CARPATHIAN J. MATH.
Volume **39** (2023), No. 1,
Pages 293 - 302

Online version at https://semnul.com/carpathian/
Print Edition: ISSN 1584 - 2851; Online Edition: ISSN 1843 - 4401
DOI: https://doi.org/10.37193/CJM.2023.01.20

In memoriam Professor Charles E. Chidume (1947- 2021)

The affine Orlicz log-Minkowki inequality

CHANG-JIAN ZHAO

ABSTRACT. In this paper, we establish an affine Orlicz log-Minkowki inequality for the affine quermassintegrals by introducing new concepts of affine measures and Orlicz mixed affine measures, and using the newly established Orlicz affine Minkowski inequality for the affine quermassintegrals. The affine Orlicz log-Minkowski inequality in special case yields L_p-affine log-Minkowski inequality. The affine log-Minkowski inequality is also derived .

1. INTRODUCTION

In 2016, Stancu [16] established the following logarithmic Minkowski inequality.

The log-Minkowski inequality *If K and L are convex bodies in $\mathbb{R}^n$ containing the origin in their interior, then*

$$\int_{S^{n-1}} \ln\left(\frac{h(K,u)}{h(L,u)}\right) d\bar{v}_1 \geq \frac{1}{n}\ln\left(\frac{V(K)}{V(L)}\right), \tag{1.1}$$

with equality if and only if K and L are homothetic, where dv_1 is the mixed volume measure $dv_1 = \frac{1}{n}h(K,u)dS(L,u)$, and $d\bar{v}_1 = \frac{1}{V_1(L,K)}dv_1$ is its normalization, and $V_1(L,K)$ denotes the usual mixed volume of L and K, is defined by (*see* [2])

$$V_1(L,K) = \frac{1}{n}\int_{S^{n-1}} h(K,u)dS(L,u),$$

where $S(L,u)$ is the affine surface area measure of convex body L, and the support function $h(K,x)$ of K is defined by

$$h(K,x) = \max\{x\cdot y : y \in K\},$$

for $x \in \mathbb{R}^n$.

Lutwak defined the affine quermassintegrals for a convex body K, $\Phi_0(K)$, $\Phi_1(K)$, ..., $\Phi_n(K)$, by taking $\Phi_0(K) := V(K)$, $\Phi_n(K) := \omega_n$ and for $0 < j < n$ (also see [11]),

$$\Phi_{n-j}(K) := \omega_n \left[\int_{G_{n,j}} \left(\frac{\mathrm{vol}_j(K|\xi)}{\omega_j}\right)^{-n} d\mu_j(\xi)\right]^{-1/n}, \tag{1.2}$$

where $G_{n,j}$ denotes the Grassman manifold of j-dimensional subspaces in $\mathbb{R}^n$, and μ_j denotes the probability Haar measure on $G_{n,j}$, and $\mathrm{vol}_j(K|\xi)$ denotes the j-dimensional volume of the orthogonal projection of K on j-dimensional subspace $\xi \subset \mathbb{R}^n$ and ω_j denotes the volume of j-dimensional unit ball.

Recently, the logarithmic Minkowski inequality and its dual form have attracted extensive attention and research, and the recent research can be found in the references [1], [3],

Received: 21.10.2021. In revised form: 04.01.2022. Accepted: 29.05.2022
2010 *Mathematics Subject Classification.* 46E30, 52C07.
Key words and phrases. *affine quermassintegral, Orlicz affine quermassintegral, log-Minkowski inequality, Orlicz affine Minkowski inequality.*

[4], [6], [7], [8], [12], [13], [14], [17], [18], [19], [20], [21], [22] and [23]. In the paper, we generalize the usual volumes and log-Minkowski inequality (1.1) to the affine quermassintegrals and Orlicz space, respectively. The following affine Orlicz log-Minkowski inequality is established by introducing the concepts of affine and Orlicz affine measures and using the newly established Orlicz Minkowski inequality for affine quermassintegrals. The new affine Orlicz log-Minkowski inequality which in special case yields the following L_p-affine log-Minkowski inequality. The affine log-Minkowski inequality is also derived.

Theorem 1.1. *(The affine Orlicz log-Minkowski inequality) If K and L are convex bodies in $\mathbb{R}^n$ containing the origin in their interior, $0 < j \leq n$ and $\varphi : [0,\infty) \to (0,\infty)$ is a convex and increasing function such that $\varphi(0) = 0$ and $\varphi(1) = 1$, then*

$$\int_{G_{n,j}} \ln\left(\frac{V_{\varphi}^{(j)}(L|\xi, K|\xi)}{\mathrm{vol}_j(L|\xi)}\right) d\varphi^{-n}_{\varphi,n-j}(L,K) \geq \ln\left(\varphi\left(\left(\frac{\Phi_{n-j}(K)}{\Phi_{n-j}(L)}\right)^{1/j}\right)\right). \tag{1.3}$$

If φ is strictly convex, equality holds if and only if L and K are homothetic. Here $V_{\varphi}^{(j)}(L|\xi, K|\xi)$ denotes the Orlicz mixed volume of j-dimensional $L|\xi$ and $K|\xi$ in j-dimensional subspace ξ, $d\Phi_{\varphi,n-j}(L,K)$ denotes a new Orlicz affine probability measure of L and K, is defined by (see Section 3)

$$d\Phi^{-n}_{\varphi,n-j}(L,K) = \frac{1}{\Phi^{-n}_{\varphi,n-j}(L,K)} d\varphi^{-n}_{\varphi,n-j}(L,K), \tag{1.4}$$

where $d\varphi_{\varphi,n-j}(L,K)$ denotes the Orlicz affine measure, is defined by

$$d\varphi^{-n}_{\varphi,n-j}(L,K) = \frac{V_{\varphi}^{(j)}(L|\xi, K|\xi)}{\mathrm{vol}_j(L|\xi)}\left(\frac{\omega_n \mathrm{vol}_j(L|\xi)}{\omega_j}\right)^{-n} d\mu_j(\xi), \tag{1.5}$$

and $\Phi_{\varphi,n-j}(L,K)$ is the Orlicz affine quermissintegral of L and K, is defined by (see [24])

$$\Phi_{\varphi,n-j}(L,K) = \omega_n\left[\int_{G_{n,j}} \frac{V_{\varphi}^{(j)}(L|\xi, K|\xi)}{\mathrm{vol}_j(L|\xi)}\left(\frac{\mathrm{vol}_j(L|\xi)}{\omega_j}\right)^{-n} d\mu_j(\xi)\right]^{-1/n}. \tag{1.6}$$

A special case of (1.3) is the following L_p-affine log-Minkowski inequality for the affine quermassintegrals.

The L_p-affine log-Minkowski inequality *If K and L are convex bodies in $\mathbb{R}^n$ containing the origin in their interior, $0 < j \leq n$ and $p \geq 1$, then*

$$\int_{G_{n,j}} \ln\left(\frac{V_p^{(j)}(L|\xi, K|\xi)}{\mathrm{vol}_j(L|\xi)}\right) d\Phi^{-n}_{p,n-j}(L,K) \geq \frac{p}{j}\ln\left(\frac{\Phi_{n-j}(K)}{\Phi_{n-j}(L)}\right), \tag{1.7}$$

with equality if and only if L and K are homothetic. Here $V_p^{(j)}(L|\xi, K|\xi)$ denotes the L_p-dual mixed volume of j-dimensional convex bodies $L|\xi$ and $K|\xi$ in j-dimensional subspace ξ, $d\Phi_{p,n-j}(L,K)$ denotes a new L_p-affine probability measure of convex bodies L and K, is defined by

$$d\Phi^{-n}_{p,n-j}(L,K) = \frac{1}{\Phi^{-n}_{p,n-j}(L,K)} d\varphi^{-n}_{p,n-j}(L,K), \tag{1.8}$$

where $d\varphi_{p,n-j}(L,K)$ denotes the L_p-affine measure, is defined by

$$d\varphi^{-n}_{p,n-j}(L,K) = \frac{V_p^{(j)}(L|\xi, K|\xi)}{\mathrm{vol}_j(L|\xi)}\left(\frac{\omega_n \mathrm{vol}_j(L|\xi)}{\omega_j}\right)^{-n} d\mu_j(\xi), \tag{1.9}$$

and $\Phi_{p,n-j}(L,K)$ *is the* L_p*-affine quermissintegral of* L *and* K*, is defined by*

$$\Phi_{p,n-j}(L,K)=\omega_n\left[\int_{G_{n,j}}\frac{V_p^{(j)}(L|\xi,K|\xi)}{\mathrm{vol}_j(L|\xi)}\left(\frac{\mathrm{vol}_j(L|\xi)}{\omega_j}\right)^{-n}d\mu_j(\xi)\right]^{-1/n}, \tag{1.10}$$

which is as in (1.6).

The L_p log-Minkowski inequality has been established in [22]. Obviously, the following affine log-Minkowski inequality for the affine quermassintegrals can be derived from (1.3).

The affine log-Minkowski inequality *If* K *and* L *are convex bodies in* $\mathbb{R}^n$ *containing the origin in their interior and* $0<j\leq n$*, then*

$$\int_{G_{n,j}}\ln\left(\frac{V_1^{(j)}(L|\xi,K|\xi)}{\mathrm{vol}_j(L|\xi)}\right)d\Phi_{n-j}^{-n}(L,K)\geq\frac{1}{j}\ln\left(\frac{\Phi_{n-j}(K)}{\Phi_{n-j}(L)}\right), \tag{1.11}$$

with equality if and only if L *and* K *are homothetic. Here* $V_1^{(j)}(L|\xi,K|\xi)$ *denotes the mixed volume of* j*-dimensional convex bodies* $L|\xi$ *and* $K|\xi$ *in* j*-dimensional subspace* ξ*,* $d\Phi_{n-j}(L,K)$ *denotes a new affine probability measure of convex bodies* L *and* K*, is defined by*

$$d\Phi_{n-j}^{-n}(L,K)=\frac{1}{\Phi_{n-j}^{-n}(L,K)}d\varphi_{n-j}^{-n}(L,K), \tag{1.12}$$

where $d\varphi_{n-j}(L,K)$ *denotes the affine measure, is defined by*

$$d\varphi_{n-j}^{-n}(L,K)=\frac{V_1^{(j)}(L|\xi,K|\xi)}{\mathrm{vol}_j(L|\xi)}\left(\frac{\omega_n\mathrm{vol}_j(L|\xi)}{\omega_j}\right)^{-n}d\mu_j(\xi), \tag{1.13}$$

and $\Phi_{n-j}(L,K)$ *is the mixed affine quermissintegral of* L *and* K*, is defined by*

$$\Phi_{n-j}(L,K)=\omega_n\left[\int_{G_{n,j}}\frac{V_1^{(j)}(L|\xi,K|\xi)}{\mathrm{vol}_j(L|\xi)}\left(\frac{\mathrm{vol}_j(L|\xi)}{\omega_j}\right)^{-n}d\mu_j(\xi)\right]^{-1/n}. \tag{1.14}$$

Obviously, when $L=K$, $\Phi_{n-j}(L,K)$ becomes the well-known affine quermissintegral $\Phi_{n-j}(K)$.

It's worth mentioning that here we are establishing the affine Orlicz log-Minkowki inequality. In fact, the Orlicz log-Minkowki inequality, which ia a special case of (1.3), has been established in [22] as follows:

The Orlicz log-Minkowski inequality *If* K *and* L *are convex bodies in* $\mathbb{R}^n$ *that containing the origin in their interior, and* $\varphi:[0,\infty)\to(0,\infty)$ *is a convex and increasing function such that* $\varphi(0)=0$ *and* $\varphi(1)=1$*, then*

$$\int_{S^{n-1}}\ln\left(\varphi\left(\frac{h(K,u)}{h(L,u)}\right)\right)d\overline{v}_\varphi\geq\ln\left(\varphi\left(\left(\frac{V(K)}{V(L)}\right)^{1/n}\right)\right).$$

If φ *is strictly convex, equality holds if and only if* K *are* L *homothetic, where* dv_φ *is the Orlicz mixed volume measure* $dv_\varphi=\frac{1}{n}\varphi\left(\frac{h(K,u)}{h(L,u)}\right)h(L,u)dS(L,\cdot)$*, and* $d\overline{v}_\varphi=\frac{1}{V_\varphi(L,K)}dv_\varphi$ *is its normalization, and* $V_\varphi(L,K)$ *denotes the usual Orlicz mixed volume of* L *and* K*, defined by (see [5])*

$$V_\varphi(L,K)=\frac{1}{n}\int_{S^{n-1}}\varphi\left(\frac{h(K,u)}{h(L,u)}\right)h(L,u)dS(L,u).$$

where $S(L,\cdot)$ *is the mixed surface area measure of* L*.*

2. Notations and preliminaries

The setting for this paper is n-dimensional Euclidean space $\mathbb{R}^n$. A body in $\mathbb{R}^n$ is a compact set equal to the closure of its interior. A set K is called a convex body, if it is compact and convex subsets with non-empty interiors. Let $\mathcal{K}^n$ denote the class of convex bodies in $\mathbb{R}^n$. Let $\mathcal{K}_o^n$ denote the class of convex bodies containing the origin in their interiors in $\mathbb{R}^n$. We reserve the letter $u \in S^{n-1}$ for unit vectors, and the letter B for the unit ball centered at the origin. The surface of B is S^{n-1}. For a compact set K, we write $V(K)$ for the (n-dimensional) Lebesgue measure of K and call this the volume of K. Let d denote the Hausdorff metric on $\mathcal{K}^n$, i.e., for $K, L \in \mathcal{K}^n$,

$$d(K, L) = |h(K, u) - h(L, u)|_\infty,$$

where $|\cdot|_\infty$ denotes the sup-norm on the space of continuous functions $C(S^{n-1})$. Let $K \subset \mathbb{R}^n$ be a nonempty closed convex set. If ξ is a subspace of $\mathbb{R}^n$, then it is easy to show that

$$h(K \mid \xi, x) = h(K, x \mid \xi),$$

for $x \in \mathbb{R}^n$. Let $\varphi : [0, \infty) \to (0, \infty)$ be a convex and increasing function such that $\varphi(1) = 1$ and $\varphi(0) = 0$. Let Φ denote the set of convex functions $\varphi : [0, \infty) \to [0, \infty)$ that is increasing and satisfies $\varphi(0) = 0$ and $\varphi(1) = 1$.

2.1 Mixed volumes

If $K_i \in \mathcal{K}^n$ $(i = 1, 2, \ldots, r)$ and λ_i $(i = 1, 2, \ldots, r)$ are nonnegative real numbers, then the volume of $\sum_{i=1}^r \lambda_i K_i$ is a homogeneous polynomial in λ_i given by (see e.g. [9])

$$V(\lambda_1 K_1 + \cdots + \lambda_n K_n) = \sum_{i_1, \ldots, i_n} \lambda_{i_1} \ldots \lambda_{i_n} V_{i_1 \ldots i_n}, \tag{2.15}$$

where the sum is taken over all n-tuples $(i_1, \ldots, i_n)$ of positive integers not exceeding r. The coefficient $V_{i_1 \ldots i_n}$ depends only on the bodies $K_{i_1}, \ldots, K_{i_n}$ and is uniquely determined by (2.15). It is called the mixed volume of $K_{i_1}, \ldots, K_{i_n}$, and is written as $V(K_{i_1}, \ldots, K_{i_n})$. Let $K_1 = \ldots = K_{n-i} = K$ and $K_{n-i+1} = \ldots = K_n = L$. Then the mixed volume $V(K_1, \ldots, K_n)$ is written as $V_i(K, L)$. If $K_1 = \cdots = K_{n-i} = K$, $K_{n-i+1} = \cdots = K_n = B$. The mixed volume $V_i(K, B)$ is written as $W_i(K)$ and called as quermassintegrals (or ith mixed quermassintegrals) of K. We write $W_i(K, L)$ for the mixed volume $V(K, \ldots, K, \underbrace{B, \ldots, B}_{i}, L)$ and call as mixed quermassintegrals, and

$$W_i(K, L) = \frac{1}{n} \int_{S^{n-1}} h(L, u) dS_i(K, u).$$

Associated with $K_1, \ldots, K_n \in \mathcal{K}^n$ is a Borel measure $S(K_1, \ldots, K_{n-1}, \cdot)$ on S^{n-1}, called the mixed surface area measure of $K_1, \ldots, K_{n-1}$, which has the property that for each $K \in \mathcal{K}^n$,

$$V(K_1, \ldots, K_{n-1}, K) = \frac{1}{n} \int_{S^{n-1}} h(K, u) dS(K_1, \ldots, K_{n-1}, u).$$

Let $K_1 = \ldots = K_{n-i-1} = K$ and $K_{n-i} = \ldots = K_{n-1} = L$, then the mixed surface area measure $S(K_1, \ldots, K_{n-1}, \cdot)$ is written as $S(K[n-i], L[i], \cdot)$. When $L = B$, $S(K[n-i], L[i], \cdot)$ is written as $S_i(K, \cdot)$ and called as ith mixed surface area measure.

2.2 L_p-mixed volumes

Mixed quermassintegrals are the first variation of the ordinary quermassintegrals with respect to Minkowski addition. The p-mixed quermassintegrals $W_{p,0}(K, L)$, $W_{p,1}(K, L), \ldots,$

$W_{p,n-1}(K,L)$, as the first variation of the ordinary quermassintegrals, with respect to Firey addition, for $K,L\in\mathcal{K}^n$, and real $p\geq 1$, is defined by (see e.g. [10])

$$W_{p,i}(K,L)=\frac{p}{n-i}\lim_{\varepsilon\to 0^+}\frac{W_i(K+_p\varepsilon\cdot L)-W_i(K)}{\varepsilon}.$$

The mixed p-quermassintegrals $W_{p,i}(K,L)$, for all $K,L\in\mathcal{K}^n$, has the following integral representation:

$$W_{p,i}(K,L)=\frac{1}{n}\int_{S^{n-1}}h(L,u)^p dS_{p,i}(K,u), \tag{2.16}$$

where $S_{p,i}(K,\cdot)$ denotes the Boel measure on S^{n-1}. The measure $S_{p,i}(K,\cdot)$ is absolutely continuous with respect to $S_i(K,\cdot)$, and has Radon-Nikodym derivative

$$\frac{dS_{p,i}(K,\cdot)}{dS_i(K,\cdot)}=h(K,\cdot)^{1-p},$$

where $S_i(K,\cdot)$ is a regular Boel measure on S^{n-1}. The measure $S_{n-1}(K,\cdot)$ is independent of the body K, and is just ordinary Lebesgue measure, S, on S^{n-1}. $S_i(B,\cdot)$ denotes the i-th surface area measure of the unit ball in $\mathbb{R}^n$. In fact, $S_i(B,\cdot)=S$ for all i. The surface area measure $S_0(K,\cdot)$ just is $S(K,\cdot)$. When $i=0$, $S_{p,i}(K,\cdot)$ is written as $S_p(K,\cdot)$. Obviously, putting $i=0$ in (2.16), the mixed p-quermassintegrals $W_{p,i}(K,L)$ become the well-known L_p-mixed volume $V_p(K,L)$, is defined by

$$V_p(K,L)=\frac{1}{n}\int_{S^{n-1}}h(L,u)^p dS_p(K,u). \tag{2.17}$$

2.3 Orlicz mixed volumes

The Orlicz mixed volume was introduced by Gardner, Hug and Weil [5], as follows: for $K,L\in\mathcal{K}^n_o$ and $\varphi\in\Phi$, the Orlicz mixed volume of L and K, is denoted by $V_\varphi(L,K)$, is defined by

$$V_\varphi(L,K)=\frac{1}{n}\int_{S^{n-1}}\varphi\left(\frac{h(K,u)}{h(L,u)}\right)h(L,u)dS(L,u). \tag{2.18}$$

If $\varphi(t)=t^p$ and $p\geq 1$, then the Orlicz mixed volume $V_\varphi(L,K)$ becomes the classical L_p-mixed volume $V_p(L,K)$. The Orlicz-Minkowski inequality is the following: for $K,L\in\mathcal{K}^n_o$ and $\varphi\in\Phi$, then

$$V_\varphi(L,K)\geq V(L)\cdot\varphi\left(\left(\frac{V(K)}{V(L)}\right)^{1/n}\right).$$

If φ is strictly convex, equality holds if and only if K and L are homothetic.

2.4 Orlicz mixed affine quermassintegrals

The Orlicz mixed affine querlmassintegral of convex bodies K and L, is denoted by $\Phi_{\varphi,n-j}(K,L)$, is defined by (see [24])

$$\Phi_{\varphi,n-j}(K,L):=\omega_n\left[\int_{G_{n,j}}\frac{V_\varphi^{(j)}(K|\xi,L|\xi)}{\mathrm{vol}_j(K|\xi)}\left(\frac{\mathrm{vol}_j(K|\xi)}{\omega_j}\right)^{-n}d\mu_j(\xi)\right]^{-1/n}, \tag{2.19}$$

where $\varphi\in\Phi$, $K,L\in\mathcal{K}^n_o$ and $0<j\leq n$. Specifically, for $j=n$, it follows that

$$\Phi_{\varphi,0}(K,L)=\left(\frac{V(K)}{V_\varphi(K,L)}\right)^{1/n}V(K). \tag{2.20}$$

3. Orlicz affine log-Minkowski inequality

In the section, in order to derive the Orlicz affine log-Minkowski inequality, we need to define some new affine measures. Lutwak [11] defined the affine quermassintegrals for a convex body K, $\Phi_{n-j}(K)$, for $0 \leq j \leq n$,

$$\Phi_{n-j}(K) := \omega_n \left[\int_{G_{n,j}} \left(\frac{\mathrm{vol}_j(K|\xi)}{\omega_j} \right)^{-n} d\mu_j(\xi) \right]^{-1/n}. \tag{3.21}$$

From (3.21), we introduce the following affine measure of star body K.

Definition 3.1. (The affine measure) For $K \in \mathcal{K}_o^n$ and $0 < j \leq n$, the affine measure of K, is denoted by $d\varphi_{n-j}^{-n}(K)$, is defined by

$$d\varphi_{n-j}^{-n}(K) = \left(\frac{\omega_n \mathrm{vol}_j(K|\xi)}{\omega_j} \right)^{-n} d\mu_j(\xi). \tag{3.22}$$

From Definition 3.1, we find the following affine probability measure.

$$d\Phi_{n-j}^{-n}(K) = \frac{1}{d\Phi_{n-j}^{-n}(K)} d\varphi_{n-j}^{-n}(K). \tag{3.23}$$

For $\varphi \in \Phi$ and $0 < j \leq n$, Orlicz mixed affine quermasintegral of L and K, is denoted by $\Phi_{\varphi,n-j}(L,K)$, is defined by (see [24])

$$\Phi_{\varphi,n-j}(L,K) = \omega_n \left[\int_{G_{n,j}} \frac{V_\varphi^{(j)}(L|\xi, K|\xi)}{\mathrm{vol}_j(L|\xi)} \left(\frac{\mathrm{vol}_j(L|\xi)}{\omega_j} \right)^{-n} d\mu_j(\xi) \right]^{-1/n}. \tag{3.24}$$

From (3.24), we introduce the following Orlicz affine measure of convex bodies L and K.

Definition 3.2. (Orlicz mixed affine measure) For $L, K \in \mathcal{K}_o^n$, $0 < j \leq n$ and $\varphi \in \Phi$, the Orlicz mixed affine measure of L and K, is denoted by $d\varphi_{\varphi,n-j}^{-n}(L,K)$, is defined by

$$d\varphi_{\varphi,n-j}^{-n}(L,K) = \frac{V_\varphi^{(j)}(L|\xi, K|\xi)}{\mathrm{vol}_j(L|\xi)} \left(\frac{\omega_n \mathrm{vol}_j(L|\xi)}{\omega_j} \right)^{-n} d\mu_j(\xi), \tag{3.25}$$

From Definition 3.2, Orlicz mixed affine probability measure is defined by

$$d\Phi_{\varphi,n-j}^{-n}(L,K) = \frac{1}{\Phi_{\varphi,n-j}^{-n}(L,K)} d\varphi_{\varphi,n-j}^{-n}(L,K). \tag{3.26}$$

Lemma 3.1. ([24]) (The Orlicz affine Minkowski inequlity for mixed affine quermassintegrals) *If $\varphi \in \Phi$, $0 < j \leq n$ and $K, L \in \mathcal{K}_o^n$, then*

$$\left(\frac{\Phi_{\varphi,n-j}(K,L)}{\Phi_{n-j}(K)} \right)^{-n} \geq \varphi\left(\left(\frac{\Phi_{n-j}(L)}{\Phi_{n-j}(K)} \right)^{1/j} \right).$$

If φ is strictly convex, equality holds if and only if K and L are homothetic.

Theorem 3.2. *(The affine Orlicz log-Minkowski inequality) If $L, K \in \mathcal{K}_o^n$, $0 < j \leq n$ and $\varphi \in \Phi$, then*

$$\int_{G_{n,j}} \ln \left(\frac{V_\varphi^{(j)}(L|\xi, K|\xi)}{\mathrm{vol}_j(L|\xi)} \right) d\Phi_{\varphi,n-j}^{-n}(L,K) \geq \ln \left(\frac{\Phi_{n-j}^n(L)}{\Phi_{\varphi,n-j}^n(L,K)} \right)$$
$$\geq \ln \left(\varphi \left(\left(\frac{\Phi_{n-j}(K)}{\Phi_{n-j}(L)} \right)^{1/j} \right) \right). \tag{3.27}$$

If φ is strictly convex, each equality holds if and only if L and K are homothetic.

Proof. From (3.22) and (3.25), we have

$$\int_{G_{n,j}} \frac{V_\varphi^{(j)}(L|\xi, K|\xi)}{\mathrm{vol}_j(L|\xi)} \ln\left(\frac{V_\varphi^{(j)}(L|\xi, K|\xi)}{\mathrm{vol}_j(L|\xi)}\right) d\varphi_{n-j}^{-n}(L)$$

$$= \int_{G_{n,j}} \ln\left(\frac{V_\varphi^{(j)}(L|\xi, K|\xi)}{\mathrm{vol}_j(L|\xi)}\right) d\varphi_{\varphi,n-j}^{-n}(L,K). \tag{3.28}$$

Note the following equality

$$\Phi_{\varphi,n-j}^{-n}(L,K) = \int_{G_{n,j}} \frac{V_\varphi^{(j)}(L|\xi, K|\xi)}{\mathrm{vol}_j(L|\xi)} \left(\frac{\omega_n \mathrm{vol}_j(L|\xi)}{\omega_j}\right)^{-n} d\mu_j(\xi).$$

From Lebesgue's dominated convergence theorem, we obtain

$$\int_{G_{n,j}} \left(\frac{V_\varphi^{(j)}(L|\xi, K|\xi)}{\mathrm{vol}_j(L|\xi)}\right)^{\frac{q}{q+n}} d\varphi_{n-j}^{-n}(L) \to \Phi_{\varphi,n-j}^{-n}(L,K),$$

as $q \to \infty$, and

$$\int_{G_{n,j}} \left(\frac{V_\varphi^{(j)}(L|\xi, K|\xi)}{\mathrm{vol}_j(L|\xi)}\right)^{\frac{q}{q+n}} \ln\left(\frac{V_\varphi^{(j)}(L|\xi, K|\xi)}{\mathrm{vol}_j(L|\xi)}\right) d\varphi_{n-j}^{-n}(L)$$

$$\to \int_{G_{n,j}} \ln\left(\frac{V_\varphi^{(j)}(L|\xi, K|\xi)}{\mathrm{vol}_j(L|\xi)}\right) d\varphi_{\varphi,n-j}^{-n}(L,K),$$

as $q \to \infty$.

Given the function $g_{L,K}(q) : [1,\infty] \to \mathbb{R}$, is defined by

$$g_{L,K}(q) = \frac{1}{\Phi_{\varphi,n-j}^{-n}(L,K)} \int_{G_{n,j}} \left(\frac{V_\varphi^{(j)}(L|\xi, K|\xi)}{\mathrm{vol}_j(L|\xi)}\right)^{\frac{q}{q+n}} d\varphi_{n-j}^{-n}(L). \tag{3.29}$$

From (3.29), we obtain

$$\frac{dg_{L,K}(q)}{dq} = \frac{n}{(q+n)^2 \Phi_{\varphi,n-j}^{-n}(L,K)} \times$$

$$\times \int_{G_{n,j}} \left(\frac{V_\varphi^{(j)}(L|\xi, K|\xi)}{\mathrm{vol}_j(L|\xi)}\right)^{\frac{q}{q+n}} \ln\left(\frac{V_\varphi^{(j)}(L|\xi, K|\xi)}{\mathrm{vol}_j(L|\xi)}\right) d\varphi_{n-j}^{-n}(L). \tag{3.30}$$

and

$$\lim_{q\to\infty} g_{L,K}(q) = 1. \tag{3.31}$$

From (3.29), (3.30) and (3.31), we have

$$\begin{aligned}
\lim_{q\to\infty}\ln(g_{L,K}(q))^{q+n} &= -(q+n)^2\lim_{q\to\infty}\frac{1}{g_{L,K}(q)}\frac{dg_{L,K}(q)}{dq}\\
&= -\frac{n}{\Phi^{-n}_{\varphi,n-j}(L,K)}\times\\
&\times \lim_{q\to\infty}\frac{\int_{G_{n,j}}\left(\frac{V^{(j)}_{\varphi}(L|\xi,K|\xi)}{\mathrm{vol}_j(L|\xi)}\right)^{\frac{q}{q+n}}\ln\left(\frac{V^{(j)}_{\varphi}(L|\xi,K|\xi)}{\mathrm{vol}_j(L|\xi)}\right)d\varphi^{-n}_{n-j}(L)}{g_{L,K}(q)}\\
&= -\frac{n}{\Phi^{-n}_{\varphi,n-j}(L,K)}\int_{G_{n,j}}\frac{V^{(j)}_{\varphi}(L|\xi,K|\xi)}{\mathrm{vol}_j(L|\xi)}\times\\
&\times \ln\left(\frac{V^{(j)}_{\varphi}(L|\xi,K|\xi)}{\mathrm{vol}_j(L|\xi)}\right)d\varphi^{-n}_{n-j}(L).
\end{aligned}$$

Hence

$$\begin{aligned}
&\exp\left(-\frac{n}{\Phi^{-n}_{\varphi,n-j}(L,K)}\int_{G_{n,j}}\frac{V^{(j)}_{\varphi}(L|\xi,K|\xi)}{\mathrm{vol}_j(L|\xi)}\ln\left(\frac{V^{(j)}_{\varphi}(L|\xi,K|\xi)}{\mathrm{vol}_j(L|\xi)}\right)d\varphi^{-n}_{n-j}(L)\right)\\
&= \lim_{q\to\infty}(g_{L,K})^{q+n}
\end{aligned}$$

$$= \lim_{q\to\infty}\left(\frac{1}{\Phi^{-n}_{\varphi,n-j}(L,K)}\int_{G_{n,j}}\left(\frac{V^{(j)}_{\varphi}(L|\xi,K|\xi)}{\mathrm{vol}_j(L|\xi)}\right)^{\frac{q}{q+n}}d\varphi^{-n}_{n-j}(L)\right)^{q+n}. \tag{3.32}$$

On the other hand, from Hölder's inequality

$$\left(\int_{G_{n,j}}\left(\frac{V^{(j)}_{\varphi}(L|\xi,K|\xi)}{\mathrm{vol}_j(L|\xi)}\right)^{\frac{q}{q+n}}d\varphi^{-n}_{n-j}(L)\right)^{(q+n)/q}\left(\int_{G_{n,j}}d\varphi^{-n}_{n-j}(L)\right)^{-n/q}$$

$$\le \int_{G_{n,j}}\frac{V^{(j)}_{\varphi}(L|\xi,K|\xi)}{\mathrm{vol}_j(L|\xi)}d\varphi^{-n}_{n-j}(L)=\Phi^{-n}_{\varphi,n-j}(L,K). \tag{3.33}$$

If φ is strictly convex, from the equality of Hölder's inequality, it follows that the equality in (3.33) holds if and only if $K|\xi$ and $L|\xi$ are homothetic. This yields that equality in (3.33) holds if and only if K and L are homothetic. Hence

$$\left(\frac{1}{\Phi^{-n}_{\varphi,n-j}(L,K)}\int_{G_{n,j}}\left(\frac{V^{(j)}_{\varphi}(L|\xi,K|\xi)}{\mathrm{vol}_j(L|\xi)}\right)^{\frac{q}{q+n}}d\varphi^{-n}_{n-j}(L)\right)^{q+n}\le\left(\frac{\Phi^{n}_{n-j}(L)}{\Phi^{-n}_{\varphi,n-j}(L,K)}\right)^{-n}.$$

If φ is strictly convex, equality holds if and only if K and L are homothetic. Therefore

$$\begin{aligned}
&\exp\left(-\frac{n}{\Phi^{-n}_{\varphi,n-j}(L,K)}\int_{G_{n,j}}\frac{V^{(j)}_{\varphi}(L|\xi,K|\xi)}{\mathrm{vol}_j(L|\xi)}\ln\left(\frac{V^{(j)}_{\varphi}(L|\xi,K|\xi)}{\mathrm{vol}_j(L|\xi)}\right)d\varphi^{-n}_{n-j}(L)\right)\\
&\le\left(\frac{\Phi^{-n}_{n-j}(L)}{\Phi^{-n}_{\varphi,n-j}(L,K)}\right)^{n}.
\end{aligned}$$

Hence

$$\frac{1}{\Phi^{n}_{\varphi,n-j}(L,K)}\int_{G_{n,j}}\frac{V^{(j)}_{\varphi}(L|\xi,K|\xi)}{\mathrm{vol}_j(L|\xi)}\ln\left(\frac{V^{(j)}_{\varphi}(L|\xi,K|\xi)}{\mathrm{vol}_j(L|\xi)}\right)d\varphi^{-n}_{n-j}(L)\geq \ln\left(\frac{\Phi^{-n}_{\varphi,n-j}(L,K)}{\Phi^{-n}_{n-j}(L)}\right).$$

That is

$$\int_{G_{n,j}}\ln\left(\frac{V^{(j)}_{\varphi}(L|\xi,K|\xi)}{\mathrm{vol}_j(L|\xi)}\right)d\Phi^{-n}_{\varphi,n-j}(L,K)\geq \ln\left(\frac{\Phi^{-n}_{\varphi,n-j}(L,K)}{\Phi^{-n}_{n-j}(L)}\right). \tag{3.34}$$

If φ is strictly convex, equality holds if and only if K and L are homothetic. The completes proof of the first inequality in (3.27).

Furthermore, by using the Orlicz affine Minkowski inequlity for mixed affine quermassintegrals in Lemma 3.1, we obtain

$$\int_{G_{n,j}}\ln\left(\frac{V^{(j)}_{\varphi}(L|\xi,K|\xi)}{\mathrm{vol}_j(L|\xi)}\right)d\Phi^{-n}_{\varphi,n-j}(L,K)\geq \ln\left(\varphi\left(\left(\frac{\Phi_{n-j}(K)}{\Phi_{n-j}(L)}\right)^{1/j}\right)\right). \tag{3.35}$$

If φ is strictly convex, from the equality of Orlicz affine Minkowski inequlity, the equality in (3.35) holds if and only if K and L are homothetic.

This completes the proof. □

When $\varphi(t)=t$, (3.27) becomes the affine log-Minkowski inequality (1.11) stated in the introduction. When $\varphi(t)=t^p$ and $p\geq 1$, (3.27) becomes the L_p-affine log-Minkowski inequality (1.7) stated in the introduction.

Moreover, when $j=n$, some new Orlicz log-Minkowski inequalities for the Orlicz mixed volumes are derived. Here we omit the details.

Competing interests The author declare that he has no competing interests.

Acknowledgements. The author expresses his thanks to the two anonymous referees for their very good suggestions and comments. The author expresses also his thanks to National Natural Science Foundation of China (11371334, 10971205).

REFERENCES

[1] Böröczky K. J.; Lutwak, E. ; Yang, D.; Zhang, G. The log-Brunn-Minkowski inequality. *Adv. Math.* **231** (2012), 1974–1997.

[2] Burago, Y. D.; Zalgaller, V. A. *Geometric Inequalities*, Springer-Verlag, Berlin, 1988.

[3] Colesanti, A.; Cuoghi, P. The Brunn-Minkowski inequality for the n-dimensional logarithmic capacity of convex bodies. *Potential Math.* **22** (2005), 289–304.

[4] Fathi, M.; Nelson, B. Free Stein kernels and an improvement of the free logarithmic Sobolev inequality. *Adv. Math.* **317** (2017), 193–223.

[5] Gardner, R. J.; Hug, D.; Weil, W. The Orlicz-Brunn-Minkowski theory: a general framework, additions, and inequalities. *J. Diff. Geom.* **97** (2014), no. 3, 427-476.

[6] Henk, M.; Pollehn, H. *On the log-Minkowski inequality for simplices and parallelepipeds* Acta Math. Hungarica, **155** (2018), 141-157.

[7] Hou, S.; Xiao, J. A mixed volumetry for the anisotropic logarithmic potential. *J. Geom Anal.* **28** (2018), 2018–2049.

[8] Li, C.; Wang, W. Log-Minkowski inequalities for the L_p-mixed quermassintegrals. *J. Inequal. Appl.* **2019** (2019): 85.

[9] Lutwak, E. Volume of mixed bodies. *Trans. Amer. Math. Soc* **294** (1986), no. 2, 487–500.

[10] Lutwak, E. The Brunn-Minkowski-Firey theory I: mixed volumes and the Minkowski problem. *J. Diff. Geom.* **38** (1975), 131–150.

[11] Lutwak, E. A general isepiphanic inequality. *roc. Amer. Math. Soc.* **90** (1984), 451–421.

[12] Lv, S.-J. The φ-Brunn–Minkowski inequality. *Acta Math. Hungarica* **156** (2018), 226–239.

[13] Ma, L. A new proof of the Log-Brunn–Minkowski inequality. *Geom. Dedicata* **177** (2015), 75–82.

[14] Saroglou, C. Remarks on the conjectured log-Brunn-Minkowski inequality. *Geometriae Dedicata* **177** (2015), 353–365.
[15] Schneider, R. *Convex Bodies: The Brunn-Minkowski Theory*. Cambridge University Press, 2014.
[16] Stancu, A. The logarithmic Minkowski inequality for non-symmetric convex bodies. *Adv. Appl. Math.* **73** (2016), 43–58.
[17] Wang, W.; Feng, M. The log-Minkowski inequalities for quermassintegrals. *J. Math. Inequal.* **11** (2017), 983-995.
[18] Wang, H.; Fang, N.; Zhou, J. Continuity of the solution to the even logarithmic Minkowski problem in the plane. *Sci. China Ser. A* **62** (2019), 1419-1428.
[19] Wang, W.; Liu, L. The dual log-Brunn-Minkowski inequalities. *Taiwanese J. Math.* **20** (2016), 909–919.
[20] Zhao, C.-J. The log-Aleksandrov-Fenchel inequality. *Mediterr. J. Math.* (2020), 17: 96.
[21] Zhao, C.-J. The dual logarithmic Aleksandrov-Fenchel inequality. *Balkan J. Geom. Appl.* **25** (2020), 157–169.
[22] Zhao, C.-J. Orlicz log-Minkowki inequality. *Diff. Geom. Appl.* **74** (2021): 101695.
[23] Zhao, C.-J. Orlicz dual affine quermassintegrals. *Forum Math.* **30** (2018), no. 4. 929-945.
[24] Zou, D. *Affine extremum problems in the Orlicz Brunn-Minkowski theory*, PhD Thesis (in Chinese), Shanghai University, Shanghai, 2015, 91-106.

DEPARTMENT OF MATHEMATICS
CHINA JILIANG UNIVERSITY
NO 258 STREET XUEYUAN, 310018, HANGZHOU, P. R. CHINA
Email address: chjzhao@163.com chjzhao@cjlu.edu.cn