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In memoriam Professor Charles E. Chidume (1947- 2021)

A Hybrid Steepest-Descent Algorithm for Convex Minimization Over the Fixed Point Set of Multivalued Mappings

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ABSTRACT. In the setting of Hilbert spaces, we show that a hybrid steepest-descent algorithm converges strongly to a solution of a convex minimization problem over the fixed point set of a finite family of multivalued demicontractive mappings. We also provide numerical results concerning the viability of the proposed algorithm with possible applications.

1. INTRODUCTION AND PRELIMINARIES

Throughout this paper, we assume that $\mathcal{H}$ is a real Hilbert space, and $\mathcal{D} \subseteq \mathcal{H}$ is nonempty, closed and convex. Let $S : \mathcal{D} \to \mathcal{D}$ be a nonexpansive mapping (i.e., , a mapping satisfying $\|Su - Sv\| \leq \|u - v\|$) and let $\Phi : \mathcal{H} \to \mathbb{R}$ be a convex and bounded below function. The minimization problem over the fixed point set of a mapping is defined as:

$$\text{find } u \in Fix(S) \text{ such that } \Phi(u) = \inf \Phi(Fix(S)), \tag{1.1}$$

where $Fix(S) = \{u \in \mathcal{D} : \ Su = u\}$ denotes the fixed point set of S.

It is remarked that the problem (1.1) is equivalent to the following variational inequality problem VIP$(\acute{\Phi}, Fix(S))$ ([13]):

$$\text{find } u \in Fix(S) \text{ such that } \langle v - u, \acute{\Phi}(u) \rangle \geq 0, \ \forall \ v \in Fix(S), \tag{1.2}$$

provided that Φ is Gâteaux differentiable over an open set including $Fix(S)$ where $\acute{\Phi}$ denotes the derivative of Φ.

For a slow decreasing sequence $\alpha_k^* \subset (0, 1)$, the following class of hybrid steepest-descent algorithm (HSDA):

$$y_{k+1} = S(y_k) - \alpha_{k+1}^* \acute{\Phi}(S(y_k)), \tag{1.3}$$

is prominent for solving (1.2). The Algorithm (1.3) converges strongly to the set of solutions of (1.2), involving a (quasi-)nonexpansive mapping S, under suitable set of conditions on $\Phi, \acute{\Phi}$ and (α_k^*) [17, 18]. A robust variant of HSDA, involving (asymptotically) quasi-shrinking operators, was analyzed in [19](see also [1–11]).

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In 2008, Maing$\acute{e}$ [15] studied the problem (1.1) involving a more general class of demicontractive and demiclosed mapping via the following Mann-type variant of the HSDA:

$$\left\{ \begin{array}{l} y_k := x_k - \alpha_k^* \acute{\Phi}(x_k); \\ x_{k+1} := (1-\beta)y_k + \beta S y_k. \end{array} \right. \tag{1.4}$$

The following compact form of (1.4) coincides with the HSDA:

$$y_{k+1} = S_\beta y_k - \alpha_{k+1}^* \acute{\Phi}(S_\beta(y_k)), \tag{1.5}$$

where $S_\beta := (1-\beta)Id + \beta S$ and Id denotes the identity mapping. Thus the following natural question arises in view of the architecture of the Algorithm (1.4):

Can one modify the Algorithm (1.4) to solve the convex minimization problem (1.1) over the fixed point set of multivalued mappings? Answering this question in the affirmative, we propose a HSDA for the convex minimization problem over the fixed point set of a finite family of multivalued demicontractive mappings in Hilbert spaces. As far as we know, such results have not so far appeared in the literature.

The rest of the paper is organized as follows: Section 2 contains some relevant preliminary concepts and results for convex minimization problem and fixed point problem. Section 3 comprises of strong convergence results of the proposed HSDA whereas Section 4 provides numerical results concerning the viability of the proposed algorithm with respect to various real world applications.

2. Preliminaries

Let $\mathcal{CB}(\mathcal{D})$ denote the family of nonempty bounded and closed subsets of $\mathcal{D}$. The Hausdorff metric on $\mathcal{CB}(\mathcal{D})$ is defined as:

$$\mathcal{H}(\tilde{A}, \tilde{B}) := \max\Big\{ \sup_{u\in\tilde{A}} d(u, \tilde{B}), \sup_{v\in\tilde{B}} d(v, \tilde{A}) \Big\},$$

for all $\tilde{A}, \tilde{B} \in \mathcal{CB}(\mathcal{D})$ where $d(u, \tilde{B}) = \inf_{a\in\tilde{B}} \|u - a\|$.

Let $S : \mathcal{D} \to \mathcal{CB}(\mathcal{D})$ be a multivalued mapping, then u is said to be: (i) a fixed point of S if $u \in S(u)$ and (ii) an endpoint of S if $S(u) = \{u\}$. If S satisfies the endpoint condition then $Fix(S)$ is convex. Recall that the multivalued mapping S is said to be (i) nonexpansive if $\mathcal{H}(Su, Sv)^2 \leq \|u - v\|^2$ for all $(u, v) \in \mathcal{D} \times \mathcal{D}$, (ii) quasi-nonexpansive if $Fix(S) \neq \emptyset$ and $\mathcal{H}(Sv, u)^2 \leq \|v - u\|^2$ for all $v \in \mathcal{D}$ and $u \in Fix(S)$ and (iii) demicontractive [12] if $Fix(S) \neq \emptyset$ and there exists $\eta \in [0, 1)$ such that $\mathcal{H}(Su, Sv)^2 \leq \|u - v\|^2 + \eta d(v, Sv)^2$ for all $v \in \mathcal{D}$ and $u \in Fix(S)$. The class of multivalued demicontractive mappings contains properly the class of multivalued quasi-nonexpansive mappings [14].

In this paper we are interested in the following multivalued variant of (1.1):

Let $x_0 \in \mathcal{H}$ and let $S_i : \mathcal{D} \to \mathcal{CB}(\mathcal{D})$, $i \in \{1, 2, \cdots, N\}$ be a finite family of η-demicontractive multivalued mappings satisfying the endpoint condition. Let $\Phi : \mathcal{D} \to \mathbb{R} \cup (-\infty, +\infty]$ be a convex, bounded below and G$\hat{a}$teaux differentiable function on $\mathcal{H}$ with the derivative $\acute{\Phi}$. Assume that $\Pi := \bigcap_{i=1}^N Fix(S_i) \neq \emptyset$ then we aim to compute $u \in \Pi$ such that $\Phi(u) = \inf \Phi(\Pi)$, via the algorithm generated as follows:

$$\left\{ \begin{array}{l} z_k := x_k - \alpha_k^* \acute{\Phi}(x_k); \\ x_{k+1} := (1-\beta_k)z_k + \beta_k \sum_{i=1}^N \hbar_i w_k^{(i)}, \ i = \{1, 2, \cdots, N\} \end{array} \right. \tag{2.6}$$

where $w_k^{(i)} \in S_i z_k$, for each $i = \{1, 2, \cdots, N\}$, $\hbar_i \in (0, 1)$ such that $\sum_{j=i}^N \hbar_i = 1$, $0 < \tau < \min(\frac{1}{2b_1}, \frac{1}{2b_2})$, $\alpha_k^* \subset [0, 1)$ and $\beta_k \in (0, 1)$.

In order to establish the convergence results of the Algorithm (2.6), we use the following required conditions throughout the rest of the paper:

(C1) $\beta_k \in (0, \frac{1-\eta_i}{2}]$ and $\beta_k(1-\beta_k-\eta_i) > 0$, $i \in \{1, 2, \cdots, N\}$;
(C2) $\lim_{k\to\infty} \alpha_k^* = 0$;
(C3) $\sum_{k\geq 0} \alpha_k^* = +\infty$;
(C4) $\acute{\Phi}$ is L-Lipschitz continuous on $\mathcal{H}$ (for some $L \geq 0$); i.e.,

$$\|\acute{\Phi}(u) - \acute{\Phi}(v)\| \leq L\|u - v\|, \; \forall\, u, v \in \mathcal{H}.$$

(C5) $\acute{\Phi}$ is Ψ-strongly monotone on $\mathcal{H}$ (for some $\Psi > 0$); i.e.,

$$\langle \acute{\Phi}(u) - \acute{\Phi}(v), u - v\rangle \geq \Psi\|u - v\|^2, \; \forall\, u, v \in \mathcal{H}.$$

We now enlist some useful results required in the sequel:

Definition 2.1. Let $\mathcal{D}$ be a nonempty, closed and convex subset of a Hilbert space $\mathcal{H}$ and let $S : \mathcal{D} \to \mathcal{CB}(\mathcal{D})$ be a multivalued mapping. Then $Id - S$ is said to be demiclosed at 0 if for any sequence (x_k) in $\mathcal{D}$ which converges weakly to $u \in \mathcal{D}$ and the sequence $(\|x_k - y_k\|)$ converges strongly to 0, where $y_k \in Sx_k$, then $u \in Fix(S)$.

For every point $u \in \mathcal{H}$, there exists a unique nearest point in $\mathcal{D}$, denote by $\mathcal{P}_{\mathcal{D}}u$, such that $\|u - \mathcal{P}_{\mathcal{D}}u\| \leq \|u - v\| \; \forall\, u, v \in \mathcal{D}$. The mapping $\mathcal{P}_{\mathcal{D}}$ is called the metric projection of $\mathcal{H}$ onto $\mathcal{D}$. It is well known that $\mathcal{P}_{\mathcal{D}}$ is nonexpansive and satisfies $\langle u - \mathcal{P}_{\mathcal{D}}u, b - \mathcal{P}_{\mathcal{D}}u\rangle \leq 0 \; \forall\, b \in \mathcal{D}$.

Lemma 2.1. *Let $u, v, n \in \mathcal{H}$ and $a \in [0, 1] \subset \mathbb{R}$, then*

(1) $\|u + v\|^2 \leq \|u\|^2 + 2\langle v, u + v\rangle$;
(2) $\|u - v\|^2 = \|u\|^2 - \|v\|^2 - 2\langle u - v, v\rangle$;
(3) $\|au + (1 - a)v - n\|^2 = a\|u - n\|^2 + (1 - a)\|v - n\|^2 - a(1 - a)\|u - v\|^2$.

We need the following result to establish the strong convergence results of the Algorithm (2.6).

Lemma 2.2. *Let Φ be a convex, bounded below and Gâteaux differentiable function on a real Hilbert space $\mathcal{H}$ with the derivative $\acute{\Phi}$. Let $\mathcal{D}$ be a nonempty closed and convex subset of $\mathcal{H}$ and for each $i = \{1, 2, \cdots, N\}$, let $S_i : \mathcal{D} \to \mathcal{CB}(\mathcal{D})$ be a finite family of η_i-demicontractive multivalued mappings satisfying the endpoint condition and the demiclosedness principle where $(\eta_i)_{i=1}^N \subset (0, 1)$. Let $\Pi = \bigcap_{i=1}^N Fix(S_i) \neq \emptyset$. Assume that the conditions $(C1)$, $(C2)$ and $(C5)$ hold, then for all $k \geq 0$ the sequence (x_k) given by (2.6) satisfies*

$$U_{k+1} - U_k + \frac{1}{2}(1 - 2L\alpha_k^*)\|x_{k+1} - x_k\|^2 \leq -\alpha_k^*\langle x_k - \bar{u}, \acute{\Phi}(x_k)\rangle, \tag{2.7}$$

where $\bar{u} \in \Pi$ and

$$U_k := \frac{1}{2}\|x_k - \bar{u}\|^2 + \alpha_k^*(\Phi(x_k) - \inf \Phi). \tag{2.8}$$

Proof. Let $\bar{u} \in \Pi$, then it follows from Lemma 2.1 and the Algorithm (2.6) that

$$\begin{aligned}
\|x_{k+1}-\bar{u}\|^2 &= \Big\|(1-\beta_k)(z_k-\bar{u})-\beta_k\Big(\sum_{i=1}^{N}\hbar_i w_k^{(i)}-\bar{u}\Big)\Big\|^2\\
&= \Big\|\sum_{i=1}^{N}\hbar_i\big[(1-\beta_k)(z_k-\bar{u})-\beta_k(w_k^{(i)}-\bar{u})\big]\Big\|^2\\
&\leq \sum_{i=1}^{N}\hbar_i\big[\|(1-\beta_k)(z_k-\bar{u})-\beta_k(w_k^{(i)}-\bar{u})\|^2\big]\\
&= \sum_{i=1}^{N}\hbar_i\big[(1-\beta_k)\|z_k-\bar{u}\|^2+\beta_k\|w_k^{(i)}-\bar{u}\|^2-(1-\beta_k)\beta_k\|z_k-w_k^{(i)}\|^2\big]\\
&= \sum_{i=1}^{N}\hbar_j\big[(1-\beta_k)\|z_k-\bar{u}\|^2+\beta_k d(w_k^{(i)},S_i\bar{u})^2-(1-\beta_k)\beta_k\|z_k-w_k^{(i)}\|^2\big]\\
&\leq \sum_{i=1}^{N}\hbar_i\big[(1-\beta_k)\|z_k-\bar{u}\|^2+\beta_k\mathcal{H}(S_iz_k,S_i\bar{u})^2-(1-\beta_k)\beta_k\|z_k-w_k^{(i)}\|^2\big]\\
&\leq \sum_{i=1}^{N}\hbar_i\big[(1-\beta_k)\|z_k-\bar{u}\|^2+\beta_k(\|z_k-\bar{u}\|^2+\eta_i d(z_k,S_iz_k)^2)\\
&\quad -(1-\beta_k)\beta_k\|z_k-w_k^{(i)}\|^2\big]\\
&\leq \sum_{i=1}^{N}\hbar_i\big[(1-\beta_k)\|z_k-\bar{u}\|^2+\beta_k(\|z_k-\bar{u}\|^2+\eta_i\|z_k-w_k^{(i)})\|^2)\\
&\quad -(1-\beta_k)\beta_k\|z_k-w_k^{(i)}\|^2\big]\\
&= \|z_k-\bar{u}\|^2-\beta_k(1-\eta_i-\beta_k)\sum_{i=1}^{N}\hbar_i\|z_k-w_k^{(i)}\|^2. \qquad (2.9)
\end{aligned}$$

Note that

$$\sum_{i=1}^{N}\hbar_i\|w_k^{(i)}-z_k\| = \frac{1}{\beta_k}(x_{k+1}-z_k).$$

Setting $\xi_k := \frac{1}{\beta_k}(1-\eta_i-\beta_k)$, we get

$$\|x_{k+1}-\bar{u}\|^2 \leq \|z_k-\bar{u}\|^2-\xi_k\|x_{k+1}-z_k\|^2. \qquad (2.10)$$

Since $\beta_k \in \Big(0, \frac{1-\eta_i}{2}\Big]$ (so that $\xi \geq 1$), therefore, we obtain

$$\|x_{k+1}-\bar{u}\|^2 \leq \|z_k-\bar{u}\|^2-\|x_{k+1}-z_k\|^2. \qquad (2.11)$$

From the estimate (2.11) and (C3), we have

$$\begin{aligned}
\|z_k-\bar{u}\|^2 &= \|(x_k-\bar{u})-\alpha_k^*\acute{\Phi}(x_k)\|^2\\
&= \|x_k-\bar{u}\|^2-2\alpha_k^*\langle x_k-\bar{u},\acute{\Phi}(x_k)\rangle+\alpha_k^2\|\acute{\Phi}(x_k)\|^2\\
&= \|x_k-\bar{u}\|^2. \qquad (2.12)
\end{aligned}$$

Moreover

$$\begin{aligned}\|z_k - x_{k+1}\|^2 &= \|(x_{k+1} - x_k) + \alpha_k^* \acute{\Phi}(x_k)\|^2 \\ &= \|x_{k+1} - x_k\|^2 + 2\alpha_k^* \langle x_{k+1} - x_k, \acute{\Phi}(x_k)\rangle + \alpha_k^2 \|\acute{\Phi}(x_k)\|^2 \\ &= \|x_{k+1} - x_k\|^2 + 2\alpha_k^* \langle x_{k+1} - x_k, \acute{\Phi}(x_k) - \acute{\Phi}(x_{k+1})\rangle \\ &\quad + 2\alpha_k^* \langle x_{k+1} - x_k, \acute{\Phi}(x_{k+1})\rangle + \alpha_k^2 \|\acute{\Phi}(x_k)\|^2. \end{aligned} \tag{2.13}$$

Using the L-Lipschitz continuity of $\acute{\Phi}$ and the convexity of Φ, we obtain

$$\langle x_{k+1} - x_k, \acute{\Phi}(x_k) - \acute{\Phi}(x_{k+1})\rangle \geq -L\|x_{k+1} - x_k\|^2,$$

and

$$\langle x_{k+1} - x_k, \acute{\Phi}(x_{k+1})\rangle \geq \Phi(x_{k+1}) - \Phi(x_k).$$

Utilizing the above estimates in (2.13), we get

$$\begin{aligned}\|x_{k+1} - z_k\|^2 &\geq \|x_{k+1} - x_k\|^2 - 2\alpha_k^* L\|x_k - x_{k+1}\|^2 + 2\alpha_k^*(\Phi(x_{k+1}) - \Phi(x_k)) + \alpha_k^2\|\acute{\Phi}(x_k)\|^2 \\ &= (1 - 2L\alpha_k^*)\|x_{k+1} - x_k\|^2 + 2\alpha_k^*(\Phi(x_{k+1}) - \Phi(x_k)) + \alpha_k^2\|\acute{\Phi}(x_k)\|^2. \end{aligned} \tag{2.14}$$

Hence from (2.11), (2.12) and (2.14), we get

$$\begin{aligned}\|x_{k+1} - \bar{u}\|^2 &\leq \|x_k - \bar{u}\|^2 - 2\alpha_k^*\langle x_k - \bar{u}, \acute{\Phi}(x_k)\rangle - (1 - 2L\alpha_k^*)\|x_{k+1} - x_k\|^2 \\ &\quad -2\alpha_k^*(\Phi(x_{k+1}) - \Phi(x_k)). \end{aligned}$$

Rearranging the above statement, we have

$$\begin{aligned}&\|x_{k+1} - \bar{u}\|^2 + 2\alpha_{k+1}^*(\Phi(x_{k+1}) - \inf\Phi) \\ &\quad \leq \|x_k - \bar{u}\|^2 + 2\alpha_k^*(\Phi(x_k) - \inf\Phi) - 2\alpha_k^*\langle \bar{q}_k - \bar{u}, \acute{\Phi}(x_k)\rangle \\ &\qquad - (1 - 2L\alpha_k^*)\|x_{k+1} - x_k\|^2 - 2(\alpha_k^* - \alpha_{k+1}^*)(\Phi(x_{k+1}) - \inf\Phi). \end{aligned}$$

Note that, if (α_k^*) is non-increasing then we have $(\alpha_k^* - \alpha_{k+1}^*)(\Phi(x_{k+1}) - \inf\Phi) \geq 0$. That is

$$\begin{aligned}&\frac{1}{2}\|x_{k+1} - \bar{u}\|^2 + \alpha_{k+1}^*(\Phi(x_{k+1}) - \inf\Phi) \\ &\quad \leq \frac{1}{2}\|x_k - \bar{u}\|^2 + \alpha_k^*(\Phi(x_k) - \inf\Phi) - \alpha_k^*\langle x_k - \bar{u}, \acute{\Phi}(x_k)\rangle \\ &\qquad - \frac{1}{2}(1 - 2L\alpha_k^*)\|x_{k+1} - x_k\|^2. \end{aligned}$$

This is the required result. □

The following results can easily be adopted from [15, Lemma 2.2 & 2.3].

Lemma 2.3. *Let Φ be a convex, bounded below and Gâteaux differentiable function on a real Hilbert space $\mathcal{H}$ with the derivative $\acute{\Phi}$. Let $\mathcal{D}$ be a nonempty closed and convex subset of $\mathcal{H}$ and for each $i = \{1, 2, \cdots, N\}$, let $S_i : \mathcal{D} \to \mathcal{CB}(\mathcal{D})$ be a finite family of η_i-demicontractive multivalued mappings satisfying the endpoint condition and demiclosedness principle where $(\eta_i)_{i=1}^N \subset (0, 1)$. Let $\Pi = \bigcap_{i=1}^N Fix(S_i) \neq \emptyset$. Assume that the condition (C5) holds, then for all $k \geq 0$, any $\varepsilon \in (0, 2)$ and for any $\bar{u} \in \Pi$ the sequence (x_k) given by (2.6) satisfies*

$$\langle x_k - \bar{u}, \acute{\Phi}(x_k)\rangle \geq \frac{1}{1 + \Psi\varepsilon\alpha_k^*}(\Psi\varepsilon U_k - (D_\varepsilon + d\Psi\varepsilon\alpha_k^*)), \tag{2.15}$$

where

$$U_k := \frac{1}{2}\|x_k - \bar{u}\|^2 + \alpha_k^*(\Phi(x_k) - \inf \Phi),$$
$$d := \Phi(\bar{u}) - \inf \Phi,$$
$$D_\varepsilon := \frac{\|\acute{\Phi}(\bar{u})\|^2}{2(2-\varepsilon)\Psi}.$$

Assume further that the conditions (C1) and (C4) hold and suppose $(\alpha_k^*) \subset (0, \frac{1}{2L}]$ *(when* $L \neq 0$*). Then we have for all* $k \geq 0$*,*

$$U_k \leq U_0 e^{-\frac{\Psi\varepsilon}{1+\Psi\varepsilon\alpha_0}(\sum_{r=0}^{k}\alpha_r - \alpha_0)} + (D_\varepsilon + d\Psi\varepsilon\alpha_0)\frac{1+2\Psi\varepsilon\alpha_0}{\Psi\varepsilon}e^{\frac{2\Psi\varepsilon}{1+\Psi\varepsilon\alpha_0}}. \tag{2.16}$$

Lemma 2.4. *Let* Φ *be a convex, bounded below and Gâteaux differentiable function on a real Hilbert space* $\mathcal{H}$ *with the derivative* $\acute{\Phi}$*. Let* $\mathcal{D}$ *be a nonempty closed and convex subset of* $\mathcal{H}$ *and for each* $i = \{1, 2, \cdots, N\}$*, let* $S_i : \mathcal{D} \to \mathcal{CB}(\mathcal{D})$ *be a finite family of* η_i*-demicontractive multivalued mappings satisfying the endpoint condition and demiclosedness principle where* $(\eta_i)_{i=1}^N \subset (0,1)$*. Let* $\Pi = \bigcap_{i=1}^N Fix(S_i) \neq \emptyset$*. Assume that the conditions (C1)-(C2), (C4) and (C5) hold, then the sequence* (x_k) *generated by (2.6) is bounded.*

3. Main results

In this section, we first prove some preliminary results to establish the strong convergence of the Algorithm (2.6).

Lemma 3.5. *Let* Φ *be a convex, bounded below and Gâteaux differentiable function on a real Hilbert space* $\mathcal{H}$ *with the derivative* $\acute{\Phi}$*. Let* $\mathcal{D}$ *be a nonempty closed and convex subset of* $\mathcal{H}$ *and for each* $i = \{1, 2, \cdots, N\}$*, let* $S_i : \mathcal{D} \to \mathcal{CB}(\mathcal{D})$ *be a finite family of* η_i*-demicontractive multivalued mappings satisfying the endpoint condition and the demiclosedness principle where* $(\eta_i)_{i=1}^N \subset (0,1)$*. Let* $\Pi = \bigcap_{i=1}^N Fix(S_i) \neq \emptyset$*. Assume that the conditions (C3) and (C5) hold and the sequence* (x_k) *generated by (2.6) is bounded and satisfies* $\|x_{k+1} - x_k\| \to 0$*, then* $x_k \rightharpoonup \bar{u}$ *implies* $\bar{u} \in \Pi$ *and*

$$\liminf_{k\to\infty}\langle x_k - \bar{u}, \acute{\Phi}(\bar{u})\rangle \geq 0,$$

where $\bar{u}$ *is the solution of (1.1) or (1.2).*

Proof. Let (x_{k_m}) be a subsequence of (x_k) which converges weakly to an element x_* in $\mathcal{H}$. Assume that $\|x_{k+1} - x_k\| \to 0$, $\alpha_k^* \to 0$ and (x_k) is bounded. Consequently, $z_{k_m} := x_{k_m} - \alpha_{k_m}^*\acute{\Phi}(x_{k_m})$ converges weakly to x_*. Utilizing (C3) and the boundedness of $\acute{\Phi}(x_{k_m})$, we have $\alpha_{k_m}^*\|\acute{\Phi}(x_{k_m})\| \to 0$. From (2.6), we get

$$\sum_{i=1}^N \hbar_i\|w_{k_m}^{(i)} - z_{k_m}\| = \frac{1}{\beta_k}\|x_{k_m+1} - z_{k_m}\| \to 0, \ \{i = 1, 2, \cdots, N\}.$$

From the demiclosed principle and the endpoint condition for S_i, we obtain that $\{x_*\} = S_i x_*$, $i \in \{1, 2, \cdots, N\}$. Since $x_{k_m} \rightharpoonup x_* \in \mathcal{H}$ as $m \to \infty$, therefore we have $x_{k_m+1} \rightharpoonup x_*$ and $z_{k_m} \rightharpoonup x_*$ as $m \to \infty$. Hence $x_* \in \Pi$.

The term $\langle x_k - \bar{u}, \acute{\Phi}(\bar{u})\rangle$ is bounded, as (x_k) is bounded. Observe that

$$\liminf_{k\to\infty}\langle x_k - \bar{u}, \acute{\Phi}(\bar{u})\rangle = \lim_{m\to\infty}\langle x_{k_m} - \bar{u}, \acute{\Phi}(\bar{u})\rangle,$$

and hence $\liminf_{k\to\infty}\langle x_k - \bar{u}, \acute{\Phi}(\bar{u})\rangle = \langle x_* - \bar{u}, \acute{\Phi}(\bar{u})\rangle$. As $\bar{u}$ is a solution of (1.2), we have $\langle x_* - \bar{u}, \acute{\Phi}(\bar{u})\rangle \geq 0$. This is the required result. □

Lemma 3.6. *Let Φ be a convex, bounded below and Gâteaux differentiable function on a real Hilbert space $\mathcal{H}$ with the derivative $\acute{\Phi}$. Let $\mathcal{D}$ be a nonempty closed and convex subset of $\mathcal{H}$ and for each $i = \{1, 2, \cdots, N\}$, let $S_i : \mathcal{D} \to \mathcal{CB}(\mathcal{D})$ be a finite family of η_i-demicontractive multivalued mappings satisfying the endpoint condition and the demiclosedness principle where $(\eta_i)_{i=1}^N \subset (0,1)$. Let $\Pi = \bigcap_{i=1}^N Fix(S_i) \neq \emptyset$. Assume that the condition (C2), (C4) and (C5) hold and the sequence (x_k) generated by (2.6) has a subsequence (x_{k_m}) such that:*

(I) $(x_{k_m}) \subset \Gamma := \{x \in \mathcal{H} : \langle x - \bar{u}, \acute{\Phi}(x)\rangle \leq 0\}$, *where $\bar{u}$ is the solution of (1.1) or (1.2);*

(II) $\|x_{k_m+1} - x_{k_m}\| \to 0$ *as $k \to \infty$;*

Then the sequence (x_{k_m}) converges strongly to $\bar{u}$.

Proof. Observe that the condition (C5) implies that $\Psi\|x_{k_m} - \bar{u}\|^2 \leq \langle x_{k_m} - \bar{u}, \acute{\Phi}(x_{k_m}) - \acute{\Phi}(\bar{u})\rangle$. Then (I) infers that

$$\Psi\|x_{k_m} - \bar{u}\|^2 \leq -\langle x_{k_m} - \bar{u}, \acute{\Phi}(\bar{u})\rangle. \tag{3.17}$$

From (3.17), we obtain $\|x_{k_m} - \bar{u}\| \leq \frac{\acute{\Phi}(\bar{u})}{\Psi}$ so therefore, (x_{k_m}) as well as Γ are bounded. Consequently, a subsequence $(x_{k_m}) \in \mathcal{H}$ converges weakly to a point $x_* \in \mathcal{H}$ and utilizing (II), we obtain $\|x_{k_m} - x_{k_{m+1}}\| \to 0$ as $m \to \infty$. Moreover from (2.6), we have

$$\begin{aligned} \beta_{k_m}\Big\|z_{k_m} - \sum_{i=1}^N \hbar_i w_{k_m}^{(i)}\Big\| &\leq \beta_{k_m}\sum_{i=1}^N \hbar_i\|z_{k_m} - w_{k_m}^{(i)}\| \\ &= \frac{1}{\beta_{k_m}}\|x_{k_{m+1}} - z_{k_m}\| \to 0, \ as\ m \to \infty. \end{aligned} \tag{3.18}$$

By using (C4) and since $(\alpha_k^*) \to 0$, z_{k_m} converges weakly to $\bar{u}$. Note that $x_* \in \Pi$, (as proved in Lemma 3.5) and utilizing (3.17) and (1.2) entails

$$\limsup_{k\to+\infty}\|x_{k_m} - \bar{u}\|^2 \leq -(\frac{1}{\Psi})\langle x_* - \bar{u}, \acute{\Phi}(\bar{u})\rangle \leq 0.$$

Hence $\lim_{m\to+\infty}\|x_{k_m} - \bar{u}\| = 0$. This completes the proof. □

Lemma 3.7. *Let Φ be a convex, bounded below and Gâteaux differentiable function on a real Hilbert space $\mathcal{H}$ with the derivative $\acute{\Phi}$. Let $\mathcal{D}$ be a nonempty closed and convex subset of $\mathcal{H}$ and for each $i = \{1, 2, \cdots, N\}$, let $S_i : \mathcal{D} \to \mathcal{CB}(\mathcal{D})$ be a finite family of η_i-demicontractive multivalued mappings satisfying the endpoint condition and demiclosedness principle where $(\eta_i)_{i=1}^N \subset (0,1)$. Let $\Pi = \bigcap_{i=1}^N Fix(S_i) \neq \emptyset$. Assume that the conditions (C1)-(C5) hold. Assume that the sequence (x_k) given by (2.6) satisfies:*

(I) $\|x_{k+1} - x_k\| \to 0$,

(II) $\lim_{k\to\infty}\|x_k - \bar{u}\|$ *exists,*

where $\bar{u}$ is the solution of (1.1) or (1.2). Then the sequence (x_k) converges strongly to $\bar{u}$.

Proof. It is observed from Lemma 2.4 that (x_k) is bounded. Suppose that $\lim_{k\to\infty}\|x_k - \bar{u}\| = \mu > 0$ and utilizing Lemma 3.5, we have $\liminf_{k\to\infty}\langle x_k - \bar{u}, \acute{\Phi}(\bar{u})\rangle \geq 0$ and also from (C5), we get

$$\langle x_k - \bar{u}, \acute{\Phi}(x_k)\rangle \geq \Psi\|x_k - \bar{u}\|^2 + \langle x_k - \bar{u}, \acute{\Phi}(\bar{u})\rangle.$$

After simplification, we obtain

$$\liminf_{k\to\infty}\langle x_k - \bar{u}, \acute{\Phi}(x_k)\rangle \geq \Psi\mu^2.$$

Also, from Lemma 2.2, there exists $k_0 \geq 0$ such that for $k \geq k_0$,

$$V_{k+1} - V_k \leq -\alpha_k^*(\frac{1}{2}\Psi\mu^2),$$

where $V_k := \frac{1}{2}\|x_k - \bar{u}\|^2 + \alpha_k^*(\Phi(x_k) - \inf \Phi)$. This implies that

$$(\frac{1}{2}\Psi\mu^2)\sum_{m=k_0}^{k}\alpha_k^* \leq V_{k_0} - V_{k+1}.$$

The only logical reasoning for the above estimate to be true is $\mu = 0$. Because if $\sum \alpha_k^* = \infty$ then for $k \to \infty$ implies that the right hand side of the above estimate is bounded whereas the left hand side is unbounded. This completes the proof. □

Theorem 3.1. *Let Φ be a convex, bounded below and Gâteaux differentiable function on a real Hilbert space $\mathcal{H}$ with the derivative $\acute{\Phi}$. Let $\mathcal{D}$ be a nonempty closed and convex subset of $\mathcal{H}$ and for each $i = \{1, 2, \cdots, N\}$, let $S_i : \mathcal{D} \to \mathcal{CB}(\mathcal{D})$ be a finite family of η_i-demicontractive multivalued mappings satisfying the endpoint condition and the demiclosedness principle where $(\eta_i)_{i=1}^N \subset (0,1)$. Let $\Pi = \bigcap_{i=1}^N Fix(S_i) \neq \emptyset$. Assume that (C1)-(C5) hold then the sequence (x_k) given by (2.6) converges strongly to $\bar{u}$, where $\bar{u}$ is the unique solution of (1.1) or (1.2).*

Proof. It follows from Lemma 2.3 that if $V_k = \frac{1}{2}\|x_k - \bar{u}\|^2 + \alpha_k^*(\Phi(x_k) - \inf \Phi)$, then both (V_k) and (x_k) are bounded. Hence, there exists a constant $M \geq 0$ such that $\|\langle x_k - \bar{u}, \acute{\Phi}(x_k)\rangle\| \leq M$ for all $k \geq 0$, utilizing Lemma 2.2, it yields

$$V_{k+1} - V_k + \frac{1}{2}(1 - 2L\alpha_k^*)\|x_{k+1} - x_k\|^2 \leq M\alpha_k^*. \tag{3.19}$$

For simplification, we consider the following two cases:

Case A. In the first instance, we assume that (V_k) is monotone, i.e., for large enough k_0, $(V_k)_{k\geq k_0}$ is either non-increasing or non-decreasing. In addition, (V_k) is bounded and hence it is convergent. Using (C2), that $\lim_{k\to+\infty}\|x_k - \bar{u}\|$ exists. Utilizing (3.19) and $\lim_{k\to\infty}\|V_{k+1} - V_k\| = 0$, we have

$$\lim_{k\to\infty}\|x_{k+1} - x_k\| = 0. \tag{3.20}$$

Now consider the re-arranged estimate (2.9) and using (A2), we have

$$\begin{aligned}\beta_k(1 - \eta_i - \beta_k)\sum_{i=1}^N \hbar_i\|z_k - w_k^{(i)}\|^2 &\leq \|x_k - \bar{u}\|^2 - \|x_{k+1} - \bar{u}\|^2\\ &\leq (\|x_k - \bar{u}\| + \|x_{k+1} - \bar{u}\|)\|x_{k+1} - x_k\|.\end{aligned}$$

Letting $k \to \infty$ and utilizing (3.20), we have

$$\beta_k(1 - \eta_i - \beta_k)\sum_{i=1}^N \hbar_i\|z_k - w_k^{(i)}\|^2 = 0. \tag{3.21}$$

It is observed that

$$\sum_{i=1}^N \hbar_i\|w_k^{(i)} - z_k\| = \frac{1}{\beta_k}\|x_{k+1} - z_k\|,\ i = \{1, 2, \cdots, N\}.$$

The above estimate implies that

$$\lim_{k\to\infty}\|z_k - x_{k+1}\| = 0. \tag{3.22}$$

From (3.20), (3.22) and the following triangular inequality:

$$\|z_k - x_k\| \leq \|z_k - x_{k+1}\| + \|x_{k+1} - x_k\|,$$

we get

$$\lim_{k\to\infty}\|z_k - x_k\| = 0. \tag{3.23}$$

Hence from Lemma 3.7, we deduce that $\bar{u} \in \Pi$.
Case B. Conversely, suppose (V_k) is not monotone sequence and for all $k \geq k_0$ (for some k_0 large enough) let a mapping $g : \mathbb{N} \to \mathbb{N}$ defined by

$$g(k) := \max\{m \in \mathbb{N}; m \leq k, V_k \leq V_{k+1}\}. \tag{3.24}$$

Note that, g is a non-decreasing sequence imply that $g(k) \to +\infty$ as $k \to +\infty$ and $V_{g_k} \leq V_{g(k)+1}$ for $k \geq k_0$, so therefor by using (3.19), it yields

$$\frac{1}{2}(1 - 2L\alpha^*_{g(k)})\|x_{g(k)+1} - x_{g(k)}\|^2 \leq M\alpha_{g_k} \to 0, \tag{3.25}$$

hence, $\|x_{g(k)+1} - x_{g(k)}\| \to 0$. Utilizing Lemma 2.2, for any $n \geq 0$, the inequality $V_{n+1} < V_n$ holds provided that $x_n \notin \Gamma := \{x \in \mathcal{H}; \langle x - \bar{u}, \acute{\Phi}(x)\rangle \leq 0\}$. Consequently, we have $x_{g(k)} \in \Gamma$ for all $k \geq k_0$ (since $V_{g(k)} \leq V_{g(k)+1}$). By using Lemma 3.6, we conclude that $\|x_{g(k)} - \bar{u}\| \to 0$ and it follows that

$$\lim_{k\to\infty} V_{g(k)} = \lim_{k\to\infty} V_{g(k)+1} = 0.$$

Moreover, for $k \geq k_0$, it is mention that $V_k \leq V_{g(k)+1}$ if $k \neq g(k)$ that is , if $g(k) < k$, because we have $V_n > V_{n+1}$ for $g(k) + 1 \leq n \leq k - 1$. It follows that for all $k \geq k_0$, $0 \leq V_k \leq \max\{V_{g(k)}, V_{g(k)+1}\} \to 0$, hence $\lim_{k\to\infty} V_k = 0$. This completes the proof. □

4. Numerical experiment and results

This section provides effective viability of Algorithm 2.6 supported by a suitable example.

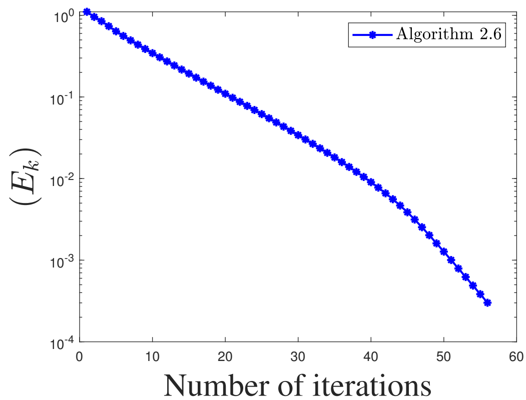
Example 4.1. Let $\mathcal{H} = \mathbb{R}$ the set of all real numbers with the inner product defined by $\langle x, y\rangle = xy$, for all $x, y \in \mathbb{R}$ and induced usual norm $|\cdot|$. Let $\mathcal{D} \subset \mathcal{H}$ and $\Phi : \mathbb{R} \to (-\infty, \infty]$ be defined by $\Phi(x) = \frac{1}{2}\|\tilde{f}x - g\|^2$ with $\tilde{A}x = 0 = g$. Then Φ is a proper, convex and lower semicontinuous function. Since $\tilde{A}$ is a continuous linear function (see [16]). Let a multivalued mapping $S : \mathbb{R} \to \mathcal{CB}(\mathcal{D})$ be defined as follows: if $x \in (-\infty, 0]$, take $Sx = [-(\frac{9x}{2}), -5x]$; if $x \in (0, \infty)$, take $Sx = [-5x, -(\frac{9x}{2})]$. In order to compute (x_{k+1}), for each $i \in \{1, 2, \cdots, N\}$, take $S_i = S$. By Example 2.2 in Ref. [14], we know that S is a multivalued demicontractive operator with a constant $\eta = \frac{96}{121}$. Choose $w_k^{(i)} = -5z_k$, $\beta_k = \frac{1}{100k+1}$, $N = 2 \times 10^5$. We compute our HSDA defined in Theorem 3.1 (i.e., Algorithm 2.6) for different initial inputs. The stopping criteria is defined as Error=$E_k = \|x_k - x_{k-1}\| < 10^{-4}$. The different cases of initial inputs are given as follows:
Case I: $x_0 = 4$, $\alpha^* = \frac{1}{6}$

Case II: $x_0 = -2$, $\alpha^* = \frac{1}{4}$

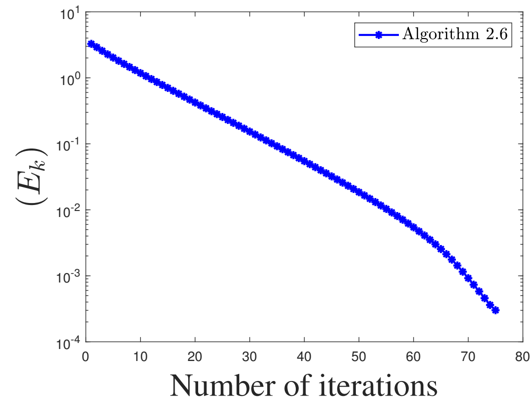
Case III: $x_0 = 2.7$, $\alpha^* = \frac{1}{2}$

Table 1. Computation of Multivalued HSDA.

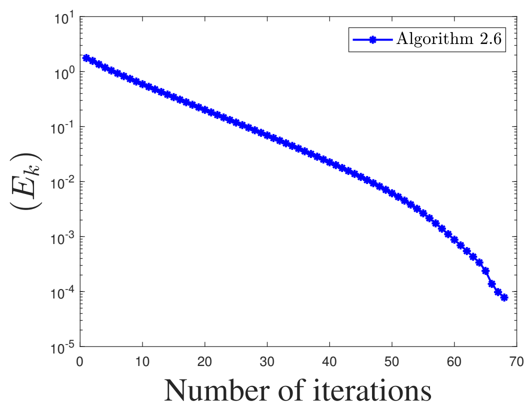
	No. of Iterations			CPU Time		
	Case I	Case II	Case III	Case I	Case II	Case III
Multivalued HSDA	57	75	68	0.049071	0.031480	0.039675



(A) Case I



(B) Case II



(C) Case III

FIGURE 1. Computational results of Multivalued HSDA

The error plotting $\|x_k - x_{k-1}\|$ of Multivalued HSDA are illustrated in Figure 1.

Conclusion. In this paper, we have devised a HSDA for computing the convex minimization problems over the set of FPP for multivalued mappings in Hilbert space. The theoretical framework of the algorithm has been strengthened with an appropriate numerical example. We would like to emphasize that the above mentioned problems occur naturally in many applications, therefore, iterative algorithms are inevitable in this field of investigation. As a consequence, our theoretical framework constitutes an important topic of future research.

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REFERENCES

[1] Arfat, Y.; Kumam, P.; Ngiamsunthorn, P. S.; Khan, M. A. A. An inertial based forward-backward algorithm for monotone inclusion problems and split mixed equilibrium problems in Hilbert spaces. *Adv. Differ. Equ.* **2020**, Paper No. 453, 25 pp.

[2] Arfat, Y.; Kumam, P.; Ngiamsunthorn, P. S.; Khan, M. A. A.; Sarwar, H.; Din, H. F. Approximation results for split equilibrium problems and fixed point problems of nonexpansive semigroup in Hilbert spaces. *Adv. Differ. Equ.* **2020**, Paper No. 512, 21 pp.

[3] Arfat, Y.; Kumam, P.; Khan, M. A. A.; Ngiamsunthorn, P. S.; Kaewkhao, A. An inertially constructed forward-backward splitting algorithm in Hilbert spaces. *Adv. Differ. Equ.* **2021**, Paper No. 124, 23 pp.

[4] Arfat, Y.; Kumam, P.,; Khan, M. A. A.; Ngiamsunthorn, P. S.; Kaewkhao, A. A parallel hybrid accelerated extragradient algorithm for pseudomonotone equilibrium, fixed point, and split null point problems. *Adv. Differ. Equ.* **2021**, Paper No. 364, 19 pp.

[5] Arfat, Y.; Kumam, P.; Khan, M. A. A.; Ngiamsunthorn, P. S. An accelerated projection based parallel hybrid algorithm for fixed point and split null point problems in Hilbert spaces. *Math. Meth. Appl. Sci.* (2021), 1–19.

[6] Arfat, Y.; Kumam, P.; Khan, M. A. A.; Ngiamsunthorn, P. S. Parallel shrinking inertial extragradient approximants for pseudomonotone equilibrium, fixed point and generalized split null point problem. *Ricerche De Math.* (2021). https://doi.org/10.1007/s11587-021-00647-4.

[7] Arfat, Y., Kumam, P., Khan, M.A.A. and Iyiola, O.S., *Multi-inertial parallel hybrid projection algorithm for generalized split null point problems*, J. Appl. Math. Comput. (2021). https://doi.org/10.1007/s12190-021-01660-4

[8] Arfat, Y.; Kumam, P.; Khan, M. A. A.; Ngiamsunthorn, P. S.; Kaewkhao, A. Shrinking approximants for fixed point problem and generalized split null point problem in Hilbert spaces. *Optim. Lett.* **16** (2022) 1895–1913.

[9] Arfat, Y.; Kumam, P.; Khan, M. A. A.; Ngiamsunthorn, P. S. An accelerated Visco-Cesaro means Tseng Type splitting method for fixed point and monotone inclusion problems. *Carpathian J. Math.* **38** (2022), no. 2, 281–297.

[10] Arfat, Y.; Iyiola, O. S.; Khan, M. A. A.; Kumam, P.; Kumam, W.; Sitthithakerngkiet, K. Convergence analysis of the shrinking approximants for fixed point problem and generalized split common null point problem. *J. Inequal. Appl.* **67** (2022), 1–21.

[11] Arfat, Y.; Khan, M. A. A.; Kumam, P.; Kumam, W.; Sitthithakerngkiet, K. Iterative solutions via some variants of extragradient approximants in Hilbert spaces. *AIMS Math.* **7** (2022), no. 8, 13910–13926.

[12] Chidume, C. E.; Bello, A. U.; Ndambomve. P. Strong and Δ-convergence theorems for common fixed points of a finite family of multivalued demicontractive mappings in CAT(0) spaces. *Abstr Appl Anal.* (2014), Article ID: 805168(2014).

[13] Ekeland, I.; Themam, R. Convex analysis and variational problems. *Classic Appl. Math.* SIAM, **28** (1999).

[14] Jailoka, P.; Suantai, Suthep. Split null point problems and fixed point problems for demicontractive multivalued mappings. *Mediterr. J. Math.* **15** (2018), 1–19.

[15] Maingé, P. E. Convex minimization over the fixed point set of Demicontractive mappings. *Positivity* **12** (2008), 269–280.

[16] Martinet, B. Regularisation d'inequalites varaiationnelles par approximations successives. *Rev. Fr. Inform. Rec. Oper.* **4** (1970), 154-158.

[17] Yamada, I. The hybrid steepest descent method for the variational inequality over the intersection of fixed point sets of nonexpansive mappings. In: Butnariu, D., Censor, Y. and Reich, S. (eds.) *Inherently Parallel Algorithm for Feasibility and Optimization and their Applications*. Elsevier, 473-504 (2001).

[18] Yamada, I.; Ogura, N. Hybrid steepest descent method for the variational inequality problem over the fixed point set of certain quasi-nonexpansive mappings. *Num. Funt. Anal. Optim.* **25** (2004), no. (7-8), 619–655.

[19] Yamada, I.; Ogura, N.; Shirakawa, N. A numerically robust hybrid steepest descent method for the convexly constrained generalized inverse problems. In: Nashed, Z., Scherzer, O. (eds) *Inverse Problems, Image Analysis and Medical imaging*. Contemporary Mathematics, 313, 269-305 (2002).

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