\right) + 3 \leq 614\log^2(1.05(y+1)) < y$$

if $y \geq 78713$, together with $F_x - 1 = p^{b_1-c_1}(P_y - 1)$, we see that the inequality $x \geq y$ holds provided that $y > 78713$. Thus, $D = x$. We also need

$$A_1 = 4h(\alpha) = 2\log(\alpha), \qquad A_2 = 4h(\gamma) = 2\log(\gamma),$$
$$A_3 = 4\log 19 \geq 4h(p), \qquad A_4 = 2h(\sqrt{8/5}) = 2\log 8.$$

Theorem 2.3 now yields

$$-2.5 \cdot 10^{16}(1 + \log x) < \log|\Gamma_1|,$$

which together with inequality (3.16) provides

$$y\log\gamma < \log 8 + 2.5 \cdot 10^{16}(1 + \log x). \tag{3.17}$$

Now the relation $F_x - 1 = p^{b_1-c_1}(P_y - 1)$ and (2.5) imply

$$(x-2)\log\alpha < b_1 \log p + (y-1)\log\gamma.$$

Combining the above inequality with (3.11), we get

$$(x-2)\log\alpha < \left(1.5\log\left(\frac{x}{2}\right) + 3\right)\log p + (y-1)\log\gamma, \tag{3.18}$$

and after some steps, using the fact that $p \leq 19$, we conclude that $x/2 < y$ for $x \geq 890$. This inequality together with (3.17) yield $x \leq 2.5 \cdot 10^{18}$. Thus, $y \leq 2.5 \cdot 10^{18}$ also holds.

The second option at (3.13) is that $b_1 \leq c_1$. The treatment is analogous to the first case. We obtain

$$p^{c_1-b_1}(F_x - 1) = P_y - 1, \tag{3.19}$$

and then the explicit formulae, and natural upper bounds on the terms provide

$$\left|\frac{\gamma^y}{\alpha^x}\frac{\sqrt{5}}{\sqrt{8}}\frac{1}{p^{c_1-b_1}} - 1\right| < \frac{8}{\alpha^x}. \tag{3.20}$$

Obviously,

$$\Gamma_2 = \frac{\gamma^y}{\alpha^x}\frac{\sqrt{5}}{\sqrt{8}}\frac{1}{p^{c_1-b_1}} - 1$$

is non-zero. Hence, we can apply Theorem 2.3 again. Now $\eta_4 := \sqrt{5}/\sqrt{8}$ and its logarithmic height is also $\log(8)/2$ (we have not changed η_1, η_2, and η_3). We claim that $D = x$ again. To see this we assume $y \geq 9 \cdot 10^5$. Consider equation (3.19) and inequalities (2.5). Combine them with (3.12). They provide

$$(y-2)\log\gamma \leq 614\log^2(1.05(y+1))\log(19) + x\log\alpha, \tag{3.21}$$

and then $y < x$ for $y \geq 9 \cdot 10^5$. From Theorem 2.3 and (3.20), we conclude $x < 2.3 \cdot 10^{18}$. Thus, $y < 2.3 \cdot 10^{18}$.

We summarise what we proved so far.

Theorem 3.4. *If x, y satisfy (3.13), then $x, y < 2.5 \cdot 10^{18}$.*

In the second part of the proof, we reduce the above bounds using the LLL algorithm. The two cases above will be considered separately.

In the first case, we know that the expression appearing at (3.15) is smaller than $8/\gamma^y$, which is smaller than $3/4$ if y is not very small. Thus,

$$\left|x\log\alpha - y\log\gamma + \log\left(\frac{\sqrt{8}}{\sqrt{5}}\right) - (b_1 - c_1)\log p\right| < \frac{16}{\gamma^y}. \tag{3.22}$$

This step is based on the fact that if real numbers x and K satisfy $|e^x - 1| < K < 3/4$, then $|x| < 2K$. We apply the LLL algorithm for each $p \in \{2, 3, 5, 7, 11, 13, 17, 19\}$ separately with the initial bounds $x, y, b_1 - c_1 < 2.5 \cdot 10^{18}$. The computations are based on Chapters 2.3.3. and 2.3.4. of Cohen's book [6]. The universal upper bound provided by the LLL algorithm is $y \le 173$. This makes it possible to bound x by (3.18), which gives $x \le 376$.

For the second case, we start with (3.20). Then it follows that

$$\left| y \log \gamma - x \log \alpha + \log \left(\frac{\sqrt{5}}{\sqrt{8}} \right) - (c_1 - b_1) \log p \right| < \frac{16}{\alpha^y}. \tag{3.23}$$

Now the LLL algorithm gives $x \le 316$ valid for each prime $p \le 19$. Suppose that $p \ne 11$ or y is odd. Then

$$\begin{aligned} (y-2)\log\gamma &< c_1 \log p + (x-1)\log\alpha \\ &\le \left(1.5 \log\left(\frac{y}{2}\right) + 3\right) \log p + (x-1)\log\alpha. \end{aligned}$$

Using $p \le 19$, we get $y \le 207$. If $p = 11$ and y is even, then, since $x \le 316$, we get

$$(y-2)\log\gamma < 614 \log^2(1.05(y+1)) \log 11 + x \log\alpha,$$

which leads to $y \le 262300$. Of course this bound does not make it possible to check the eligible cases by brute force. Therefore first we applied the LLL algorithm for (3.23), but with $x \le 316$. This reduced the bound to $x \le 89$. Note that this new estimate does not reduce essentially the upper bound on y. Then we checked the integers $P_j - 1$ for the range $1 \le j \le 262300$ modulo 11^k. The largest value k for which $11^k \mid P_j - 1$ is $k = 5$, which holds only for $j = 200192$. In the given range, there is no integer j with $11^6 \mid P_j - 1$ (the first such index is $j = 551576$). Hence, in the equation

$$11^{c_1 - b_1}(F_x - 1) = P_y - 1$$

we know $0 \le c_1 - b_1 \le 5$, $x \le 89$, and $y \le 262300$. For each possible exponent $c_1 - b_1$, and for each possible subscript x, we checked whether $11^{c_1-b_1}(F_x - 1) + 1$ is a Pell number or not. The results are included in the statement of Theorem 1.1.

The proof was completed by a direct computer verification of (3.13) for the values $x \le 376$, $y \le 207$, and for all the primes $p \ne 11$. All the solutions found are listed in Theorem 1.1.

4. Comments and generalizations

In this concluding section, we illustrate the generality of our method and point out how it compares to previous work on the topic. Let $(U_n)_{n\ge0}$, $(V_n)_{n\ge0}$ be non-degenerate binary recurrent sequences of integers whose characteristic roots are quadratic units. That is, their characteristic polynomials are $x^2 - ux - s$, $x^2 - vs - s_1$, respectively, with $s, s_1 \in \{\pm1\}$ and $\Delta := u^2 + 4s > 0$, $\Delta_1 := v^2 + 4s_1 > 0$. We assume that the characteristic roots of $(U_n)_{n\ge0}$ and $(V_n)_{n\ge0}$ are multiplicatively independent. This is equivalent to saying that the real quadratic fields $\mathbb{Q}(\sqrt{\Delta})$ and $\mathbb{Q}(\sqrt{\Delta_1})$ are distinct. Minor modifications of our method gives the following.

Theorem 4.5. *Let $(U_n)_{n\ge0}, (V_n)_{n\ge0}$ be non-degenerate binary recurrent sequences of integers satisfying the above conditions. Let d, d_1 be integers and $\mathcal{S}$ be a finite set of primes. Then there are only finitely many pairs of positive inetegers (m, n) with $U_m \ne d$, $V_n \ne d_1$ such that*

$$\frac{(U_m - d)(V_n - d_1)}{\gcd(U_m - d, V_n - d_1)^2} \quad \text{is an} \quad \mathcal{S}-\text{unit}.$$

To see the connection, write $c := \gcd(U_m - d, V_n - d_1)$. This is well-defined since the numbers $U_m - d,\ V_n - d_1$ are non-zero. Write $a := (U_m - d)/c,\ b := (V_n - d_1)/c$. Then the last condition becomes

$$ac + d = U_m, \quad bc + d_1 = V_n, \quad ab \quad \text{is an} \quad \mathcal{S} - \text{unit}.$$

The present paper explicitly computed all the solutions of the above system for the choices $d = d_1 = 1$, $(U_n)_{n\geq 0}$ and $(V_n)_{n\geq 0}$ are the sequences of Fibonacci and Pell numbers for which $u = 1, v = 2, s = s_1 = 1$ so $\Delta = 5,\ \Delta_1 = 8$, and $\mathcal{S} = \{p\}$ where $p < 20$ is a prime. Thus, our problem is a hybrid between Diophantine 3-tuples with values in binary recurrences and $\mathcal{S}$-units and our method has the advantage of being completely explicit which is rarely the case in general, as most of the available results in the literature on the topic of Diophantine triples with $\mathcal{S}$-units and/or linear recurrences are obtained using the Subspace theorem and as such are not effective except in particular special cases like the ones treated in [15] and [16].

Acknowledgments. We thank the referees for comments which improved the quality of our paper. F. Luca was supported in part by the ANR project JINVARIANT. For L. Szalay the research was supported by Hungarian National Foundation for Scientific Research Grant No. 128088, and No. 130909, furthermore by National Research, Development and Innovation Office Grant 2019-2.1.11-TÉT-2020-00165, and by the Slovak Scientific Grant Agency VEGA 1/0776/21.

REFERENCES

[1] Alp, M.: Irmak, N.; Szalay, L. Balancing diophantine triples. *Acta Univ. Sapientiae Math.* **4** (2012), 11–19

[2] Bugeaud, Y.; Laurent, M. Minoration effective de la distance p-adique entre puissances de nombres algébriques. *J. Number Theory* **61** (1996), 311–342.

[3] Bugeaud, Y.; Luca, F. A quantitative lower bound for the greatest prime factor of $(ab + 1)(bc + 1)(ca + 1)$. *Acta Arith.* **114** (2004), 275–294.

[4] Bugeaud, Y.; Maurice, M.; Siksek, S. Classical and modular approaches to exponential Diophantine equations I. Fibonacci and Lucas perfect powers. *Ann. Math.* **163** (2006), 969–1018.

[5] Carmichael, R. D. On the numerical factors of the arithmetic function $\alpha^n \pm \beta^n$. *Ann. Math.* **15** (1913-1914), 30–48.

[6] Cohen, H. Number Theory, Volume I: Tools and Diophantine Equations. *Graduate Texts in Mathematics* **239**, Springer, 2007.

[7] Corvaja, P.; Zannier, U. On the greatest prime factor of $(ab + 1)(ac + 1)$. *Proc. Amer. Math. Soc.* **131** (2003), 1705–1709.

[8] Győry, K.; Sarkozy, A.; Stewart, C. L. On the number of prime factors of integers of the form $ab + 1$. *Acta Arith.* **74** (1996), 365–385.

[9] Fuchs, C.; Heintze, S. Another S-unit variant of Diophantine tuples. *Proc. Amer. Math. Soc.* **149** (2021), 27–35.

[10] Fuchs, C.; Hutle, C.; Luca, F.; Szalay, L. Diophantine triples and k-generalized Fibonacci sequences. *Bull. Malays. Math. Sci. Soc.* **41** (2018), 1449–1465.

[11] Fuchs, C.; Hutle, C.; Luca, F. Diophantine triples in linear recurrences of Pisot type. *Res. Number Theory* **4** (2018), no. 3, Paper No. 29, 22 pp.

[12] Fuchs, C.; Hutle, C.; Irmak, N.; Luca, F.; Szalay, L. Only finitely many tribonacci Diophantine triples exist. *Math. Slovaca* **67** (2017), 853–862.

[13] Fuchs, C.; Luca, F.; Szalay, L. Diophantine triples with values in binary recurrences. *Ann. Sc. Norm. Super. Pisa Cl. Sc.* **7** (2008), 579–608.

[14] Lengyel, T. The order of the Fibonacci and Lucas numbers. *Fibonacci Quart.* **33** (1995), 234–239.

[15] Luca, F.; Szalay, L. Fibonacci diophantine triples. *Glas. Mat. Ser. III* **43** (2008), no. 63, 253–264.

[16] Luca, F.; Szalay, L. Lucas diophantine triples. *Integers* **9** (2009), Article A35, 441–457.

[17] Luca, F.; Ziegler, V. A note on the number of S-Diophantine quadruples. *Commun. Math.* **22** (2014), 49–55.

[18] Lucas, E. Théorie des fonctions numériques simplement périodiques. *Amer. J. Math.* **1** (1878), 289–321.

[19] Matveev, E. M. An explicit lower bound for a homogeneous rational linear form in logarithms of algebraic numbers, II. *Izv. Ross. Akad. Nauk Ser. Mat.* **64** (2000), 125–180. English translation in *Izv. Math.* **64** (2000), 1217–1269.

[20] Sanna, C. The p-adic valuation of Lucas sequences. *Fibonacci Quart.* **58** (2016), 118–124

[21] Szalay, L.; Ziegler, V. On an S-unit variant of Diophantine m-tuples. *Publ. Math. Debrecen* **83** (2013), 97–121.
[22] Szalay, L.; Ziegler, V. S-Diophantine quadruples with $S = \{2, q\}$. *Int. J. Number Theory* **11** (2015), 849–868.
[23] Ziegler, V. On the existence of S-Diophantine quadruples. *Glas. Mat. Ser. III* **54** (2019), no. 79 279–319.

UNIVERSITY OF THE WITWATERSRAND, SCHOOL OF MATHEMATICS
PRIVATE BAG 3, WITS 2050, JOHANNESBURG, SOUTH AFRICA

KING ABDULAZIZ UNIVERSITY, RESEARCH GROUP IN ALGEBRAIC STRUCTURES AND APPLICATIONS
JEDDAH, SAUDI ARABIA

IMB, UNIVERSITÉ DE BORDEAUX,
351, COURS DE LA LIBÉRATION, 33 405 TALENCE, FRANCE

CENTRO DE CIENCIAS MATEMÁTICAS, UNAM
MORELIA, MEXICO
Email address: Florian.Luca@wits.ac.za

J. SELYE UNIVERSITY, DEPARTMENT OF MATHEMATICS
HRADNA UL. 21, 94501 KOMÁRNO, SLOVAKIA

UNIVERSITY OF SOPRON, INSTITUTE OF MATHEMATICS
BAJCSY-ZSILINSZKY UTCA 4, H-9400, SOPRON, HUNGARY
Email address: szalay.laszlo@uni-sopron.hu