CARPATHIAN J. MATH.
Volume **39** (2023), No. 1,
Pages 325 - 334

Online version at https://semnul.com/carpathian/
Print Edition: ISSN 1584 - 2851; Online Edition: ISSN 1843 - 4401
DOI: https://doi.org/10.37193/CJM.2023.01.23

In memoriam Professor Charles E. Chidume (1947- 2021)

A novel approach for solving simultaneously one-parameter nonexpansive semigroup, convex minimization and fixed point problems involving set-valued operators

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ABSTRACT. In this paper, we introduce a new iterative process for solving simultaneously one-parameter nonexpansive semigroup, convex minimization and fixed point problems involving set-valued operators in real Hilbert spaces and establish strong convergence theorems for the proposed iterative process. There is no compactness assumption. Our results improve important recent results.

1. INTRODUCTION

The theory of one-parameter semigroups of linear operators on in Hilbert spaces started in the first half of this century, acquired its core in 1948 with the Hille-Yosida generation theorem, and attained its first apex with the 1957 edition of Semigroups and Functional Analysis by E. Hille and R. S. Phillips. Semigroups have become important tools for integro-differential equations and functional differential equations, in quantum mechanics or in infinite-dimensional control theory. Semigroup methods are also applied with great success to concrete equations arising, e.g., in population dynamics or transport theory (see, for example, [14] and the references contained in them).

One parameter family of mappings $S := \{G(t) : 0 \leq t < \infty\}$ is called a continuous Lipschitzian semigroup on K (see e.g., [2]), if the following conditions are satisfied:

(a) $G(0)x = x$ for all $x \in K$;
(b) $G(s+t) = G(s)G(t)$ for all $s, t \geq 0$;
(c) for each $t > 0$, there exists a bounded measurable function $L_t : (0, \infty) \to [0, \infty)$ such that $\|G(t)x - G(t)y\| \leq L_t \|x - y\|, \ \forall\, x, y \in K$;
(d) for each $x \in K$, the mapping $G(.)x$ from $[0, \infty)$ into K is continuous.

A Lipschitzian semigroup S is called nonexpansive if $L_t = 1$ for all $t > 0$. Let $Fix(S)$ denote the common fixed point set of the semigroup S i.e. $Fix(S) := \{x \in K : \ G(t)x = x, \forall t > 0\}$. A simple example of nonexpansive semigroup on a Hilbert space is shown here. Let $G(t) : \mathbb{R}^3 \to \mathbb{R}^3$ be a rotation in $\mathbb{R}^3$ defined by, for $\alpha \in \mathbb{R}$ and $t \geq 0$,

$$G(t) = \begin{pmatrix} \cos(\alpha t) & -\sin(\alpha t) & 0 \\ \cos(\alpha t) & \sin(\alpha t) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.1}$$

Received: 24.04.2022. In revised form: 02.05. 2022. Accepted: 30.07.2022
2010 *Mathematics Subject Classification.* 47H09, 47H10, 65C25.
Key words and phrases. *Convex minimization problem,Proximal point algorithm, One-parameter nonexpansive semigroup,Quasi-nonexpansive mappings,Set-valued operators.*
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Then $S = \{G(t) : t \geq 0\}$ is a nonexpansive semigroup on $\mathbb{R}^3$ with the common fixed point set $Fix(S) = \{x \in \mathbb{R}^3 : \ (0, 0, x_3)^T\}$.
Let (E, d) be a metric space, K be a nonempty subset of E and $T : K \to 2^K$ be a multivalued mapping. An element $x \in K$ is called a fixed point of T if $x \in Tx$. For single valued mapping, this reduces to $Tx = x$. The fixed point set of T is denoted by $Fix(T) := \{x \in D(T) : x \in Tx\}$. For several years, the study of fixed point theory for single-valued and multivalued nonlinear mappings has attracted, and continues to attract, the interest ofseveral well-known mathematicians (see, for example, Brouwer [5], Sow, Djitte, and Chidume [23], and Gorniewicz [11]).

Many problems arising in different areas of mathematics such as optimization, variational analysis, differential equations, mathematical economics, control theory, optimization, calculus of variations, non-smooth and convex analysis, game theory, mathematical economics and in other fields can be modeled as fixed point equations of the form:

$$x \in Tx, \tag{1.2}$$

where T is a set-valued mapping. In the last decades, many effective algorithms for solving (1.2) are developed.

Let D be a nonempty subset of a normed space E. The set D is called *proximinal* (see, *e.g.*, [13]) if for each $x \in E$, there exists $u \in D$ such that

$$d(x, u) = \inf\{\|x - y\| : y \in D\} = d(x, D),$$

where $d(x, y) = \|x - y\|$ for all $x, y \in E$. Every nonempty, closed and convex subset of a real Hilbert space is proximinal. Let $CB(D)$, $K(D)$ and $P(D)$ denote the family of nonempty closed bounded subsets, nonempty compact subsets, and nonempty proximinal bounded subsets of D respectively. The Pompeiu *Hausdorff metric* on $CB(K)$ is defined by:

$$H(A, B) = \max\left\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\right\}$$

for all $A, B \in CB(K)$ (see, Berinde [3]). A multi-valued mapping $T : D(T) \subseteq E \to CB(E)$ is called *L- Lipschitzian* if there exists $L > 0$ such that

$$H(Tx, Ty) \leq L\|x - y\| \ \ \forall x, y \in D(T). \tag{1.3}$$

When $L \in (0, 1)$, we say that T is a *contraction*, and T is called *nonexpansive* if $L = 1$.
A multivalued map T is called quasi-nonexpansive if

$$H(Tx, Tp) \leq \|x - p\|$$

holds for all $x \in D(T)$ and $p \in F(T)$.

Remark 1.1. It is easy to see that the class of mulivalued quasi-nonexpansive mappings properly includes that of multivalued nonexpansive maps with fixed points.

The minimization problem (MP) is one of the most important problems in nonlinear analysis and optimization theory. The MP is defined as follows: find $x \in H$, such that

$$g(x) = \min_{y \in H} g(y),$$

where $g : H \to (-\infty, +\infty]$ is a proper convex and lower semi-continuous. The set of all minimizers of g on H is denoted by $\text{argmin}_{y \in H}\, g(y)$. A successful and powerful tool for solving this problem is the well-known Proximal Point Algorithm (shortly, the PPA) which was initiated by Martinet [15] in 1970 and later studied by Rockafellar [18] in 1976. The PPA is defined as follows:

$$\begin{cases} x_1 \in H, \\ x_{n+1} = \text{argmin}_{y \in H} \Big[g(y) + \frac{1}{2\lambda_n} \|x_n - y\|^2 \Big], \end{cases} \tag{1.4}$$

where $\lambda_n > 0$ for all $n \geq 1$. In [18] Rockafellar proved that the sequence $\{x_n\}$ given by (1.4) converges weakly to a minimizer of g. He then posed the following question:
Q1: does the sequence $\{x_n\}$ converges strongly? This question was resolved in the negative by Güler [19] (1991). He produced a proper lower semi continuous and convex function g in l_2 for which the PPA *converges weakly* but not *strongly*. This leads naturally to the following question:
Q2: Can the PPA be modified to guarantee *strong convergence*? In response to Q2, several works have been done (see, *e.g.,* Güler [19], Kamimura and Takahashi [20], Chidume and Djitte [9] and the references therein). In the recent years, the problem of finding a common element of the set of solutions of convex minimization, variational inequality and the set of fixed point problems in real Hilbert spaces, Banach spaces and complete CAT(0) (Hadamard) spaces have been intensively studied by many authors; see, for example, [10, 22] and the references therein.

Motivated and inspired by the ongoing results in this field, we introduce a new iterative approach and prove a strong convergence theorem for convex minimization, one-parameter nonexpansive semigroup and fixed point problems with multivalued quasi-nonexpansive mappings in Hilbert spaces. Finally, our method of proof is of independent interest.

2. Preliminaries

Let H be a real Hilbert space and K be a nonempty convex subset of H. Let $g : K \to (-\infty, +\infty]$ be a proper, lower semi-continuous and convex function. For every $\lambda > 0$, the Moreau-Yosida resolvent of g, J_λ^g is defined by:

$$J_\lambda^g x = \text{argmin}_{u \in K} \Big[g(u) + \frac{1}{2\lambda} \|x - u\|^2 \Big],$$

for all $x \in H$. It was shown in [19] that the set of fixed points of the resolvent associated to g coincides with the set of minimizers of g. Also, the resolvent J_λ^g of g is nonexpansive for all $\lambda > 0$.

Lemma 2.1. (Miyadera [17]) *Let H be a real Hilbert space and K be a nonempty convex subset of H. Let $g : K \to (-\infty, +\infty]$ be a proper, lower semi-continuous and convex function. For every $r > 0$ and $\mu > 0$, the following holds:*

$$J_r^g x = J_\mu^g \big(\frac{\mu}{r} x + (1 - \frac{\mu}{r}) J_r^g x\big).$$

Lemma 2.2 (Sub-differential inequality, Ambrosi *et al.,* [1])**.** *Let H be a real Hilbert space and and $g : H \to (-\infty, +\infty]$ be a proper, lower semicontinuous and convex function. Then, for every $x, y \in H$ and $\lambda > 0$, the following sub-differential inequality holds:*

$$\frac{1}{\lambda} \|J_\lambda^g x - y\|^2 - \frac{1}{\lambda} \|x - y\|^2 + \frac{1}{\lambda} \|x - J_\lambda^g x\|^2 + g(J_\lambda^g x) \leq f(y). \tag{2.5}$$

Definition 2.1. Let H be a real Hilbert space and $T : D(T) \subset H \to 2^H$ be a multivalued mapping.The multivalued map $I - T$ is said to be demiclosed at 0 if for any sequence $\{x_n\} \subset D(T)$ such that $\{x_n\}$ converges weakly to p and $d(x_n, Tx_n)$ converges to zero, then $p \in Tp$, where I is the identity map of H.

Lemma 2.3 (Chidume, [8]). *Let H be a real Hilbert space.Then,for every $x, y \in H$, and every $\lambda \in [0,1]$, the following hold:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle.$$

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - (1-\lambda)\lambda\|x-y\|^2, \ \ \lambda \in (0,1).$$

Lemma 2.4 (Xu, [24]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \leq (1-\alpha_n)a_n + \sigma_n$ for all $n \geq 0$, where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\sigma_n\}$ is a sequence in $\mathbb{R}$ such that*
(a) $\displaystyle\sum_{n=0}^{\infty} \alpha_n = \infty$, (b) $\displaystyle\limsup_{n\to\infty} \frac{\sigma_n}{\alpha_n} \leq 0$ *or* $\displaystyle\sum_{n=0}^{\infty} |\sigma_n| < \infty$. *Then* $\displaystyle\lim_{n\to\infty} a_n = 0$.

Lemma 2.5 (Mainge, [16]). *Let $\{t_n\}$ be a sequence of real numbers that does not decreases at infinity in the sense that there exists a subsequence $\{t_{n_i}\}$ of*
$\{t_n\}$ such that $t_{n_i} \leq t_{n_{i+1}}$ for all $i \geq 0$. For $n \in \mathbb{N}$, sufficiently large, let $\{\tau(n)\}$ be the sequence of integers defined as follows:

$$\tau(n) = \max\{k \leq n : \ t_k \leq t_{k+1}\}.$$

Then, $\tau(n) \to \infty$ as $n \to \infty$ and

$$\max\{t_{\tau(n)}, \ t_n\} \leq t_{\tau(n)+1}.$$

Lemma 2.6 ([21]). *Let D be a nonempty, bounded, closed and convex subset of a real Hilbert space H and let $S := \{G(u) : 0 \leq u < \infty\}$ be a one-parameter nonexpansive semigroup on D, then for any $h \geq 0$,*

$$\lim_{t\to 0} \sup_{x\in D} \Big\| G(h)\Big(\frac{1}{t}\int_0^t G(u)x du\Big) - \Big(\frac{1}{t}\int_0^t G(u)x du\Big)\Big\| = 0.$$

3. Main Results

We start by the following result.

Lemma 3.7. *Let H be a real Hilbert space and K be a nonempty closed convex subset of H. Let $F : K \to (-\infty, +\infty]$ be a proper, lower semi-continuous and convex function and $T : K \to CB(K)$ be a multivalued quasi-nonexpansive mapping such that $Tp = \{p\}$ $\forall\, p \in Fix(T)$. Then, $Fix(T \circ J_\lambda^F) = Fix(T) \cap argmin_{u\in K} F(u)$ and $T \circ J_\lambda^F$ is a multivalued quasi-nonexpansive mapping.*

Proof. We split the proof into two steps.
Step 1: First, we show that $Fix(T) \cap \text{argmin}_{u\in K} F(u) = Fix(T \circ J_\lambda^F)$. We note that $Fix(T) \cap \text{argmin}_{u\in K} F(u) = Fix(T) \cap Fix(J_\lambda^F) \subset Fix(T \circ J_\lambda^F)$. Thus, we only need to show that $Fix(T \circ J_\lambda^F) \subseteq Fix(T) \cap Fix(J_\lambda^F)$. Let $p \in Fix(T) \cap Fix(J_\lambda^F)$ and $q \in Fix(T \circ J_\lambda^F)$. By using properties of T and J_λ^F, we have

$$\begin{aligned} \|q-p\|^2 &\leq H(T \circ J_\lambda^F q, Tp)^2 \\ &\leq \|J_\lambda^F q - p\|^2. \end{aligned} \tag{3.6}$$

Using the fact that J_λ^F is firmly nonexpansive, we have

$$\begin{aligned} \|J_\lambda^F q - p\|^2 &\leq \langle J_\lambda^F q - p, q - p\rangle \\ &= \frac{1}{2}(\|J_\lambda^F q - p\|^2 + \|q-p\|^2 - \|J_\lambda^F q - q\|^2), \end{aligned}$$

which yields

$$\|J_\lambda^F q - p\|^2 \leq \|q-p\|^2 - \|J_\lambda^F q - q\|^2. \tag{3.7}$$

Using (3.6) implies that (3.7) becomes

$$\begin{aligned} \|J_\lambda^F q - p\|^2 &\leq \|q-p\|^2 - \|J_\lambda^F q - q\|^2 \\ &\leq \|J_\lambda^F q - p\|^2 - \|J_\lambda^F q - q\|^2. \end{aligned}$$

Clearly, $\|J_\lambda^F q - q\| = 0$, which implies that

$$q = J_\lambda^F q.$$

We obtain,

$$q = J_\lambda^F q \in T \circ J_\lambda^F q = Tq.$$

Thus, $q \in Fix(T) \cap Fix(J_\lambda^F)$. Hence, $Fix(T) \cap \text{argmin}_{u \in K} F(u) = Fix(T \circ J_\lambda^F)$.
Step 2: We show $T \circ J_\lambda^F$ is a quasi-nonexpansive mapping on K. Let $x \in K$ and $p \in Fix(T \circ J_\lambda^F)$. Then, $p \in Fix(T) \cap Fix(J_\lambda^F)$ by step 1. We observe that,

$$\begin{aligned} H(T \circ J_\lambda^F x, T \circ J_\lambda^F p) &= H(T \circ J_\lambda^F x, Tp) \\ &\leq \|J_\lambda^F x - p\| \\ &\leq \|x - p\|. \end{aligned}$$

This completes the proof. □

We now apply Lemma 3.7 for solving simultaneously one-parameter nonexpansive semigroup, convex minimization and fixed point problems involving set-valued operators in real Hilbert spaces.

Theorem 3.1. *Let K be a nonempty, closed convex subset of real a Hilbert space H. Let $F : K \to \mathbb{R} \cup \{+\infty\}$ be a proper, lower semi-continuous and convex function and $f : K \to K$ be an b-contraction mapping. Let $T : K \to CB(K)$ be a multivalued quasi-nonexpansive mapping and $S := \{G(u) : 0 \leq u < \infty\}$ be a one-parameter nonexpansive semigroup on K such that $\Gamma := Fix(S) \cap Fix(T) \cap argmin_{u \in K} F(u) \neq \emptyset$. Let α_n, θ_n and β_n be three sequences in $(0, 1)$. Let $\{x_n\}$ be a sequence defined as follows:*

$$\begin{cases} x_0 \in K, \\ z_n = \theta_n x_n + (1-\theta_n) u_n, \quad u_n \in T \circ J_\lambda^F x_n, \\ y_n = \beta_n z_n + (1-\beta_n) \dfrac{1}{t_n} \displaystyle\int_0^{t_n} G(u) z_n \, du, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) y_n. \end{cases} \tag{3.8}$$

Assume that $I - T \circ J_\lambda^F$ is demiclosed at origin and $Tp = \{p\}, \ \forall p \in \Gamma$. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\theta_n\}$ are the sequences such that:

(i) $\lim\limits_{n\to\infty} \alpha_n = 0, \quad \sum\limits_{n=0}^{\infty} \alpha_n = \infty,$
(ii) $\lim\limits_{n\to\infty} \inf (1-\theta_n)\theta_n > 0,$
(iii) $\lim\limits_{n\to\infty} \inf (1-\beta_n)\beta_n > 0.$

Then, the sequence $\{x_n\}$ defined by (3.8) *converges strongly to $x^* \in Fix(S) \cap Fix(T) \cap argmin_{u \in K} F(u)$, which solves the following variational inequality:*

$$\langle x^* - f(x^*), x^* - p \rangle \leq 0, \quad \forall p \in Fix(S) \cap Fix(T) \cap argmin_{u \in K} F(u). \tag{3.9}$$

Proof. From $(I-f)$ is strongly monotone, then the variational inequality (3.9) has a unique solution in Γ. In what follows, we denote x^* to be the unique solution of (3.9). Now we show that $\{x_n\}$ is bounded. Let $p \in \Gamma$. Using (3.8), the fact that $Tp = \{p\}$, Lemma 3.7 and Lemma 2.3, we have

$$\begin{aligned}
\|z_n - p\|^2 &= \|\theta_n x_n + (1-\theta_n)u_n - p\|^2 \\
&= \theta_n\|x_n - p\|^2 + (1-\theta_n)\|u_n - p\|^2 - (1-\theta_n)\theta_n\|x_n - u_n\|^2 \\
&\leq \theta_n\|x_n - p\|^2 + (1-\theta_n)H(T(J_\lambda^F x_n), Tp)^2 - (1-\theta_n)\theta_n\|x_n - u_n\|^2 \\
&\leq \theta_n\|x_n - p\|^2 + (1-\theta_n)\|J_\lambda^F x_n - p\|^2 - (1-\theta_n)\theta_n\|x_n - u_n\|^2 \\
&\leq \|x_n - p\|^2 - (1-\theta_n)\theta_n\|x_n - u_n\|^2.
\end{aligned}$$

Hence,

$$\|z_n - p\|^2 \leq \|x_n - p\|^2 - (1-\theta_n)\theta_n\|x_n - u_n\|^2 \tag{3.10}$$

From (3.8), we have

$$\|y_n - p\| = \|\beta_n z_n + (1-\beta_n)\frac{1}{t_n}\int_0^{t_n} G(u)z_n\, du - p\| = \|\beta_n(z_n - p) + (1-\beta_n)\Big(\frac{1}{t_n}\int_0^{t_n}[G(u)z_n - G(u)p]du\Big)\|$$

$$\leq \beta_n\|z_n - p\| + (1-\beta_n)\|\frac{1}{t_n}\int_0^{t_n}[G(u)z_n - G(u)p]du\| \leq \beta_n\|z_n - p\| + (1-\beta_n)\|z_n - p\| \leq \|z_n - p\|.$$

Therefore, we have

$$\|y_n - p\| \leq \|z_n - p\| \leq \|x_n - p\|. \tag{3.11}$$

By using (3.8) and (3.11), we obtain

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n f(x_n) + (1-\alpha_n)y_n - p\| \\
&\leq \alpha_n\|f(x_n) - f(p)\| + (1-\alpha_n)\|y_n - p\| + \alpha_n\|f(p) - p\| \\
&\leq (1-\alpha_n(1-b))\|x_n - p\| + \alpha_n\|f(p) - p\| \\
&\leq \max\{\|x_n - p\|, \frac{\|f(p) - p\|}{1-b}\}.
\end{aligned}$$

By induction, we conclude that

$$\|x_n - p\| \leq \max\{\|x_0 - p\|, \frac{\|f(p) - p\|}{1-b}\}, \quad n \geq 1.$$

Hence $\{x_n\}$ is bounded, also $\{y_n\}$ and $\{f(x_n)\}$ are all bounded.
Thus we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \|\alpha_n f(x_n) + (1-\alpha_n)y_n - p\|^2 \\
&\leq \alpha_n\|f(x_n) - p\|^2 + (1-\alpha_n)\|y_n - p\|^2 \\
&\leq \alpha_n\|f(x_n) - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 - (1-\alpha_n)(1-\theta_n)\theta_n\|u_n - x_n\|^2.
\end{aligned}$$

Since $\{x_n\}$ is bounded, then there exists a constant $C > 0$ such that

$$(1-\alpha_n)(1-\theta_n)\theta_n\|u_n - x_n\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n C. \tag{3.12}$$

Now, we prove that $\{x_n\}$ converges strongly to x^*. We divide the rest of the proof into two cases.

Case 1. Assume that the sequence $\{\|x_n - p\|\}$ is monotonically decreasing. Then $\{\|x_n - p\|\}$ is convergent. Clearly, we have

$$\lim_{n\to\infty}\big[\|x_n - p\|^2 - \|x_{n+1} - p\|^2\big] = 0. \tag{3.13}$$

Using the fact that $\lim_{n\to\infty}\inf(1-\theta_n)\theta_n > 0$, we have

$$\lim_{n\to\infty}\|u_n - x_n\| = 0. \tag{3.14}$$

Since $u_n \in T \circ J_\lambda^F x_n$, it follows that

$$\lim_{n\to\infty} d(x_n,\, T \circ J_\lambda^F x_n) = 0. \tag{3.15}$$

Noticing that

$$\begin{aligned}
\|z_n - x_n\| &= \|\theta_n x_n + (1-\theta_n)u_n - x_n\| \\
&= \|\theta_n x_n + (1-\theta_n)u_n - \theta_n x_n - (1-\theta_n)x_n\| \\
&= (1-\theta_n)\|u_n - x_n\| \\
&\leq \|u_n - x_n\|.
\end{aligned}$$

Therefore, from (3.16) we get that

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{3.16}$$

From Lemma 2.3, properties of one-parameter nonexpansive semigroup S and (3.11), we have

$$\begin{aligned}
\|y_n - x^*\|^2 &= \|\beta_n z_n + (1-\beta_n)\frac{1}{t_n}\int_0^{t_n} G(u)z_n\, du - x^*\| \\
&= \beta_n\|z_n - x^*\|^2 + (1-\beta_n)\|z_n - x^*\|^2 \\
&\quad -(1-\beta_n)\beta_n\|\frac{1}{t_n}\int_0^{t_n} G(u)z_n\, du - z_n\|^2 \\
&\leq \|x_n - x^*\|^2 - (1-\beta_n)\beta_n\|\frac{1}{t_n}\int_0^{t_n} G(u)z_n\, du - z_n\|^2.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|\alpha_n f(x_n) + (1-\alpha_n)y_n - x^*\|^2 \\
&= \|(1-\alpha_n)(y_n - x^*) + \alpha_n(f(x_n) - x^*)\|^2 \\
&\leq (1-\alpha_n)\|y_n - x^*\|^2 + \alpha_n\|f(x_n) - x^*\|^2 + 2\alpha_n(1-\alpha_n)\|f(x_n) - x^*\|\|y_n - x^*\| \\
&\leq (1-\alpha_n)\|x_n - x^*\|^2 - (1-\alpha_n)(1-\beta_n)\beta_n\|\frac{1}{t_n}\int_0^{t_n} G(u)z_n\, du - z_n\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n)\|y_n - x^*\|\|f(x_n) - x^*\|.
\end{aligned}$$

Since $\{x_n\}$ is bounded, then there exists a constant $B > 0$ sucht that

$$(1-\alpha_n)(1-\beta_n)\beta_n\|\frac{1}{t_n}\int_0^{t_n} G(u)z_n\, du - z_n\|^2 \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \alpha_n B. \tag{3.17}$$

It then implies from (3.17) and (3.13) that

$$\lim_{n\to\infty}(1-\beta_n)\beta_n\|\frac{1}{t_n}\int_0^{t_n} G(u)z_n\, du - z_n\|^2 = 0. \tag{3.18}$$

Since $\lim_{n\to\infty} \inf \beta_n(1-\beta_n) > 0$, we have

$$\lim_{n\to\infty}\|\frac{1}{t_n}\int_0^{t_n} G(u)z_n\, du - z_n\| = 0. \tag{3.19}$$

Put $D := \{\omega \in H : \|\omega - x^*\| \leq \{\|x_0 - x^*\|, \dfrac{\|f(x^*) - x^*\|}{1-b}\}$. Then D is a nonempty, bounded, closed and convex subset of H. Since $G(u)$ is nonexpansive for any $u \in [0,\infty)$, D is $G(u)$-invariant for each $u \in [0,\infty)$ and contains $\{z_n\}$. Without loss of generality, we may assume that $S := \{G(u) : 0 \leq u < \infty\}$ be a one-parameter nonexpansive semigroup on D. By Lemma 2.6, we get

$$\lim_{n\to\infty} \Big\| G(h)\Big(\frac{1}{t_n}\int_0^{t_n} G(u)z_n du\Big) - \Big(\frac{1}{t_n}\int_0^{t_n} G(u)z_n du\Big)\Big\| = 0, \tag{3.20}$$

for every $h \in [0, \infty)$. Furthermore, observe that

$$\begin{aligned}
\|z_n - G(h)z_n\| &\leq \|z_n - \frac{1}{t_n}\int_0^{t_n} G(u)z_n du\| + \Big\| G(h)\Big(\frac{1}{t_n}\int_0^{t_n} G(u)z_n du\Big) \\
&\quad -\Big(\frac{1}{t_n}\int_0^{t_n} G(h)z_n du\Big)\Big\| \\
&+ \Big\| G(h)\Big(\frac{1}{t_n}\int_0^{t_n} G(h)z_n du\Big) - G(h)z_n\Big\| \\
&\leq 2\|z_n - \frac{1}{t_n}\int_0^{t_n} G(u)z_n du\| \\
&\quad + \Big\| G(h)\Big(\frac{1}{t_n}\int_0^{t_n} G(u)z_n du\Big) - \Big(\frac{1}{t_n}\int_0^{t_n} G(u)z_n du\Big)\Big\|,
\end{aligned}$$

from inequalities (3.12) and (3.20), we get

$$\lim_{n\to\infty} \|z_n - G(h)z_n\| = 0. \tag{3.21}$$

We show that $\limsup_{n\to+\infty}\langle x^* - f(x^*), x^* - x_n\rangle \leq 0$. First, we note that there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that x_{n_j} converges weakly to ω in K and

$$\limsup_{n\to+\infty}\langle x^* - f(x^*), x^* - x_n\rangle = \lim_{j\to+\infty}\langle x^* - f(x^*), x^* - x_{n_j}\rangle.$$

Since $\{x_{n_j}\}$ is bounded, there exists a subsequence $\{x_{n_{j_i}}\}$ of $\{x_{n_j}\}$ which converges weakly to ω. Without loss of generality, we can assume that $\{x_{n_j}\}$ converges weakly to the point ω. From (3.15) and $I - T \circ J_\lambda^F$ is demiclosed, we obtain $\omega \in Fix(T \circ J_\lambda^F)$. Next, we show that $\omega \in Fix(S)$. Assume that $\omega \neq G(h)\omega$ for some $h \in [0, \infty)$.

$$\begin{aligned}
\liminf_{j\to\infty} \|z_{n_j} - \omega\| < \liminf_{j\to\infty} \|z_{n_j} - G(h)\omega\| &\leq \liminf_{j\to\infty} \Big(\|z_{n_j} - G(h)z_{n_j}\| + \|G(h)\omega - G(h)z_{n_j}\|\Big) \\
&\leq \liminf_{j\to\infty} \|z_{n_j} - \omega\|.
\end{aligned}$$

This is a contradiction. Hence, $\omega \in Fix(S)$. Thus $\omega \in \Gamma := Fix(S) \cap Fix(T) \cap \text{argmin}_{u\in K} F(u)$. Therefore,

$$\begin{aligned}
\limsup_{n\to+\infty}\langle x^* - f(x^*), x^* - x_n\rangle &= \lim_{j\to+\infty}\langle x^* - f(x^*), x^* - x_{n_j}\rangle \\
&= \langle x^* - f(x^*), x^* - a)\rangle \leq 0.
\end{aligned}$$

Finally, we show that $x_n \to x^*$. From (3.8) and Lemma 2.3, we get that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|\alpha_n f(x_n) + (1-\alpha_n)y_n - x^*\|^2 \\
&\leq \|\alpha_n(f(x_n) - f(x^*)) + (1-\alpha_n)(y_n - x^*)\|^2 + 2\alpha_n\langle x^* - f(x^*), x^* - x_{n+1}\rangle \\
&\leq \Big(\alpha_n\|f(x_n) - f(x^*)\| + \|(1-\alpha_n)(y_n - x^*)\|\Big)^2 + 2\alpha_n\langle x^* - f(x^*), x^* - x_{n+1}\rangle \\
&\leq \Big(\alpha_n b\|x_n - x^*\| + (1-\alpha_n)\|y_n - x^*\|\Big)^2 + 2\alpha_n\langle x^* - f(x^*), x^* - x_{n+1}\rangle \\
&\leq \Big((1 - \alpha_n(1-b))\|x_n - x^*\|\Big)^2 + 2\alpha_n\langle x^* - f(x^*), x^* - x_{n+1}\rangle \\
&\leq (1 - \alpha_n(1-b))\|x_n - x^*\|^2 + 2\alpha_n\langle x^* - f(x^*), x^* - x_{n+1}\rangle.
\end{aligned}$$

From Lemma 2.4, its follows that $x_n \to x^*$.
Case 2. Assume that the sequence $\{\|x_n - x^*\|\}$ is not monotonically decreasing. Set $B_n = \|x_n - x^*\|^2$ and $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping defined for all $n \geq n_0$ (for some n_0 large enough) by $\tau(n) = \max\{k \in \mathbb{N} : k \leq n,\ B_k \leq B_{k+1}\}$. We have τ is a non-decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and $B_{\tau(n)} \leq B_{\tau(n)+1}$ for $n \geq n_0$. Let $i \in \mathbb{N}^*$, from (3.12), we have

$$(1-\alpha_{\tau(n)})(1-\theta_{\tau(n)})\theta_{\tau(n)}\Big\|u_{\tau(n)} - x_{\tau(n)}\Big\|^2 \leq \alpha_{\tau(n)}C.$$

Furthermore, we have

$$(1-\alpha_{\tau(n)})(1-\theta_{\tau(n)})\theta_{\tau(n)}\Big\|u_{\tau(n)} - x_{\tau(n)}\Big\|^2 = 0.$$

Since $\lim_{n\to\infty} \inf(1-\theta_{\tau(n)})\theta_{\tau(n)} > 0$, we can deduce

$$\lim_{n\to\infty} \Big\|u_{\tau(n)} - x_{\tau(n)}\Big\|^2 = 0. \tag{3.22}$$

Since $u_{\tau(n)} \in T \circ J_\lambda^F x_{\tau(n)}$, it follows that

$$\lim_{n\to\infty} d\Big(x_{\tau(n)}, T \circ J_\lambda^F x_{\tau(n)}\Big) = 0. \tag{3.23}$$

By a similar argument as in case 1, we can show that $x_{\tau(n)}$ and $y_{\tau(n)}$ are bounded in K and $\limsup_{\tau(n)\to+\infty} \langle x^* - f(x^*), x^* - x_{\tau(n)})\rangle \leq 0$. We have for all $n \geq n_0$,

$$0 \leq \|x_{\tau(n)+1}-x^*\|^2-\|x_{\tau(n)}-x^*\|^2 \leq \alpha_{\tau(n)}[-(1-b)\|x_{\tau(n)}-x^*\|^2+2\langle x^*-f(x^*), x^*-x_{\tau(n)+1}\rangle],$$

which implies that

$$\|x_{\tau(n)} - x^*\|^2 \leq \frac{2}{1-b}\langle x^* - f(x^*), x^* - x_{\tau(n)+1}\rangle.$$

Then, we have

$$\lim_{n\to\infty} \|x_{\tau(n)} - x^*\|^2 = 0.$$

Therefore,

$$\lim_{n\to\infty} B_{\tau(n)} = \lim_{n\to\infty} B_{\tau(n)+1} = 0.$$

Thus, by Lemma 2.5, we conclude that

$$0 \leq B_n \leq \max\{B_{\tau(n)},\ B_{\tau(n)+1}\} = B_{\tau(n)+1}.$$

Hence, $\lim_{n\to\infty} B_n = 0$, that is $\{x_n\}$ converges strongly to x^*. This completes the proof. □

Remark 3.2. Many already studied problems in the literature can be considered as special cases of this paper; see, for example, [6, 12] and the references therein. Our results are applicable for finding a common solution of inclusion problems, convex optimization problems and fixed point problems involving set-valued operators in real Hilbert spaces (see, for example, [4] for more details).

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