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In memoriam Professor Charles E. Chidume (1947- 2021)

A hybrid scheme for fixed points of a countable family of generalized nonexpansive-type maps and finite families of variational inequality and equilibrium problems, with applications

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ABSTRACT. Let C be a nonempty closed and convex subset of a uniformly smooth and uniformly convex real Banach space E with dual space E^*. We present a novel hybrid method for finding a common solution of a family of equilibrium problems, a common solution of a family of variational inequality problems and a common element of fixed points of a family of a general class of nonlinear nonexpansive maps. The sequence of this new method is proved to converge strongly to a common element of the families. Our theorem and its applications complement, generalize, and extend various results in literature.

1. INTRODUCTION

Let E be a real Banach space with topological dual E^*. Let $C \subset E$ be closed and convex with JC also closed and convex, where J is the normalized duality map (see definition 2.1). The variational inequality problem, which has its origin in the 1964 result of Stampacchia [21], has engaged the interest of researchers in the recent past (see, e.g., [26, 27] and many others). This is concerned with the following: For a monotone operator $A : C \to E$, find a point $x^* \in C$ such that

$$\langle y - x^*, Ax^* \rangle \geq 0 \; for \; all \; y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $VI(C, A)$. This problem, which plays a crucial role in nonlinear analysis, is also related to fixed point problems, zeros of nonlinear operators, complementarity problems, and convex minimization problems (see, for example, [9, 20]).

A related problem is the equilibrium problem, which has been studied by several researchers and is mostly applied in solving optimization problems (see [3]). For a map $f : C \to E$, the equilibrium problem is concerned with finding a point $x^* \in C$ such that

$$f(x^*, y) \geq 0 \; for \; all \; y \in C. \tag{1.2}$$

The set of solutions of (1.2) is denoted by $EP(f)$. The variational inequality and equilibrium problems are special cases of the so-called generalized mixed equilibrium problem (see [18]). Another related problem is the fixed point problem. For a map $T : D(T) \subset E \to E$, the fixed points of T are the points $x^* \in D(T) \; such \; that \; Tx^* = x^*$. Recently, owing to the need to develop methods for solving fixed points of problems for functions

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from a space to its dual, a new concept of *fixed points for maps from a real normed space E to its dual space E^**, called $J-$fixed point has been introduced and studied (see [5, 15, 25]). With this evolving fixed point theory, we study the $J-$fixed points of certain maps and the following equilibrium problem. Let $f : JC \times JC \to \mathbb{R}$ be a bifunction. The equilibrium problem for f is finding

$$x^* \in C \; such \; that \; f(Jx^*, Jy) \geq 0, \forall \, y \in C. \tag{1.3}$$

We denote the solution set of (1.3) by $EP(f)$. Several problems in physics, optimization and economics reduce to finding a solution of (1.3) (see, e.g., [8, 26] and the references in them). Most of the equilibrium problems studied in the past two decades centered on their existence and applications (see, e.g., [3, 8]). However, recently, several researchers have started working on finding approximate solutions of equilibrium problems and their generalizations (see, e.g., [11, 27]). Not long ago, some researchers investigated the problem of establishing a common element in the solution set of an equilibrium problem, fixed point of a family of nonexpansive maps and solution set of a variational inequality problem for different classes of maps (see [28] and references therein).

In this paper, inspired by the above results especially the works in [4, 24, 28], we present an algorithm for finding a common element of the fixed point of an infinite family of generalized J_*-nonexpansive maps, the solution set of the variational inequality problem of a finite family of continuous monotone maps and the solution set of the equilibrium point of a finite family of bifunctions satisfying some given conditions. Our results complement, generalize and extend results in [14, 19, 17, 28] (see the section on conclusion) and other recent results in this direction. It is worth noting that very recently, the authors in [4] introduced a new class of maps which they called *relatively weak $J-$nonexpasive* and developed an algorithm for approximating a common element of the $J-$fixed point of a countable family of such maps and zeros of some other class of maps in certain Banach spaces. Previously, maps with similar requirements as these *relatively weak $J-$nonexpasive* maps have also been studied in [6] where they were called *quasi$-\phi - J-$nonexpansive*. We observe that these two sets of maps (*relatively weak $J-$nonexpasive* and *quasi $-\phi - J-$nonexpansive*) coincide in definition with the J_*-nonexpansive maps in our results.

2. Preliminaries

In this section, we present definitions and lemmas used in proving our main results.

Definition 2.1. (Normalized duality map) The map $J : E \to 2^{E^*}$ defined by

$$Jx := \{x^* \in E^* : \langle x, x^* \rangle = \|x\|.\|x^*\|, \; \|x\| = \|x^*\|\}$$

is called the *normalized duality map* on E.

It is well known that if E is smooth, strictly convex and reflexive then J^{-1} exists (see e.g., [22]); $J^{-1} : E^* \to E$ is the normalized duality mapping on E^*, and $J^{-1} = J_*, \; JJ_* = I_{E^*}$ and $J_*J = I_E$, where I_E and I_{E^*} are the identity maps on E and E^*, respectively. A well known property of J is, see e.g., [7, 22], if E is uniformly smooth, then J is uniformly continuous on bounded subsets of E.

Definition 2.2. (Lyapunov Functional) [1, 11] Let E be a smooth real Banach space with dual E^*. The *Lyapunov functional* $\phi : E \times E \to \mathbb{R}$, is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \text{ for } x, y \in E, \tag{2.4}$$

where J is the normalized duality map. If $E = H$, a real Hilbert space, then equation (2.4) reduces to $\phi(x,y) = \|x-y\|^2$ for $x, y \in H$. Additionally,

$$(\|x\| - \|y\|)^2 \leq \phi(x,y) \leq (\|x\| + \|y\|)^2 \text{ for } x, y \in E. \tag{2.5}$$

Definition 2.3. (Generalized nonexpansive) [12, 13] Let C be a nonempty closed and convex subset of a real Banach space E and T be a map from C to E. The map T is called *generalized nonexpansive* if $F(T) := \{x \in C : Tx = x\} \neq \emptyset$ and $\phi(Tx, p) \leq \phi(x, p)$ for all $x \in C, p \in F(T)$.

Definition 2.4. (Retraction) [12, 13] A map R from E onto C is said to be a retraction if $R^2 = R$. The map R is said to be *sunny* if $R(Rx + t(x - Rx)) = Rx$ for all $x \in E$ and $t \leq 0$.

A nonempty closed subset C of a smooth Banach space E is said to be a *sunny generalized nonexpansive retract* of E if there exists a sunny generalized nonexpansive retraction R from E onto C.

NST-condition. Let C be a closed subset of a Banach space E. Let $\{T_n\}$ and Γ be two families of generalized nonexpansive maps of C into E such that $\cap_{n=1}^{\infty} F(T_n) = F(\Gamma) \neq \emptyset$, where $F(T_n)$ is the set of fixed points of $\{T_n\}$ and $F(\Gamma)$ is the set of common fixed points of Γ.

Definition 2.5. [12] The sequence $\{T_n\}$ satisfies the NST-condition (see e.g., [16]) with Γ if for each bounded sequence $\{x_n\} \subset C$,

$$\lim_{n\to\infty} ||x_n - T_n x_n|| = 0 \Rightarrow \lim_{n\to\infty} ||x_n - Tx_n|| = 0, \ for\ all\ T \in \Gamma.$$

Remark 2.1. If $\Gamma = \{T\}$ a singleton, $\{T_n\}$ satisfies the NST-condition with $\{T\}$. If $T_n = T$ for all $n \geq 1$, then, $\{T_n\}$ satisfies the NST-condition with $\{T\}$.

Let C be a nonempty closed and convex subset of a uniformly smooth and uniformly convex real Banach space E with dual space E^*. Let J be the normalized duality map on E and J_* be the normalized duality map on E^*. Observe that under this setting, J^{-1} exists and $J^{-1} = J_*$. With these notations, we have the following definitions.

Definition 2.6. (Closed map) [24] A map $T : C \to E^*$ is called J_*-*closed* if $(J_* \circ T) : C \to E$ is a closed map, i.e., if $\{x_n\}$ is a sequence in C such that $x_n \to x$ and $(J_* \circ T)x_n \to y$, then $(J_* \circ T)x = y$.

Definition 2.7. ($J-$fixed Point) [5] A point $x^* \in C$ is called a $J-$*fixed point of* T if $Tx^* = Jx^*$. The set of $J-$fixed points of T will be denoted by $F_J(T)$.

Definition 2.8. (Generalized J_* nonexpansive) [24] A map $T : C \to E^*$ will be called *generalized* J_*-*nonexpansive* if $F_J(T) \neq \emptyset$, and $\phi(p, (J_* \circ T)x) \leq \phi(p, x)$ for all $x \in C$ and for all $p \in F_J(T)$.

Remark 2.2. Exampes of generalized J_*-nonexpansive maps in Hilbert and more general Banach spaces were given in [4, 24].

Let C be a nonempty closed subset of a smooth, strictly convex and reflexive Banach space E such that JC is closed and convex. For solving the equilibrium problem, let us assume that a bifunction $f : JC \times JC \to \mathbb{R}$ satisfies the following conditions:

(A1) $f(x^*, x^*) = 0$ for all $x^* \in JC$;
(A2) f is monotone, i.e. $f(x^*, y^*) + f(y^*, x^*) \leq 0$ for all $x^*, y^* \in JC$;
(A3) for all $x^*, y^*, z^* \in JC$, $\limsup_{t\downarrow 0} f(tz^* + (1-t)x^*, y^*) \leq f(x^*, y^*)$;
(A4) for all $x^* \in JC$, $f(x^*, \cdot)$ is convex and lower semicontinuous.

With the above definitions, we now provide the lemmas we shall use.

Lemma 2.1. [29] Let E be a uniformly convex Banach space, $r > 0$ be a positive number, and $B_r(0)$ be a closed ball of E. For any given points $\{x_1, x_2, \cdots, x_N\} \subset B_r(0)$ and any given positive numbers $\{\lambda_1, \lambda_2, \cdots, \lambda_N\}$ with $\sum_{n=1}^{N} \lambda_n = 1$, there exists a continuous strictly increasing and convex function $g : [0, 2r) \to [0, \infty)$ with $g(0) = 0$ such that, for any $i, j \in \{1, 2, \cdots N\},\ i < j$,

$$\|\sum_{n=1}^{N} \lambda_n x_n\|^2 \leq \sum_{n=1}^{N} \lambda_n \|x_n\|^2 - \lambda_i \lambda_j g(\|x_i - x_j\|). \tag{2.6}$$

Lemma 2.2. [11] Let X be a real smooth and uniformly convex Banach space, and let $\{x_n\}$ and $\{y_n\}$ be two sequences of X. If either $\{x_n\}$ or $\{y_n\}$ is bounded and $\phi(x_n, y_n) \to 0$ as $n \to \infty$, then $\|x_n - y_n\| \to 0$ as $n \to \infty$.

Lemma 2.3. [1] Let C be a nonempty closed and convex subset of a smooth, strictly convex and reflexive Banach space E. Then, the following are equivalent.
(i) C is a sunny generalized nonexpansive retract of E,
(ii) C is a generalized nonexpansive retract of E,
(iii) JC is closed and convex.

Lemma 2.4. [1] Let C be a nonempty closed and convex subset of a smooth and strictly convex Banach space E such that there exists a sunny generalized nonexpansive retraction R from E onto C. Then, the following hold.
(i) $z = Rx$ iff $\langle x - z, Jy - Jz\rangle \leq 0$ for all $y \in C$,
(ii) $\phi(x, Rx) + \phi(Rx, z) \leq \phi(x, z)$.

Lemma 2.5. [10] Let C be a nonempty closed sunny generalized nonexpansive retract of a smooth and strictly convex Banach space E. Then the sunny generalized nonexpansive retraction from E to C is uniquely determined.

Lemma 2.6. [3] Let C be a nonempty closed subset of a smooth, strictly convex and reflexive Banach space E such that JC is closed and convex, let f be a bifunction from $JC \times JC$ to $\mathbb{R}$ satisfying $(A1) - (A4)$. For $r > 0$ and let $x \in E$. Then there exists $z \in C$ such that $f(Jz, Jy) + \frac{1}{r}\langle z - x, Jy - Jz\rangle \geq 0,\ \forall\, y \in C$.

Lemma 2.7. [23] Let C be a nonempty closed subset of a smooth, strictly convex and reflexive Banach space E such that JC is closed and convex, let f be a bifunction from $JC \times JC$ to $\mathbb{R}$ satisfying $(A1) - (A4)$. For $r > 0$ and let $x \in E$, define a mapping $T_r(x) : E \to C$ as follows:

$$T_r(x) = \{z \in C : f(Jz, Jy) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0,\ \forall\, y \in C\}.$$

Then the following hold:
- (i) T_r is single valued;
- (ii) for all $x, y \in E$, $\langle T_r x - T_r y, JT_r x - JT_r y\rangle \leq \langle x - y, JT_r x - JT_r y\rangle$;
- (iii) $F(T_r) = EP(f)$;
- (iv) $\phi(p, T_r(x)) + \phi(T_r(x), x) \leq \phi(p, x)$ for all $p \in F(T_r)$.
- (v) $JEP(f)$ is closed and convex.

Lemma 2.8. [24] Let C be a nonempty closed subset of a smooth, strictly convex and reflexive Banach space E. Let $A : C \to E^*$ be a continuous monotone mapping. For $r > 0$ and let $x \in E$, define a mapping $F_r(x) : E \to C$ as follows:

$$F_r(x) = \{z \in C : \langle y - z, Az\rangle + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0,\ \forall\, y \in C\}.$$

Then the following hold:

(i) F_r is single valued;
(ii) for all $x, y \in E$, $\langle F_r x - T_r y, JF_r x - JF_r y\rangle \leq \langle x - y, JF_r x - JF_r y\rangle$;
(iii) $F(F_r) = VI(C, A)$;
(iv) $\phi(p, F_r(x)) + \phi(F_r(x), x) \leq \phi(p, x)$ for all $p \in F(F_r)$.
(v) $JVI(C, A)$ is closed and convex.

Lemma 2.9. [24] Let E be a uniformly convex and uniformly smooth real Banach space with dual space E^* and let C be a closed subset of E such that JC is closed and convex. Let T be a generalized J_*-nonexpansive map from C to E^* such that $F_J(T) \neq \emptyset$, then $F_J(T)$ and $JF_J(T)$ are closed. Additionally, if $JF_J(T)$ is convex, then $F_J(T)$ is a sunny generalized nonexpansive retract of E.

3. MAIN RESULTS

Let E be a uniformly smooth and uniformly convex real Banach space with dual space E^* and let C be a nonempty closed and convex subset of E such that JC is closed and convex. Let $f_l, l = 1, 2, 3, ..., L$ be a family of bifunctions from $JC \times JC$ to $\mathbb{R}$ satisfying $(A1)-(A4)$, $T_n : C \to E^*, n = 1, 2, 3, ...$ be an infinite family of generalized J_*-nonexpansive maps, and $A_k : C \to E^*, k = 1, 2, 3, ..., N$ be a finite family of continuous monotone mappings. Let the sequence $\{x_n\}$ be generated by the following iteration process:

$$(3.7) \qquad \begin{cases} x_1 = x \in C; C_1 = C, \\ z_n := \{z \in C : f_n(Jz, Jy) + \frac{1}{r_n}\langle y - z, Jz - Jx_n\rangle \geq 0, \ \forall\, y \in C\}, \\ u_n := \{z \in C : \langle y - z, A_n z\rangle + \frac{1}{r_n}\langle y - z, Jz - Jx_n\rangle \geq 0, \ \forall\, y \in C\}, \\ y_n = J^{-1}(\alpha_1 Jx_n + \alpha_2 Jz_n + \alpha_3 T_n u_n), \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = R_{C_{n+1}} x, \end{cases}$$

for all $n \in \mathbb{N}$, with $\alpha_1, \alpha_2, \alpha_3 \in (0, 1)$ satisfying $\alpha_1 + \alpha_2 + \alpha_3 = 1$, $\{r_n\} \subset [a, \infty)$ for some $a > 0$, $A_n = A_{n(mod\ N)}$ and $f_n(\cdot, \cdot) = f_{n(mod\ L)}(\cdot, \cdot)$.

Lemma 3.10. *The sequence $\{x_n\}$ generated by (3.7) is well defined.*

Proof. Observe that JC_1 is closed and convex. Moreover, it is easy to see that $\phi(z, y_n) \leq \phi(z, x_n)$ is equivalent to

$$0 \leq ||x_n||^2 - ||y_n||^2 - 2\langle z, Jx_n - Jy_n\rangle,$$

which is affine in z. Hence, by induction JC_n is closed and convex for each $n \geq 1$. Therefore, from Lemma 2.3, we have that C_n is a sunny generalized retract of E for each $n \geq 1$. This shows that $\{x_n\}$ is well defined. □

Theorem 3.1. *Let E be a uniformly smooth and uniformly convex real Banach space with dual space E^* and let C be a nonempty closed and convex subset of E such that JC is closed and convex. Let $f_l, l = 1, 2, 3, ..., L$ be a family of bifunctions from $JC \times JC$ to $\mathbb{R}$ satisfying $(A1) - (A4)$, $T_n : C \to E^*, n = 1, 2, 3, ...$ be an infinite family of generalized J_*-nonexpansive maps, $A_k : C \to E^*, k = 1, 2, 3, ..., N$ be a finite family of continuous monotone mappings and Γ be a family of J_*-closed and generalized J_*-nonexpansive maps from C to E^* such that $\cap_{n=1}^{\infty} F_J(T_n) = F_J(\Gamma) \neq \emptyset$ and $B := F_J(\Gamma) \cap \Big[\cap_{l=1}^{L} EP(f_l)\Big] \cap \Big[\cap_{k=1}^{N} VI(C, A_k)\Big] \neq \emptyset$. Assume that $JF_J(\Gamma)$ is convex and $\{T_n\}$ satisfies the NST-condition with Γ. Then, $\{x_n\}$ generated by (3.7) converges strongly to $R_B x$, where R_B is the sunny generalized nonexpansive retraction of E onto B.*

Proof. The proof is given in 6 steps.
Step 1: We show that the expected limit $R_B x$ exists as a point in C_n for all $n \geq 1$.
First, we show that $B \subset C_n$ for all $n \geq 1$ and B is a sunny generalized retract of E.
Since $C_1 = C$, we have $B \subset C_1$. Suppose $B \subset C_n$ for some $n \in \mathbb{N}$. Let $u \in B$. We observe from algorithm (3.7) that $u_n = F_{r_n} x_n$ and $z_n = T_{r_n} x_n$ for all $n \in \mathbb{N}$, using this and the fact that $\{T_n\}$ is an infinite family of generalized J_*-nonexpansive maps, the definition of y_n, Lemmas 2.7, 2.8, and 2.1, we compute as follows:

$$\begin{aligned}
\phi(u, y_n) &= \phi(u, J^{-1}(\alpha_1 J x_n + \alpha_2 J z_n + \alpha_3 T_n u_n) \\
&\leq \alpha_1 \left[||u||^2 - 2\langle u, Jx_n\rangle + ||x_n||^2\right] + \alpha_2 \left[||u||^2 - 2\langle u, Jz_n\rangle + ||z_n||^2\right] \\
&\quad + \alpha_3 \left[||u||^2 - 2\langle u, J(J_* \circ T_n)u_n\rangle + ||T_n u_n||^2\right] \\
&\quad - \alpha_1 \alpha_3 g(||Jx_n - J(J_* \circ T_n)u_n||) \\
&= \alpha_1 \phi(u, x_n) + \alpha_2 \phi(u, z_n) + \alpha_3 \phi(u, (J_* \circ T_n)u_n) - \alpha_1 \alpha_3 g(||Jx_n - T_n u_n||) \\
&\leq \alpha_1 \phi(u, x_n) + \alpha_2 \phi(u, z_n) + \alpha_3 \phi(u, u_n) - \alpha_1 \alpha_3 g(||Jx_n - T_n u_n||) \qquad (3.8) \\
&= \alpha_1 \phi(u, x_n) + \alpha_2 \phi(u, T_{r_n} x_n) + \alpha_3 \phi(u, u_n) - \alpha_1 \alpha_3 g(||Jx_n - T_n u_n||) \\
&\leq \alpha_1 \phi(u, x_n) + \alpha_2 \phi(u, x_n) + \alpha_3 \phi(u, u_n) - \alpha_1 \alpha_3 g(||Jx_n - T_n u_n||),
\end{aligned}$$

which yields

$$\phi(u, y_n) \leq \phi(u, x_n) - \alpha_1 \alpha_3 g(||Jx_n - T_n u_n||). \qquad (3.9)$$

Hence, $\phi(u, y_n) \leq \phi(u, x_n)$ and we have that $u \in C_{n+1}$, which implies that $B \subset C_n$ for all $n \geq 1$. Moreover, From Lemma 2.7 and 2.8 both $JVI(C, A_k)$ and $JEP(f_l)$ are closed and convex for each l and for each k. Also, using our assumption and lemma 2.9, we have that $J(F_J(\Gamma)$ is closed and convex. Since E is uniformly convex, J is one-to-one. Thus, we have that,
$J\Big(F_J(\Gamma) \cap \Big[\cap_{l=1}^{L} EP(f_l)\Big] \cap \Big[\cap_{k=1}^{N} VI(C, A_k)\Big]\Big) = JF_J(\Gamma) \cap J\Big[\cap_{l=1}^{L} EP(f_l)\Big] \cap J\Big[\cap_{k=1}^{N} VI(C, A_k)\Big]$
so $J(B)$ is closed and convex. Using Lemma 2.3, we obtain that B is a sunny generalized retract of E. Therefore, from Lemma 2.5 , we have that $R_B x$ exists as a point in C_n for all $n \geq 1$. This completes step 1.
Step 2: We show that the sequence $\{x_n\}$ defined by (3.7) converges to some $x^* \in C$.
Using the fact that $x_n = R_{C_n} x$ and Lemma 2.4(ii), we obtain

$$\phi(x, x_n) = \phi(x, R_{C_n} x) \leq \phi(x, u) - \phi(R_{C_n} x, u) \leq \phi(x, u),$$

for all $u \in F_J(\Gamma) \cap EP(f_l) \cap VI(C, A_k) \subset C_n; (l = 1, 2, \ldots, L;\ k = 1, 2, \ldots, K)$. This implies that $\{\phi(x, x_n)\}$ is bounded. Hence, from equation (2.5), $\{x_n\}$ is bounded. Also, since $x_{n+1} = R_{C_{n+1}} x \in C_{n+1} \subset C_n$, and $x_n = R_{C_n} x \in C_n$, applying Lemma 2.4(ii) gives

$$\phi(x, x_n) \leq \phi(x, x_{n+1})\ \forall\ n \in \mathbb{N}.$$

So, $\lim_{n\to\infty} \phi(x, x_n)$ exists. Again, using Lemma 2.4(ii) and $x_n = R_{C_n} x$, we obtain that for all $m, n \in \mathbb{N}$ with $m > n$,

$$\begin{aligned}
\phi(x_n, x_m) &= \phi(R_{C_n} x, x_m) \leq \phi(x, x_m) - \phi(x, R_{C_n} x) \\
&= \phi(x, x_m) - \phi(x, x_n) \to 0\ as\ n \to \infty. \qquad (3.10)
\end{aligned}$$

From Lemma 2.2, we conclude that $||x_n - x_m|| \to 0,\ as\ m,\ n \to \infty$. Hence, $\{x_n\}$ is a Cauchy sequence in C, and so, there exists $x^* \in C$ such that $x_n \to x^*$ completing step 2.
Step 3: We prove $x^* \in \cap_{k=1}^{N} VI(C, A_k)$.
From the definitions of C_{n+1} and x_{n+1}, we obtain that $\phi(x_{n+1}, y_n) \leq \phi(x_{n+1}, x_n) \to 0$ as $n \to \infty$. Hence, by Lemma 2.2 , we have that

$$\lim_{n\to\infty} ||x_n - y_n|| = 0. \qquad (3.11)$$

Since from step 2 $x_n \to x^*\ as\ n \to \infty$, equation (3.11) implies that $y_n \to x^*\ as\ n \to \infty$. Using the fact that $u_n = F_{r_n}x_n$ for all $n \in \mathbb{N}$ and Lemma 2.2, we get for $u \in B$,

$$\begin{aligned} \phi(u_n, x_n) &= \phi(F_{r_n}x_n, x_n) \\ &\leq \phi(u, x_n) - \phi(u, F_{r_n}x_n) \\ &= \phi(u, x_n) - \phi(u, u_n). \end{aligned} \tag{3.12}$$

From equations (3.8) and (3.9) we have

$$\phi(u, y_n) \leq \alpha_1\phi(u, x_n) + \alpha_2\phi(u, x_n) + \alpha_3\phi(u, u_n) \leq \phi(u, x_n). \tag{3.13}$$

Since $x_n, y_n \to x^*$ as $n \to \infty$, equation (3.13) implies that $\phi(u, u_n) \to \phi(u, x^*)$ as $n \to \infty$. Therefore, from (3.12), we have $\phi(u, x_n) - \phi(u, u_n) \to 0$ as $n \to \infty$ which implies that $\lim_{n\to\infty} \phi(u_n, x_n) = 0$. Hence, from Lemma 2.2, we have

$$\lim_{n\to\infty} ||u_n - x_n|| = 0. \tag{3.14}$$

Observe that since J is uniformly continuous on bounded subsets of E, it follows from (3.14) that $||Ju_n - Jx_n|| \to 0$.
Again, since $r_n \in [a, \infty)$, we have that

$$\lim_{n\to\infty} \frac{||Ju_n - Jx_n||}{r_n} = 0. \tag{3.15}$$

From $u_n = F_{r_n}x_n$, we have

$$\langle y - u_n, A_n u_n\rangle + \frac{1}{r_n}\langle y - u_n, Ju_n - Jx_n\rangle \geq 0,\ \forall\, y \in C. \tag{3.16}$$

Let $\{n_l\}_{l=1}^{\infty} \subset \mathbb{N}$ be such that $A_{n_l} = A_1\ \forall\, l \geq 1$. Then, from (3.16), we obtain

$$\langle y - u_{n_l}, A_1 u_{n_l}\rangle + \frac{1}{r_{n_l}}\langle y - u_{n_l}, Ju_{n_l} - Jx_{n_l}\rangle \geq 0,\ \forall\, y \in C. \tag{3.17}$$

If we set $v_t = ty + (1-t)x^*$ for all $t \in (0, 1]$ and $y \in C$, then we get that $v_t \in C$. Hence, it follows from (3.17) that

$$\langle v_t - u_{n_l}, A_1 u_{n_l}\rangle + \langle y - u_{n_l}, \frac{Ju_{n_l} - Jx_{n_l}}{r_{n_l}}\rangle \geq 0. \tag{3.18}$$

This implies that

$$\begin{aligned} \langle v_t - u_{n_l}, A_1 v_t\rangle &\geq \langle v_t - u_{n_l}, A_1 v_t\rangle - \langle v_t - u_{n_l}, A_1 u_{n_l}\rangle - \langle y - u_{n_l}, \frac{Ju_{n_l} - Jx_{n_l}}{r_{n_l}}\rangle \\ &= \langle v_t - u_{n_l}, A_1 v_t - A_1 u_{n_l}\rangle - \langle y - u_{n_l}, \frac{Ju_{n_l} - Jx_{n_l}}{r_{n_l}}\rangle. \end{aligned}$$

Since A_1 is monotone, $\langle v_t - u_{n_l}, A_1 v_t - Au_{n_l}\rangle \geq 0$. Thus, using (3.15), we have that

$$0 \leq \lim_{l\to\infty}\langle v_t - u_{n_l}, A_1 v_t\rangle = \langle v_t - x^*, A_1 v_t\rangle,$$

therefore,

$$\langle y - x^*, A_1 v_t\rangle \geq 0,\ \forall\, y \in C.$$

Letting $t \to 0$ and using continuity of A_1, we have that

$$\langle y - x^*, A_1 x^*\rangle \geq 0,\ \forall\, y \in C.$$

This implies that $x^* \in VI(C, A_1)$. Similarly, if $\{n_i\}_{i=1}^{\infty} \subset \mathbb{N}$ is such that $A_{n_i} = A_2$ for all $i \geq 1$, then we have again that $x^* \in VI(C, A_2)$. If we continue in similar manner, we obtain that $x^* \in \cap_{k=1}^{N} VI(C, A_k)$.

Step 4: We prove that $x^* \in F_J(\Gamma)$.
First, we show that $\lim_{n\to\infty} ||Jx_n - Tu_n|| = 0 \, \forall \, T \in \Gamma$.
From inequality (3.9) and the fact that g is nonnegative, we obtain

$$0 \leq \alpha_1\alpha_3 g(||Jx_n - T_nu_n||) \leq \phi(u, x_n) - \phi(u, y_n) \leq 2||u||.||Jx_n - Jy_n|| + ||x_n - y_n||M,$$

for some $M > 0$. Thus, using (3.11) and properties of g, we obtain that $\lim_{n\to\infty} ||Jx_n - T_nu_n|| = 0$. Using the above and triangle inequality gives $\|Ju_n - T_nu_n| \to 0 \; as \; n \to \infty$. Since $\{T_n\}_{n=1}^{\infty}$ satisfies the NST condition with Γ, we have that

$$\lim_{n\to\infty} ||Ju_n - Tu_n|| = 0 \, \forall \, T \in \Gamma. \tag{3.19}$$

Now, from equation (3.14), we have $u_n \to x^* \in C$. Assume that $(J_* \circ T)u_n \to y^*$. Since T is J_*-closed, we have $y^* = (J_* \circ T)x^*$. Furthermore, by the uniform continuity of J on bounded subsets of E, we have: $Ju_n \to Jx^*$ and $J(J_* \circ T)u_n \to Jy^*$ as $n \to \infty$. Hence, we have

$$\lim_{n\to\infty} ||Ju_n - J(J_* \circ T)u_n|| = \lim_{n\to\infty} ||Ju_n - Tu_n|| = 0, \; \forall \, T \in \Gamma,$$

which implies $||Jx^* - Jy^*|| = ||Jx^* - J(J_* \circ T)x^*|| = ||Jx^* - Tx^*|| = 0$. So, $x^* \in F_J(\Gamma)$.
Step 5: We prove that $x^* \in \cap_{l=1}^{L} EP(f_l)$.
This follows by similar argument as in step 3 but for the sake of completeness we provide the details. Using the fact that $z_n = T_{r_n}x_n$ and Lemma 2.7, we obtain that for $u \in F_J(\Gamma) \cap EP(f_l) \cap VI(C, A_k)$ for all i, k,

$$\begin{aligned} \phi(z_n, x_n) &= \phi(T_{r_n}x_n, x_n) \\ &\leq \phi(u, x_n) - \phi(u, T_{r_n}x_n) \\ &= \phi(u, x_n) - \phi(u, z_n). \end{aligned} \tag{3.20}$$

From equations (3.8) and (3.9), we have

$$\phi(u, y_n) \leq \alpha_1\phi(u, x_n) + \alpha_2\phi(u, z_n) + \alpha_3\phi(u, x_n) \leq \phi(u, x_n). \tag{3.21}$$

Since $x_n, \; y_n, \; u_n \to x^*$ as $n \to \infty$, from equation (3.21) we have $\phi(u, z_n) \to \phi(u, x^*)$ as $n \to \infty$. Therefore, from (3.20), we have $\phi(u, x_n) - \phi(u, u_n) \to 0$ as $n \to \infty$. Hence $\lim_{n\to\infty} \phi(z_n, x_n) = 0$. From Lemma 2.2, we have

$$\lim_{n\to\infty} ||z_n - x_n|| = 0, \tag{3.22}$$

which implies that $z_n \to x^* \; as \; n \to \infty$. Again, since J is uniformly continuous on bounded subsets of E, (3.22) implies $\|Jz_n - Jx_n\| \to 0$. Since $r_n \in [a, \infty)$, we have that

$$\lim_{n\to\infty} \frac{||Jz_n - Jx_n||}{r_n} = 0. \tag{3.23}$$

Since $z_n = T_{r_n}x_n$, we have that

$$\frac{1}{r_n}\langle y - z_n, Jz_n - Jx_n\rangle \geq -f_n(Jz_n, Jy), \; \forall \, y \in C.$$

Let $\{n_l\}_{l=1}^{\infty} \subset \mathbb{N}$ be such that $f_{n_l} = f_1 \, \forall \, l \geq 1$. Then, using (A2), we have

$$\langle y - z_n, \frac{Jz_n - Jx_n}{r_n}\rangle \geq -f_1(Jz_n, Jy) \geq f_1(Jy, Jz_n), \; \forall \, y \in C. \tag{3.24}$$

Since $f_1(x, \cdot)$ is convex and lower-semicontinuous and $z_n \to x^*$, it follows from equation (3.23) and inequality (3.24) that

$$f_1(Jy, Jx^*) \leq 0, \; \forall \, y \in C.$$

For $t \in (0,1]$ and $y \in C$, let $y_t^* = tJy + (1-t)Jx^*$. Since JC is convex, we have that $y_t^* \in JC$ and hence $f_1(y_t^*, Jx^*) \leq 0$. Applying (A1) gives,

$$0 = f_1(y_t^*, y_t^*) \leq t f_1(y_t^*, Jy) + (1-t) f_1(y_t^*, Jx^*) \leq t f_1(y_t^*, Jy), \ \forall\, y \in C.$$

This implies that

$$f_1(y_t^*, Jy) \geq 0, \ \forall\, y \in C.$$

Letting $t \downarrow 0$ and using (A3), we get

$$f_1(Jx^*, Jy) \geq 0, \ \forall\, y \in C.$$

Therefore, we have that $Jx^* \in JEP(f_1)$. This implies that $x^* \in EP(f_1)$. Applying similar argument, we can show that $x^* \in EP(f_l)$ for $l = 2, 3, \ldots, L$. Hence, $x^* \in \cap_{l=1}^{L} EP(f_l)$.
Step 6: Finally, we show that $x^* = R_B x$.
From Lemma 2.4(ii), we obtain that

$$\phi(x, R_B x) \leq \phi(x, x^*) - \phi(R_B x, x^*) \leq \phi(x, x^*). \tag{3.25}$$

Again, using Lemma 2.4(ii), definition of x_{n+1}, and $x^* \in B \subset C_n$, we compute as follows:

$$\begin{aligned} \phi(x, x_{n+1}) &\leq \phi(x, x_{n+1}) + \phi(x_{n+1}, R_B x) \\ &= \phi(x, R_{C_{n+1}} x) + \phi(R_{C_{n+1}} x, R_B x) \leq \phi(x, R_B x). \end{aligned}$$

Since $x_n \to x^*$, taking limits on both sides of the last inequality, we obtain

$$\phi(x, x^*) \leq \phi(x, R_B x). \tag{3.26}$$

Using inequalities (3.25) and (3.26), we obtain that $\phi(x, x^*) = \phi(x, R_B x)$. By the uniqueness of R_B(Lemma 2.5), we obtain that $x^* = R_B x$. This completes proof of the theorem. □

4. Applications

Corollary 4.1. *Let E be a uniformly smooth and uniformly convex real Banach space with dual space E^* and let C be a nonempty closed and convex subset of E such that JC is closed and convex. Let f be a bifunction from $JC \times JC$ to $\mathbb{R}$ satisfying $(A1) - (A4)$, $A : C \to E^*$, be a continuous monotone mapping, $T : C \to E^*$, be a generalized J_*-nonexpansive and J_*-closed map such that $B := F_J(T) \cap EP(f) \cap VI(C, A) \neq \emptyset$. Assume that $JF_J(T)$ is convex. Then, $\{x_n\}$ generated by (3.7) converges strongly to $R_B x$, where R_B is the sunny generalized nonexpansive retraction of E onto B.*

Proof. Set $T_n := T$ for all $n \in \mathbb{N}$, $A := A_i$ for any $i = 1, 2, \cdots, N$, and $f := f_l$ for any $l = 1, 2, \cdots, L$. Then, from remark 2.1, $\{T_n\}$ satisfies the NST-condition with $\{T\}$. The conclusion follows from Theorem 3.1. □

Corollary 4.2. *Let E be a uniformly smooth and uniformly convex real Banach space with dual space E^* and let C be a nonempty closed and convex subset of E such that JC is closed and convex. Let $f_l, l = 1, 2, 3, ..., L$ be a family of bifunctions from $JC \times JC$ to $\mathbb{R}$ satisfying $(A1) - (A4)$, $T_n : C \to E^*, n = 1, 2, 3, ...$ be an infinite family of generalized J_*-nonexpansive maps and Γ be a family of J_*-closed and generalized J_*-nonexpansive maps from C to E^* such that $\cap_{n=1}^{\infty} F_J(T_n) = F_J(\Gamma) \neq \emptyset$ and $B := F_J(\Gamma) \cap \Big[\cap_{l=1}^{L} EP(f_l) \Big] \neq \emptyset$. Assume that $JF_J(\Gamma)$ is convex and $\{T_n\}$ satisfies the NST-condition with Γ. Then, $\{x_n\}$ generated by (3.7) converges strongly to $R_B x$, where R_B is the sunny generalized nonexpansive retraction of E onto B.*

Proof. Setting $A_k = 0$ for any $k = 1, 2, 3, ..., N$, then result follows from Theorem 3.1. □

Remark 4.3. *We note here that the theorem and corollaries presented above are applicable in classical Banach spaces, such as L_p, l_p, or $W_p^m(\Omega)$, $1 < p < \infty$, where $W_p^m(\Omega)$ denotes the usual Sobolev space.*

Remark 4.4. *([2]; p. 36) The analytical representations of duality maps are known in a number of Banach spaces, for example, in the spaces L_p, l_p, and $W_m^p(\Omega)$, $p \in (1,\infty)$, $p^{-1} + q^{-1} = 1$.*

Corollary 4.3. *Let $E = H$, a real Hilbert space and let C be a nonempty closed and convex subset of H. Let $f_l, l = 1,2,3,...,L$ be a family of bifunctions from $C \times C$ to $\mathbb{R}$ satisfying $(A1)-(A4)$, $T_n : C \to H, n = 1,2,3,...$ be an infinite family of nonexpansive maps, $A_k : C \to H, k = 1,2,3,...,N$ be a finite family of continuous monotone mappings and Γ be a family of closed and generalized nonexpansive maps from C to H such that $\cap_{n=1}^{\infty} F(T_n) = F(\Gamma) \neq \emptyset$ and $B := F(\Gamma) \cap \Big[\cap_{l=1}^{L} EP(f_l)\Big] \cap \Big[\cap_{k=1}^{N} VI(C, A_k)\Big] \neq \emptyset$. Assume that $\{T_n\}$ satisfies the NST-condition with Γ. Let $\{x_n\}$ be generated by:*

$$
(4.27)\qquad \begin{cases}
x_1 = x \in C; C_1 = C,\\
z_n := \{z \in C : f_n(z,y) + \frac{1}{r_n}\langle y - z, z - x_n\rangle \geq 0, \ \forall\, y \in C\},\\
u_n := \{z \in C : \langle y - z, A_n z\rangle + \frac{1}{r_n}\langle y - z, z - x_n\rangle \geq 0, \ \forall\, y \in C\},\\
y_n = \alpha_1 J x_n + \alpha_2 z_n + \alpha_3 T_n u_n,\\
C_{n+1} = \{z \in C_n : ||z - y_n|| \leq ||z - x_n||\},\\
x_{n+1} = P_{C_{n+1}} x,
\end{cases}
$$

for all $n \in \mathbb{N}$, $\alpha_1, \alpha_2, \alpha_3 \in (0,1)$ such that $\alpha_1 + \alpha_2 + \alpha_3 = 1$, $\{r_n\} \subset [a, \infty)$ for some $a > 0$, $A_n = A_{n(mod\ N)}$ and $f_n(\cdot,\cdot) = f_{n(mod\ L)}(\cdot,\cdot)$. Then, $\{x_n\}$ converges strongly to $P_B x$, where P_B is the metric projection of H onto B.

Proof. In a Hilbert space, J is the identity operator and $\phi(x,y) = ||x-y||^2$ for all $x, y \in H$. The result follows from Theorem 3.1. □

Example 4.1. *Let $E = l_p$, $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, and $C = \overline{B_{l_p}}(0,1) = \{x \in l_p : ||x||_{l_p} \leq 1\}$. Then $JC = \overline{B_{l_q}}(0,1)$. Let $f : JC \times JC \longrightarrow \mathbb{R}$ defined by $f(x^*, y^*) = \langle J^{-1}x^*, x^* - y^*\rangle\ \forall\, x^* \in JC$, $A : C \longrightarrow l_q$ defined by $Tx = J(x_1, x_2, x_3, \cdots)\ \forall\, x = (x_1, x_2, x_3, \cdots) \in C$, $T : C \longrightarrow l_q$ defined by $Tx = J(0, x_1, x_2, x_3, \cdots)\ \forall\, x = (x_1, x_2, x_3, \cdots) \in C$, and $T_n : C \longrightarrow l_q$ defined by $T_n x = \alpha_n Jx + (1-\alpha_n)Tx$, $\forall n \geq 1$, $\forall\, x \in C, \alpha_n \in (0,1)$ such that $1 - \alpha_n \geq \frac{1}{2}$. Then C, JC, f, A, T, and T_n satisfy the conditions of Theorem 3.1. Moreover, $0 \in F_J(\Gamma) \cap EP(f) \cap VI(C, A)$.*

5. Conclusion

Our theorem and its applications complement, generalize, and extend results of Uba et al. [24], Zegeye and Shahzad [28], Kumam [14], Qin and Su [19], and Nakajo and Takahashi [17]. Theorem 3.1 is a complementary analogue and extension of Theorem 3.2 of [28] in the following sense: while Theorem 3.2 of [28] is proved for a finite family of *self-maps* in uniformly smooth and strictly convex real Banach space which has the Kadec–Klee property, Theorem 3.1 is proved for countable family of *non-self maps* in uniformly smooth and uniformly convex real Banach space; in Hilbert spaces, Corollary 4.3 is an extension of Corollary 3.5 of [28] from *finite family of nonexpansive self-maps* to *countable family of nonexpansive non-self maps*. Additionally, Theorem 3.1 extends and generalizes Theorem 3.7 of [24] in the following sense: while Theorem 3.7 of [24] studied equilibrium problem and countable family of *generalized J_*−nonexpansive non-self maps*, Theorem 3.1 studied finite family of equilibrium and variational inequality problems and countable family of *generalizes J_*−nonexpansive non-self maps*; corollary 4.2 generalized Theorem 3.7 of [24] to a finite family of equilibrium problems and countable family of *generalized J_*-nonexpansive non-self maps*. Furthermore, Corollary 4.1 extends Theorem 3.1 of [14] from Hilbert spaces to a more general uniformly smooth and uniformly convex Banach spaces and to a more general class of continuous monotone mappings. Finally, Corollary 4.1 improves and extends the results in [19, 17] from *a nonexpansive self-map* to *a generalized J_*-nonexpansive non-self map*.

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