CARPATHIAN J. MATH.
Volume **39** (2023), No. 1,
Pages 265-280

Online version at https://semnul.com/carpathian/
Print Edition: ISSN 1584 - 2851; Online Edition: ISSN 1843 - 4401
DOI: https://doi.org/10.37193/CJM.2023.01.18

In memoriam Professor Charles E. Chidume (1947- 2021)

Unpredictable solutions of quasilinear differential equations with generalized piecewise constant arguments of mixed type

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ABSTRACT. An unpredictable solution is found for a quasilinear differential equation with generalized piecewise constant argument (EPCAG). Sufficient conditions are provided for the existence, uniqueness and exponential stability of the unpredictable solution. The theoretical results are confirmed by examples and illustrated by simulations.

1. INTRODUCTION

It is worth noting that numerous results, which include the most effective methods and important applications, are obtained for periodic, quasi-periodic and almost periodic solutions in the theory of differential equations [27, 31, 33, 34, 35, 36, 37, 38, 39, 40]. On the other hand, Poisson stable solutions are also crucial for the theory of differential equations [45]. In our research [12, 13], we have developed the recurrence in functional spaces to a more refined level, where the Poisson stable functions are assigned the unpredictability. Our proposal can revive interests of mathematicians in sophisticated oscillations for two reasons. The first one is related to the verification of the unpredictability, which requires a more developed technique than that for other oscillations. Thus, the problem of the existence of unpredictable solutions is a challenging one. In paper [46], a method of comparability of functions by the character of their recurrence was suggested, which is suitable for applications in the theory of differential equations. In particular, it is useful for Poisson stable solutions [30, 46]. In our papers [14, 15, 16], we have applied a new approach, which is different from the one used in [30, 46] to prove the Poisson stability. It can be utilized for various types of dynamical equations in the future. Moreover, we introduced and developed an entirely new method that shows how to verify the unpredictability property for solutions of differential equations and oscillations in neural networks [9, 17, 18, 19, 20, 21, 22, 23]. It promises to be universal and can be applied for various types of differential equations. Partial differential equations, evolution equations, impulsive systems and hybrid systems are among them. Another reason to consider our proposals is the phenomenon of chaos, for which the unpredictability is a criterion [12, 13]. In other words, the proof of unpredictability simultaneously verifies the Poincaré chaos of the Bebutov dynamics in the functional space with the topology of uniform convergence on compact sets of the real axis. This opens new prospects for control and synchronization

Received: 03.03.2022. In revised form: 28.04.2022. . Accepted: 23.05.2022
2010 *Mathematics Subject Classification.* 34E15, 34B15, 34E05.
Key words and phrases. *quasilinear differential equation, unpredictable solution, piecewise constant argument of generalized type, delayed and advanced argument, Poincaré chaos, exponential stability.*
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of chaos in differential equations. This time, we proceed the initial steps of constructing the basics of the theory and prove the existence of the unpredictable solution for a special type of hybrid systems, where discontinuities appear in the time-argument of the solution of a differential equation. Differential equations with generalized piecewise constant functions as arguments (EPCAG) have been introduced and developed in papers [1, 2, 3, 4, 5, 6, 10]. The ideas suggested in these papers became very useful not only in modeling but also in methodological sense, since the construction of equivalent integral equations for EPCAG has opened the research gate for methods of operator theory and functional analysis [7, 9, 24, 26, 28, 29, 32, 43, 48, 49, 50, 52, 53]. This was also confirmed with applications in neuroscience [8, 9, 11, 25, 41, 42, 44, 47, 50, 51]. In the present research, we have joined the chaos concept with the most flexible and convenient functional differential equations for applications. It should be emphasized that the models under research are suitable for adaptation of methods and tools of discrete dynamics, which are still the main source of sophisticated motions.

2. Preliminaries

Denote by $\mathbb{N}, \mathbb{R}, \mathbb{Z}$ the set of all natural numbers, real numbers and integers, respectively. Introduce a norm for the vector $x = (x_1, \cdots, x_m)$, $x_i \in \mathbb{R}$, $i = 1, \cdots, m$, as $||x||_1 = \max_{1\le i\le m} |x_i|$, where $|\cdot|$ is the absolute value. Let $\|A\| = \max_{1\le i\le m} \sum_{j=1}^{m} |a_{ij}|$ denote the norm for a square matrix $A = (a_{ij})_{m\times m}$. Fix two real valued sequences $\theta_i,\ \xi_i,\ i \in \mathbb{Z}$, such that $\theta_i < \theta_{i+1},\ \theta_i \le \xi_i \le \theta_{i+1}$ for all $i \in \mathbb{Z}$, $|\theta_i| \to \infty$ as $|i| \to \infty$.

We will consider the following quasilinear system with generalized piecewise constant argument of mixed type

$$x'(t) = Ax(t) + f(x(t)) + g(x(\gamma(t))) + h(t), \tag{2.1}$$

where $t \in \mathbb{R}$, $x \in \mathbb{R}^m$ for a fixed $m \in \mathbb{N}$, $A \in \mathbb{R}^{m\times m}$ is a constant matrix and $\gamma(t) = \xi_i$ if $\theta_i \le t < \theta_{i+1}$, $i \in \mathbb{Z}$. Throughout this paper, we assume that the functions $f, g : D \to \mathbb{R}^m$ are continuous on a bounded domain $D = \{x \in \mathbb{R}^m : \|x\| < H\}$, where H is a positive constant. $h : \mathbb{R} \to \mathbb{R}^m$ is a uniformly continuous and bounded function. Moreover, it is assumed that all eigenvalues of the matrix A have negative real parts and $\|A\| = \overline{\lambda}$. In this case, it can be concluded that there exist real numbers $\sigma \ge 1$ and $\lambda > 0$ such that $\|e^{At}\| \le \sigma e^{-\lambda t}$ for all $t \ge 0$.

Definition 2.1. [13] *A uniformly continuous and bounded function $v : \mathbb{R} \to \mathbb{R}^m$ is unpredictable if there exist positive numbers ϵ_0, δ and sequences t_n, u_n both of which diverge to infinity such that $v(t + t_n) \to v(t)$ as $n \to \infty$ uniformly on compact subsets of $\mathbb{R}$ and $\|v(t + t_n) - v(t)\| \ge \epsilon_0$ for each $t \in [u_n - \delta, u_n + \delta]$ and $n \in \mathbb{N}$.*

The following conditions will be required in the present paper:

(C1) functions f and g satisfy a Lipschitz condition with constants L_f, L_g : $\|f(u_1) - f(u_2)\| \le L_f\|u_1 - u_2\|$ and $\|g(u_1) - g(u_2)\| \le L_g\|u_1 - u_2\|$ for all $u_1, u_2 \in D$;

(C2) $\exists\, m_f > 0, m_g > 0$ such that $\sup_{\|x\|<H} \|f(x)\| \le m_f$ and $\sup_{\|x\|<H} \|g(x)\| \le m_g$;

(C3) $\exists\, m_h > 0$ such that $\sup_{t\in\mathbb{R}} \|h(t)\| \le m_h$;

(C4) $\dfrac{\sigma}{\lambda}(m_f + m_g + m_h) < H$;

(C5) $\dfrac{\sigma}{\lambda}(L_f + L_g) < 1$;

(C6) $\exists\, \theta > 0$ such that $\theta_{i+1} - \theta_i \le \theta$ for all $i \in \mathbb{Z}$.

In what follows we will use the following notation

$$B = \left(1 - \theta[(\overline{\lambda} + L_f)(1 + L_g\theta)e^{(\overline{\lambda}+L_f)\theta} + L_g]\right)^{-1}.$$

(C7) $-\lambda + \sigma(L_f + BL_g) < 0$;
(C8) $\theta[(\overline{\lambda} + L_f)(1 + L_g\theta)e^{(\overline{\lambda}+L_f)\theta} + L_g] < 1$;
(C9) $\exists\ \{\eta_n\}$ with $\eta_n \to \infty$ as $n \to \infty$ such that

$$\theta_{i-\eta_n} + t_n - \theta_i \to 0 \text{ and } \xi_{i-\eta_n} + t_n - \xi_i \to 0$$

as $n \to \infty$ on each finite interval of integers, where t_n is the sequence defined in Definition 2.1.

3. Main result

Let $\mathcal{P}$ be defined as the space of m-dimensional vector-functions $\phi : \mathbb{R} \to \mathbb{R}^m$, $\phi = (\phi_1, \phi_2, ..., \phi_m)$ with $\|\phi\|_1 = \sup_{t\in\mathbb{R}} \|\phi(t)\|$. A function ϕ that belongs to the space $\mathcal{P}$ has the following properties:

(P1) it is uniformly continuous;
(P2) $\|\phi\|_1 < H$;
(P3) $\exists\ \{t_n\}$, $t_n \to \infty$ as $n \to \infty$ such that $\phi(t+t_n) \to \phi(t)$ uniformly on each closed and bounded interval of the real axis.

It is well known by the theory of differential equations that [37], a function $x(t)$ which is bounded on the whole real axis is a solution of system (2.1) if and only if it satisfies the following integral equation

$$x(t) = \int_{-\infty}^{t} e^{A(t-s)} \left[f(x(s)) + g(x(\gamma(s))) + h(s)\right] ds. \tag{3.2}$$

Define an operator Π on $\mathcal{P}$ as follows

$$\Pi\phi(t) = \int_{-\infty}^{t} e^{A(t-s)} \left[f(\phi(s)) + g(\phi(\gamma(s))) + h(s)\right] ds.$$

Lemma 3.1. *The operator* Π *is invariant in* $\mathcal{P}$.

Proof. We need to show that $\Pi\mathcal{P} \subseteq \mathcal{P}$. First, we differentiate $\Pi\phi(t)$ with respect to t as follows:

$$\frac{d\Pi\phi(t)}{dt} = f(\phi(t)) + g(\phi(\gamma(t))) + h(t) + A\int_{-\infty}^{t} e^{A(t-s)} \left[f(\phi(s)) + g(\phi(\gamma(s))) + h(s)\right] ds.$$

From this we can find for all $t \in \mathbb{R}$ that

$$\begin{aligned} \left\|\frac{d\Pi\phi(t)}{dt}\right\| &\leq \|f(\phi(t))\| + \|g(\phi(\gamma(t)))\| + \|h(t)\| \\ &+ \overline{\lambda}\int_{-\infty}^{t} \sigma e^{-\lambda(t-s)} \left(\|f(\phi(t))\| + \|g(\phi(\gamma(t)))\| + \|h(t)\|\right) ds \\ &\leq m_f + m_g + m_h + \frac{\sigma\overline{\lambda}}{\lambda}\left(m_f + m_g + m_h\right) = (1 + \frac{\sigma\overline{\lambda}}{\lambda})\left(m_f + m_g + m_h\right). \end{aligned}$$

Thus, we see that the derivative $\dfrac{d\Pi\phi(t)}{dt}$ is bounded and hence $\Pi\phi$ is uniformly continuous. As a result of this discussion, it is seen that $\Pi\phi$ satisfies the property (P1).

Additionally, we can find for $\phi \in \mathcal{P}$ that

$$\begin{aligned}
\|\Pi\phi(t)\| &= \Big\| \int_{-\infty}^{t} e^{A(t-s)} \left(f(\phi(s)) + g(\phi(\gamma(s))) + h(s) \right) ds \Big\| \\
&\leq \int_{-\infty}^{t} \sigma e^{-\lambda(t-s)} \left(\|f(\phi(s))\| + \|g(\phi(\gamma(s)))\| + \|h(s)\| \right) ds \\
&\leq \int_{-\infty}^{t} \sigma e^{-\lambda(t-s)} \left(m_f + m_g + m_h \right) ds = \frac{\sigma}{\lambda} \left(m_f + m_g + m_h \right).
\end{aligned}$$

It follows from the last inequality and condition (C4) that $||\Pi\phi||_1 < H$. Therefore, $\Pi\phi$ satisfies the property (P2).

We are now in a position to prove the last property (P3).That is to say, we need to show that there exists a sequence t_n which diverges to infinity such that for each $\Pi\phi \in \mathcal{P}$, $\Pi\phi(t+t_n) \to \Pi\phi(t)$ uniformly on each closed and bounded interval of the real axis. For this aim, we fix an arbitrary positive number ε and a closed interval $[a,b]$, where $a, b \in \mathbb{R}$ with $a < b$. It is enough to show that $||\Pi\phi(t+t_n) - \Pi\phi(t)|| < \varepsilon$ for sufficiently large n and $t \in [a,b]$. Let us take two numbers $c < a$ and $\epsilon > 0$ such that

$$\frac{2\sigma}{\lambda} \left(L_f H + L_g H + m_h \right) e^{-\lambda(a-c)} < \frac{\varepsilon}{4}, \tag{3.3}$$

$$\frac{\sigma\epsilon}{\lambda}(1 + L_f) < \frac{\varepsilon}{4}. \tag{3.4}$$

We choose n large enough such that $\|\phi(t+t_n) - \phi(t)\| < \epsilon$ and $\|h(t+t_n) - h(t)\| < \epsilon$ on $[c,b]$, and $\theta_{j-\eta_n} + t_n - \theta_j < \epsilon$ for $\theta_j \in [c,b], j \in \mathbb{Z}$. Then, we can write the following inequality

$$\begin{aligned}
\|\Pi\phi(t+t_n) - \Pi\phi(t)\| &= \Big\| \int_{-\infty}^{t+t_n} e^{A(t+t_n-s)} \left[f(\phi(s)) + g(\phi(\gamma(s))) + h(s) \right] ds \\
&- \int_{-\infty}^{t} e^{A(t-s)} \left(f(\phi(s)) + g(\phi(\gamma(s))) + h(s) \right) ds \Big\| \\
&= \Big\| \int_{-\infty}^{t} e^{A(t-s)} ([f(\phi(s+t_n)) - f(\phi(s))] \\
&+ [g(\phi(\gamma(s+t_n))) - g(\phi(\gamma(s)))] + h(s+t_n) - h(s)) ds \Big\| \\
&\leq \int_{-\infty}^{t} \sigma\, e^{-\lambda(t-s)} (L_f \|\phi(s+t_n) - \phi(s)\| \\
&+ L_g \|\phi(\gamma(s+t_n)) - \phi(\gamma(s))\| + \|h(s+t_n) - h(s)\|) ds.
\end{aligned}$$

Now, let us rewrite the last integral as a sum of two integrals. We obtain that

$$\begin{aligned}
\|\Pi\phi(t+t_n)-\Pi\phi(t)\| &\leq \int_{-\infty}^{c} \sigma e^{-\lambda(t-s)}(L_f\|\phi(s+t_n)-\phi(s)\| \\
&+ L_g\|\phi(\gamma(s+t_n))-\phi(\gamma(s))\| + \|h(s+t_n)-h(s)\|)ds \\
&+ \int_{c}^{t} \sigma e^{-\lambda(t-s)}(L_f\|\phi(s+t_n)-\phi(s)\| \\
&+ L_g\|\phi(\gamma(s+t_n))-\phi(\gamma(s))\| + \|h(s+t_n)-h(s)\|)ds \\
&\leq \frac{2\sigma}{\lambda}\left(L_fH + L_gH + m_h\right)e^{-\lambda(a-c)} + \int_{c}^{t} \sigma e^{-\lambda(t-s)}(1+L_f)\epsilon ds \\
&+ \int_{c}^{t} \sigma e^{-\lambda(t-s)} L_g\|\phi(\gamma(s+t_n))-\phi(\gamma(s))\|ds \\
&\leq \frac{2\sigma}{\lambda}\left(L_fH + L_gH + m_h\right)e^{-\lambda(a-c)} + \frac{\sigma}{\lambda}(1+L_f)\epsilon \\
&+ \sigma L_g \int_{c}^{t} e^{-\lambda(t-s)}\|\phi(\gamma(s+t_n))-\phi(\gamma(s))\|ds.
\end{aligned}$$

For a fixed $t \in [a,b]$, we assume without loss of generality that $\theta_k \leq \theta_{k-\eta_n} + t_n$ and $\theta_k \leq \theta_{k-\eta_n} + t_n = c < \theta_{k+1} < \theta_{k+2} < \cdots < \theta_{k+p} \leq \theta_{k+p-\eta_n} + t_n \leq t < \theta_{k+p+1}$ so that there exist exactly p discontinuity moments in the interval $[c,t]$.

Let the following inequalities

$$\sigma L_g \frac{2pH}{\lambda}(e^{\lambda\epsilon}-1) < \frac{\varepsilon}{4}, \tag{3.5}$$

$$\sigma L_g \frac{2(p+1)\epsilon}{\lambda}(1-e^{-\lambda\theta}) < \frac{\varepsilon}{4}. \tag{3.6}$$

be satisfied for the given $\epsilon > 0$.

We aim to obtain an upper bound for the last integral which will be denoted by

$$I = \int_{c}^{t} e^{-\lambda(t-s)}\|\phi(\gamma(s+t_n))-\phi(\gamma(s))\|ds.$$

We evaluate I by considering it on finite number of subintervals as described below:

$$\begin{aligned}
I &= \int_{c}^{\theta_{k+1}} e^{-\lambda(t-s)}\|\phi(\gamma(s+t_n))-\phi(\gamma(s))\|ds \\
&+ \int_{\theta_{k+1}}^{\theta_{k+1-\eta_n}+t_n} e^{-\lambda(t-s)}\|\phi(\gamma(s+t_n))-\phi(\gamma(s))\|ds \\
&+ \int_{\theta_{k+1-\eta_n}+t_n}^{\theta_{k+2}} e^{-\lambda(t-s)}\|\phi(\gamma(s+t_n))-\phi(\gamma(s))\|ds \\
&+ \int_{\theta_{k+2}}^{\theta_{k+2-\eta_n}+t_n} e^{-\lambda(t-s)}\|\phi(\gamma(s+t_n))-\phi(\gamma(s))\|ds \\
&+ \int_{\theta_{k+2-\eta_n}+t_n}^{\theta_{k+3}} e^{-\lambda(t-s)}\|\phi(\gamma(s+t_n))-\phi(\gamma(s))\|ds \\
&\ \ \vdots \\
&+ \int_{\theta_{k+p-\eta_n}+t_n}^{t} e^{-\lambda(t-s)}\|\phi(\gamma(s+t_n))-\phi(\gamma(s))\|ds \\
&= \sum_{i=k}^{k+p-1} \int_{\theta_{i-\eta_n}+t_n}^{\theta_{i+1}} e^{-\lambda(t-s)}\|\phi(\gamma(s+t_n))-\phi(\gamma(s))\|ds \\
&+ \sum_{i=k}^{k+p-1} \int_{\theta_{i+1}}^{\theta_{i+1-\eta_n}+t_n} e^{-\lambda(t-s)}\|\phi(\gamma(s+t_n))-\phi(\gamma(s))\|ds \\
&+ \int_{\theta_{k+p-\eta_n}+t_n}^{t} e^{-\lambda(t-s)}\|\phi(\gamma(s+t_n))-\phi(\gamma(s))\|ds.
\end{aligned}$$

Let us define the integrals in the above expression as

$$A_i = \int_{\theta_{i-\eta_n}+t_n}^{\theta_{i+1}} e^{-\lambda(t-s)}\|\phi(\gamma(s+t_n))-\phi(\gamma(s))\|ds$$

and

$$B_i = \int_{\theta_{i+1}}^{\theta_{i+1-\eta_n}+t_n} e^{-\lambda(t-s)}\|\phi(\gamma(s+t_n))-\phi(\gamma(s))\|ds,$$

for $i = k, k+1, \cdots, k+p-1$.

Using the notations A_i and B_i, we can write

$$I = \sum_{i=k}^{k+p-1} A_i + \sum_{i=k}^{k+p-1} B_i + \int\limits_{\theta_{k+p-\eta_n}+t_n}^{t} e^{-\lambda(t-s)} \|\phi(\gamma(s+t_n)) - \phi(\gamma(s))\| ds.$$

For $t \in [\theta_{i-\eta_n}+t_n, \theta_{i+1})$, $i \in \mathbb{Z}$, it is clear that $\gamma(t) = \xi_i$ and it follows from the condition (C9) that $\gamma(t+t_n) = \xi_{i+\eta_n}$. Using this result, we reach the following estimation:

$$\begin{aligned} A_i &= \int\limits_{\theta_{i-\eta_n}+t_n}^{\theta_{i+1}} e^{-\lambda(t-s)} \|\phi(\xi_{i+\eta_n}) - \phi(\xi_i)\| ds \\ &= \int\limits_{\theta_{i-\eta_n}+t_n}^{\theta_{i+1}} e^{-\lambda(t-s)} \|\phi(\xi_i + t_n + o(1)) - \phi(\xi_i)\| ds \\ &= \int\limits_{\theta_{i-\eta_n}+t_n}^{\theta_{i+1}} e^{-\lambda(t-s)} \|\phi(\xi_i + t_n) - \phi(\xi_i) + \phi(\xi_i + t_n + o(1)) - \phi(\xi_i + t_n)\| ds \\ &\leq \int\limits_{\theta_{i-\eta_n}+t_n}^{\theta_{i+1}} e^{-\lambda(t-s)} \Big[\|\phi(\xi_i + t_n) - \phi(\xi_i)\| + \|\phi(\xi_i + t_n + o(1)) - \phi(\xi_i + t_n)\|\Big] ds \\ &\leq \int\limits_{\theta_{i-\eta_n}+t_n}^{\theta_{i+1}} e^{-\lambda(t-s)} \Big[\epsilon + \|\phi(\xi_i + t_n + o(1)) - \phi(\xi_i + t_n)\|\Big] ds. \end{aligned}$$

We already know that ϕ is a uniformly continuous function. Thus, for $\epsilon > 0$ and sufficiently large n we can find a $\rho > 0$ such that $\|\phi(\xi_i + t_n + o(1)) - \phi(\xi_i + t_n)\| < \epsilon$ if $|\xi_{i+\eta_n} - \xi_i - t_n| < \rho$. This implies in turn that

$$A_i \leq 2\epsilon \int\limits_{\theta_{i-\eta_n}+t_n}^{\theta_{i+1}} e^{-\lambda(t-s)} ds \leq \frac{2\epsilon}{\lambda}(1 - e^{-\lambda\theta}).$$

On the other hand, condition (C9) gives us that

$$B_i \leq 2H \int\limits_{\theta_{i+1}}^{\theta_{i+1-\eta_n}+t_n} e^{-\lambda(t-s)} ds \leq \frac{2H}{\lambda}(e^{\lambda\epsilon} - 1).$$

If we use a similar approach used for the estimation of the integral A_i, then it follows that

$$\int\limits_{\theta_{k+p-\eta_n}+t_n}^{t} e^{-\lambda(t-s)} \|\phi(\gamma(s+t_n)) - \phi(\gamma(s))\| \leq \frac{2\epsilon}{\lambda}(1 - e^{-\lambda\theta}).$$

Therefore, it can be seen that

$$I \leq \frac{2(p+1)\epsilon}{\lambda}(1 - e^{-\lambda\theta}) + \frac{2pH}{\lambda}(e^{\lambda\epsilon} - 1).$$

As a result of these computations, we get

$$\begin{aligned}\|\Pi\phi(t+t_n)-\Pi\phi(t)\| &\leq \frac{2\sigma}{\lambda}\left(L_f H+L_g H+m_h\right)e^{-\lambda(a-c)}+\frac{\sigma\epsilon}{\lambda}(1+L_f)\\ &+ \sigma L_g\frac{2(p+1)\epsilon}{\lambda}(1-e^{-\lambda\theta})+\sigma L_g\frac{2pH}{\lambda}(e^{\lambda\epsilon}-1)\end{aligned}$$

for all $t\in[a,b]$. In consequence, the inequalities (3.3) -(3.6) give that

$$||\Pi\phi(t+t_n)-\Pi\phi(t)||<\varepsilon$$

for $t\in[a,b]$. Thus, the function $\Pi\phi$ satisfies the property (P3). Finally, it turns out that the operator Π is invariant in $\mathcal{P}$. □

Lemma 3.2. *The operator* Π *is contractive on the space* $\mathcal{P}$.

Proof. Let the functions ϕ_1 and ϕ_2 lie in $\mathcal{P}$. For all $t\in\mathbb{R}$, we have

$$\begin{aligned}\|\Pi\phi_1(t)-\Pi\phi_2(t)\| &= \Big\|\int_{-\infty}^{t} e^{A(t-s)}\big[\,[f(\phi_1(s))-f(\phi_2(s))]+[g(\phi_1(\gamma(s))-g(\phi_2(\gamma(s))]\big]ds\Big\|\\ &\leq \int_{-\infty}^{t}\sigma e^{-\lambda(t-s)}\big[L_f\|\phi_1(s)-\phi_2(s)\|+L_g\|\phi_1(\gamma(s))-\phi_2(\gamma(s))\|\big]ds\\ &\leq \int_{-\infty}^{t}\sigma e^{-\lambda(t-s)}\big[L_f\|\phi_1(s)-\phi_2(s)\|_1+L_g\|\phi_1(s)-\phi_2(s)\|_1\big]ds\\ &\leq \frac{\sigma}{\lambda}\left(L_f+L_g\right)\|\phi_1(t)-\phi_2(t)\|_1.\end{aligned}$$

Then,

$$\|\Pi\phi_1-\Pi\phi_2\|_1\leq\frac{\sigma}{\lambda}\left(L_f+L_g\right)\|\phi_1-\phi_2\|_1$$

holds true for all $t\in\mathbb{R}$. In conclusion, the condition (C5) implies that the $\Pi:\mathcal{P}\to\mathcal{P}$ is a contraction operator. □

The following result will be useful in the proof of the stability of the solution.

Lemma 3.3. [6] *Assume that the conditions (C1),(C6),(C8) hold true and* $y(t)$ *is a continuous function with* $\|y(t)\|_1<H$. *If* $v(t)$ *is a solution of the following differential equation with piecewise constant argument of generalized type*

(3.7) $$v'(t)=Av(t)+f(v(t)+y(t))-f(y(t))+g(v(\gamma(t))+y(\gamma(t)))-g(y(\gamma(t))),$$

then the inequality given by

(3.8) $$||v(\gamma(t))||\leq B||v(t)||$$

is satisfied for all $t\in\mathbb{R}$.

Proof. Fix $i\in\mathbb{Z}$ such that $t\in[\theta_i,\theta_{i+1})$, and consider the cases:
(a) $\theta_i\leq\xi_i\leq t<\theta_{i+1}$ and (b) $\theta_i\leq t<\xi_i<\theta_{i+1}$.

(a) For the case $t \geq \xi_i$, we can write that

$$\begin{aligned} ||v(t)|| &\leq ||v(\xi_i)|| + \int_{\xi_i}^{t} (||A||||v(s)|| + L_f||v(s)|| + L_g||v(\xi_i)||)\, ds \\ &\leq ||v(\xi_i)|| + \int_{\xi_i}^{t} \left(\overline{\lambda}||v(s)|| + L_f||v(s)|| + L_g||v(\xi_i)||\right) ds \\ &\leq ||v(\xi_i)||(1 + L_g\theta) + \int_{\xi_i}^{t} \left(\overline{\lambda} + L_f\right) ||v(s)||ds. \end{aligned}$$

If we use the Gronwall-Bellman Lemma [37], we get

$$||v(t)|| \leq ||v(\xi_i)||(1 + L_g\theta)e^{(\overline{\lambda}+L_f)\theta}.$$

In other respects, we have that

$$\begin{aligned} ||v(\xi_i)|| &\leq ||v(t)|| + \int_{\xi_i}^{t} [||A||||v(s)|| + L_f||v(s)|| + L_g||v(\xi_i)||]\, ds \\ &\leq ||v(t)|| + \int_{\xi_i}^{t} [(\overline{\lambda} + L_f)||v(s)|| + L_g||v(\xi_i)||]ds \\ &\leq ||v(t)|| + \int_{\xi_i}^{t} \left[(\overline{\lambda} + L_f)(1 + L_g\theta)e^{(\overline{\lambda}+L_f)\theta}||v(\xi_i)|| + L_g||v(\xi_i)||\right] ds \\ &\leq ||v(t)|| + \theta \left[(\overline{\lambda} + L_f)(1 + L_g\theta)e^{(\overline{\lambda}+L_f)\theta} + L_g\right] ||v(\xi_i)||. \end{aligned}$$

Therefore, condition (C8) yields that $\|v(\xi_i)\| \leq B\|v(t)\|$, for $t \in [\theta_i, \theta_{i+1})\,, i \in \mathbb{Z}$. Hence, (3.8) holds for all $\theta_i \leq \xi_i \leq t < \theta_{i+1}$, $i \in \mathbb{Z}$. The second case (b) where $\theta_i \leq t < \xi_i < \theta_{i+1}, i \in \mathbb{Z}$ can be proved by using a similar approach.

Thus, the inequality (3.8) holds true for all $t \in \mathbb{R}$. The lemma is proved. □

The next theorem states the most important result of the present paper.

Theorem 3.1. *Assume that the conditions (C1)-(C9) are fulfilled. If the function h is unpredictable, then the system (2.1) has a unique exponentially stable unpredictable solution.*

Proof. First, we aim to show that the space $\mathcal{P}$ is complete. Let $\pi_k(t)$ be a Cauchy sequence in $\mathcal{P}$ with $\pi_k(t) \to \pi(t)$ on $\mathbb{R}$ as $k \to \infty$. It is clear that the limit function $\pi(t)$ is uniformly continuous and bounded [37]. Thus, properties (P2) and (P3) are satisfied by $\pi(t)$. We need to show that property (P3) is also satisfied by $\pi(t)$. Let I be a closed and bounded interval on $\mathbb{R}$. One can write

$$\|\pi(t + t_n) - \pi(t)\| \leq \|\pi(t + t_n) - \pi_k(t + t_n)\| + \|\pi_k(t + t_n) - \pi_k(t)\| + \|\pi_k(t) - \pi(t)\|$$

by means of the triangle inequality.

If we take sufficiently large n and k such that each term on the right hand side of last the inequality is less than $\frac{\varepsilon}{3}$ for sufficiently small $\varepsilon > 0$ and $t \in I$, then the inequality $\|\pi(t + t_n) - \pi(t)\| < \varepsilon$ is satisfied on I. This implies that the sequence of the functions $\pi(t + t_n)$ converges to $\pi(t)$ uniformly on I. Therefore, $\mathcal{P}$ is a complete space. We know that the operator Π is invariant and contractive in $\mathcal{P}$ according to Lemma 3.1 and Lemma

3.2, respectively. The contraction mapping theorem implies that the operator Π has a unique fixed point $y(t) \in \mathcal{P}$, which is the unique solution of the system (2.1). Hence, the uniqueness of the solution is proved. We need to show that this unique solution is unpredictable.

Let $l, k \in \mathbb{N}$ and κ be a positive number satisfying the following inequalities

$$\kappa < \delta, \tag{3.9}$$

$$\kappa\Big[-(\overline{\lambda}+L_f)(\frac{1}{l}+\frac{2}{k})-2L_g+\frac{1}{2}\Big] \geq \frac{4}{3l}, \tag{3.10}$$

and

$$\|y(t+s)-y(t)\| < \epsilon_0 \min\{\frac{1}{k}, \frac{1}{3l}\},\ t \in \mathbb{R},\ |s| < \kappa. \tag{3.11}$$

Assume that the numbers κ, l, k and $n \in \mathbb{N}$ are fixed. We will use the symbol Δ to denote the value $\|y(u_n+t_n)-y(u_n)\|$, then consider the two cases (i) $\Delta \geq \frac{\epsilon_0}{l}$ and (ii) $\Delta < \frac{\epsilon_0}{l}$.

(i) If $\Delta \geq \frac{\epsilon_0}{l}$, one can conclude that

$$\begin{aligned} \|y(t+t_n)-y(t)\| &\geq \|y(u_n+t_n)-y(u_n)\| - \|y(u_n)-y(t)\| \\ &- \|y(t+t_n)-y(u_n+t_n)\| > \frac{\epsilon_0}{l} - \frac{\epsilon_0}{3l} - \frac{\epsilon_0}{3l} = \frac{1}{3l}\epsilon_0 \end{aligned}$$

for $t \in [u_n-\kappa, u_n+\kappa]$, $n \in \mathbb{N}$.

(ii) If $\Delta < \frac{\epsilon_0}{l}$, (3.11) gives that

$$\begin{aligned} \|y(t+t_n)-y(t)\| &\leq \|y(u_n+t_n)-y(u_n)\| + \|y(u_n)-y(t)\| \\ &+ \|y(t+t_n)-y(u_n+t_n)\| < \frac{\epsilon_0}{l} + \frac{\epsilon_0}{k} + \frac{\epsilon_0}{k} = (\frac{1}{l}+\frac{2}{k})\epsilon_0 \end{aligned}$$

for $t \in [u_n, u_n+\kappa]$. Take the following integral equations

$$y(t) = y(u_n) + \int_{u_n}^{t} \Big[Ay(s) + f(y(s)) + g(y(\gamma(s))) + h(s)\Big] ds$$

and

$$y(t+t_n) = y(u_n+t_n) + \int_{u_n}^{t} \Big[Ay(s+t_n) + f(y(s+t_n)) + g(y(\gamma(s+t_n))) + h(s+t_n)\Big] ds$$

into consideration. If we subtract the first equation from the second one, we get

$$\begin{aligned} y(t+t_n)-y(t) &= y(u_n+t_n)-y(u_n) \\ &+ \int_{u_n}^{t}\Big[A[y(s+t_n)-y(s)]+[f(y(s+t_n))-f(y(s))] \\ &+ [g(y(\gamma(s+t_n)))-g(y(\gamma(s)))]+[h(s+t_n)-h(s)]\Big]ds \\ &= y(u_n+t_n)-y(u_n)-\int_{u_n}^{t}A[y(s+t_n)-y(s)]ds \\ &+ \int_{u_n}^{t}[f(y(s+t_n))-f(y(s))]ds \\ &+ \int_{u_n}^{t}[g(y(\gamma(s+t_n)))-g(y(\gamma(s)))]ds+\int_{u_n}^{t}[h(s+t_n)-h(s)]ds. \end{aligned}$$

By taking the norm of both sides and using the triangle inequlity, it is seen that

$$\begin{aligned} \|y(t+t_n)-y(t)\| &\geq -\|y(u_n+t_n)-y(u_n)\| \\ &- \int_{u_n}^{t}\overline{\lambda}\|y(s+t_n)-y(s)\|ds-\int_{u_n}^{t}\|f(y(s+t_n))-f(y(s))\|ds \\ &- \int_{u_n}^{t}\|g(y(\gamma(s+t_n)))-g(y(\gamma(s)))\|ds+\int_{u_n}^{t}\|h(s+t_n)-h(s)\|ds \\ &\geq -\frac{\epsilon_0}{l}-\overline{\lambda}\kappa(\frac{1}{l}+\frac{2}{k})\epsilon_0-L_f\kappa(\frac{1}{l}+\frac{2}{k})\epsilon_0 \\ &- L_g\int_{u_n}^{t}\|y(\gamma(s+t_n))-y(\gamma(s))\|ds+\frac{\kappa}{2}\epsilon_0 \end{aligned}$$

for $t\in[u_n+\frac{\kappa}{2},u_n+\kappa]$.

Define the last integral above as

$$J=\int_{u_n}^{t}\|y(\gamma(s+t_n))-y(\gamma(s))\|ds.$$

For a fixed $t\in[u_n+\frac{\kappa}{2},u_n+\kappa]$, choose κ sufficiently small so that $\theta_{i-\eta_n}+t_n\leq u_n<u_n+\frac{\kappa}{2}\leq t\leq u_n+\kappa<\theta_{i+1}$ for some $i\in\mathbb{Z}$. Thus, we have $\gamma(t)=\xi_i$ for $t\in[u_n+\frac{\kappa}{2},u_n+\kappa]$ and $\gamma(t+t_n)=\xi_{i+\eta_n}$ due to the condition (C9). Since $y(t)\in\mathcal{P}$, it is a uniformly continuous function. Hence, for $\epsilon_0>0$ and large n, we can find a $\rho>0$ such that $\|y(\xi_{i+\eta_n})-y(\xi_i)\|\leq\|y(\xi_i+t_n)-y(\xi_i)\|+\|y(\xi_i+t_n+o(1))-y(\xi_i+t_n)\|<2\epsilon_0$ if $\|\xi_{i+\eta_n}-\xi_i-t_n\|<\rho$.

So, we have $J\leq 2\kappa\epsilon_0$. As a result, inequality (3.10) implies that

$$\begin{aligned} \|y(t+t_n)-y(t)\| &\geq -\frac{\epsilon_0}{l}-\overline{\lambda}(\frac{1}{l}+\frac{2}{k})\kappa\epsilon_0-L_f(\frac{1}{l}+\frac{2}{k})\kappa\epsilon_0-2L_g\kappa\epsilon_0+\frac{\kappa}{2}\epsilon_0 \\ &\geq -\frac{\epsilon_0}{l}+\frac{4\epsilon_0}{3l}\geq\frac{\epsilon_0}{3l}. \end{aligned}$$

Based on the inequalities obtained in cases (i) and (ii), we see that the solution $y(t)$ is unpredictable.

Lastly, let us give our attention to the stability analysis of the solution $y(t)$. Denote $v(t) = y(t) - z(t)$, where $z(t)$ is another solution of the system (2.1). Then $v(t)$ will be a solution of the system (3.7) and thus it is true that

$$||v(t)|| \leq \sigma e^{-\lambda(t-t_0)}\|v(t_0)\| + \int_{t_0}^{t} \sigma e^{-\lambda(t-s)}[L_f||v(s)|| + L_g||v(\gamma(s))||]ds. \tag{3.12}$$

Using Lemma 3.3 in (3.12), we obtain that

$$||v(t)|| \leq \sigma e^{-\lambda(t-t_0)}||v(t_0)|| + \int_{t_0}^{t} \sigma e^{-\lambda(t-s)}(L_f + BL_g)\|v(s)\|ds.$$

The last inequality leads to

$$e^{\lambda t}\|v(t)\| \leq \sigma e^{\lambda t_0}||v(t_0)|| + \sigma(L_f + BL_g)\int_{t_0}^{t} e^{\lambda s}\|v(s)\|ds.$$

If the Gronwall-Bellman Lemma [37] is applied for the last inequality, it is seen that

$$||v(t)|| \leq \sigma||v(t_0)||e^{(-\lambda+\sigma(L_f+BL_g))(t-t_0)}.$$

This inequality means that

$$||y(t) - z(t)|| \leq \sigma||y(t_0) - z(t_0)||e^{(-\lambda+\sigma(L_f+BL_g))(t-t_0)}. \tag{3.13}$$

From the condition (C7), we reach the conclusion that the unpredictable solution $y(t)$ of (2.1) is uniformly exponentially stable. The theorem is proved. □

4. Examples and numerical simulations

We give examples with numerical simulations to illustrate the theoretical results of this research. To investigate the presence of an unpredictable solution, we need to consider the following logistic map [12]

$$\lambda_{i+1} = \mu\lambda_i(1 - \lambda_i), \tag{4.14}$$

where $i \in \mathbb{Z}$. By virtue of Theorem 4.1 [12], for each $\mu \in [3 + (\frac{2}{3})^{1/2}, 4]$, the system (4.14) possesses an unpredictable solution. Let φ_i, $t \in [i, i+1)$, $i \in \mathbb{Z}$, be an unpredictable solution of (4.14) with $\mu = 3.92$.

In what follows, we will utilize the unpredictable function

$$\Theta(t) = \int_{-\infty}^{t} e^{-3(t-s)}\Omega(s)ds,\ t \in \mathbb{R},$$

with $\Omega(t) = \varphi_i$ for $t \in [i, i+1)$, $i \in \mathbb{Z}$, which was introduced in the paper [18].

Furthermore, the argument function $\gamma(t) = \xi_k$ is defined by the sequences $\theta_k = \frac{3}{4}k$, $\xi_k = \frac{\theta_k+\theta_{k+1}}{2} + \varphi_k = \frac{3(2k+1)}{8} + \varphi_k,\ k \in \mathbb{Z}$.

Consider the following quasilinear system with the generalized piecewise constant argument of mixed type

$$x'(t) = \begin{pmatrix} 0.1 & -0.6 & 0 \\ 0.1 & -0.4 & 0 \\ 0 & 0 & -0.3 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} + \begin{pmatrix} 0.01\tanh(\frac{x_1(t)}{25}) \\ 0.01\tanh(\frac{x_2(t)}{25}) \\ 0.01\tanh(\frac{x_3(t)}{25}) \end{pmatrix} + \begin{pmatrix} 0.01\tanh(\frac{x_1(\gamma(t))}{20}) \\ 0.01\tanh(\frac{x_2(\gamma(t))}{20}) \\ 0.01\tanh(\frac{x_3(\gamma(t))}{20}) \end{pmatrix} + \begin{pmatrix} -4\Theta^3(t) + 0.02 \\ 0.5\Theta(t) - 0.03 \\ 3\Theta^3(t) + 0.01 \end{pmatrix}. \tag{4.15}$$

Moreover, $h_1(t) = -4\Theta^3(t) + 0.02,\ h_2(t) = 0.5\Theta(t) - 0.03,\ h_3(t) = 3\Theta^3(t) + 0.01$ are unpredictable functions in accordance with Lemmas 1.4 and 1.5 given in [16].

We can see that the conditions (C1)-(C9) are valid for the system (4.15) with $\lambda = 0.1, \bar{\lambda} = 0.7, L_f = 0.0004, L_g = 0.0005, m_f = m_g = 0.01$, and moreover $m_h = 0.19, \sigma = 20, H = 38$. Thus, by the Theorem 3.1, system (4.15) has a unique exponentially stable unpredictable solution $x(t)$.

To imagine the behavior of the unpredictable oscillation $x(t)$, we consider the simulation of another solution $\psi(t)$, with initial values $\psi_1(0) = -1.1951, \psi_2(0) = -0.2828, \psi_3(0) = 0.1587$. Applying (3.13), one can obtain that

$$||\psi(t) - x(t)|| \leq 20e^{-0.002t}||\psi(0) - x(0)||,\ t \geq 0.$$

The last inequality demonstrates that the difference $\psi(t) - x(t)$ diminishes exponentially. Consequently, the graph of the function $\psi(t)$ approaches to the unpredictable solution $x(t)$ of the system (4.15), as time increases. Thus, instead of the curve describing the unpredictable solution, one can consider the graph of $\psi(t)$.

The coordinates and trajectory of the solution $\psi(t)$, which exponentially converges to the unpredictable solution $x(t)$, are shown in Figures 1 and 2, respectively. Moreover, in Figure 1 you can see that the solution of system (4.15) is continuous function with discontinuous derivatives, and it continuously differentiable within intervals $[\theta_k, \theta_{k+1}), k \in \mathbb{Z}$.

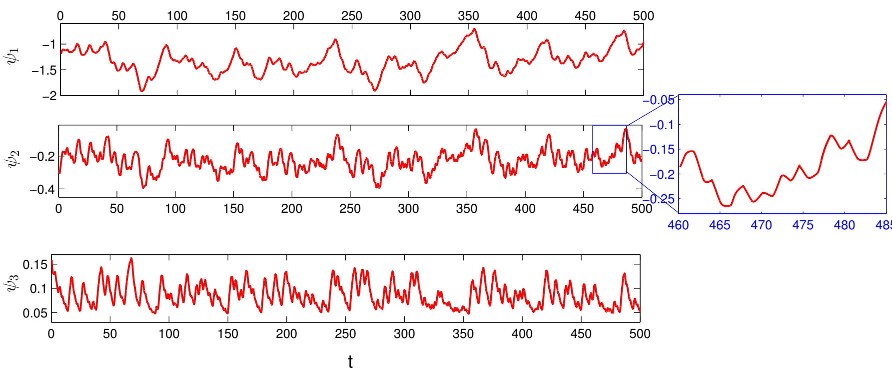

FIGURE 1. The coordinates of the function $\psi(t)$.

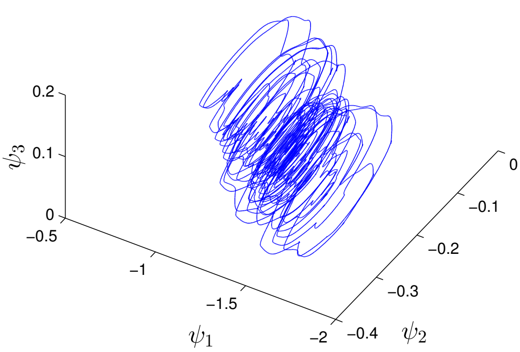

FIGURE 2. The trajectory of the function $\psi(t)$.

Acknowledgements. The authors wish to express their sincere gratitude to the referees for the helpful criticism and valuable suggestions.

M. Tleubergenova and Z. Nugayeva are supported by the Science Committee of the Ministry of Education and Science of the Republic of Kazakhstan (Grants No. AP09258737 and No. AP08856170). M. Akhmet is supported by 2247-A National Leading Researchers Program of TUBITAK, Turkey, N 120C138.

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