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In memoriam Professor Charles E. Chidume (1947- 2021)

An iterative method for variational inclusions and fixed points of total uniformly L-Lipschitzian mappings

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ABSTRACT. The characterizations of m-relaxed monotone and maximal m-relaxed monotone operators are presented and by defining the resolvent operator associated with a maximal m-relaxed monotone operator, its Lipschitz continuity is proved and an estimate of its Lipschitz constant is computed. By using resolvent operator associated with a maximal m-relaxed monotone operator, an iterative algorithm is constructed for approximating a common element of the set of fixed points of a total uniformly L-Lipschitzian mapping and the set of solutions of a variational inclusion problem involving maximal m-relaxed monotone operators. By employing the concept of graph convergence for maximal m-relaxed monotone operators, a new equivalence relationship between the graph convergence of a sequence of maximal m-relaxed monotone operators and their associated resolvent operators, respectively, to a given maximal m-relaxed monotone operator and its associated resolvent operator is established. At the end, we study the strong convergence of the sequence generated by the proposed iterative algorithm to a common element of the above mentioned sets.

1. INTRODUCTION

Many mathematical problems arising in different fields, such as optimization theory, game theory, economic equilibrium, mechanics and social sciences can be formulated as variational inequality / inclusion problems. During the past few decades, significant efforts have been made by many authors to propose and develop several effective methods to find the solutions of these problems, namely, projection method and its variant forms such as resolvent operator method, see, for example, [4, 5, 9, 10, 15] and the references therein.

It is well known truth that there is a close relation between the variational inequality / inclusion problems and the fixed point problems. This fact has motivated many authors to present a unified approach for these two different problems. For more information and relevant commentaries, we refer [4–7, 11] and the references therein. Due to the fact that the study of nonexpansive mappings is a very interesting research area in the fixed point theory, in the last five decades, various extensions of this notion have been proposed and analyzed. In 2006, with the aim of presenting a unifying framework for generalized nonexpansive mappings appeared in the literature and verifying a general convergence theorem applicable to all these classes of nonlinear mappings, Alber et al. [1] introduced a more general class of asymptotically nonexpansive mappings called total asymptotically nonexpansive mappings and studied methods for approximation of fixed points of such mappings. Recently, Kiziltunc and Purtas [14] succeeded to introduce the concept of total uniformly L-Lipschitzian mappings which extends and unifies the classes of generalized nonexpansive mappings existing in the literature. They also studied the approximation

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methods for fixed points of such kind of mappings. Further extensions along with several interesting illustrative examples can be found in [1, 5, 8, 13, 14, 18] and the references therein.

On the other hand, the concept of graph convergence which was originally introduced by Attouch [2] in 1984, has received a great deal of interest from the researchers. It is pointed out in [2] that this notion is limited to maximal monotone operators, and the equivalence between the graph convergence and resolvent operator convergence is established for such kind of operators. But, recent interests are focused on presenting various extensions of the notion of graph convergence and obtaining new equivalence relations for the generalized monotone (accretive) operators existing in the literature.

The main objectives of this paper are (i) to propose an iterative method for finding a common element of the set of fixed points of a total uniformly L-Lipschitzian mapping and the set of solutions of a variational inclusion problem involving maximal m-relaxed monotone multi-valued operator, and (ii) to study the strong convergence of the sequence generated by the proposed iterative algorithm.

The layout of the paper is as follows. Section 2 provides the basic definitions and preliminaries concerning maximal m-relaxed monotone operators. We give characterizations of m-relaxed monotone and maximal m-relaxed monotone operators and define the resolvent operator associated with a maximal m-relaxed monotone operator. We also prove the Lipschitz continuity of the resolvent operator associated with a maximal m-relaxed monotone operator and compute an estimate of its Lipschitz constant. In Section 3, under sufficient conditions, the existence of a unique solution to the variational inclusion problem involving a maximal m-relaxed monotone operator is proved in the setting of Hilbert spaces. We recall the definition of a total asymptotically nonexpansive mapping and total uniformly L-Lipschtizian mapping and give an example of a total uniformly L-Lipschtizian mapping which is not total asymptotically nonexpansive. In the last section, we propose an iterative algorithm for approximating a common element of the set of solutions of the variational inclusion problem and the set of fixed points of a total uniformly L-Lipschitzian mapping. The notion of graph convergence for maximal m-relaxed monotone operators is recalled and a new equivalence relation between the graph convergence of a sequence of maximal m-relaxed monotone operators and their associated resolvent operators, respectively, to a given maximal m-relaxed monotone operator and its associated resolvent operator is established. Finally, we prove the strong convergence of the sequence generated by the proposed iterative algorithm to a common element of the above mentioned sets.

2. Basic Definitions and technical preliminaries

Let X be a real Hilbert space endowed with a norm $\|\cdot\|$ and an inner product $\langle .,. \rangle$. We denote by 2^X the family of all nonempty subsets of X. Given a multi-valued operator $M : X \to 2^X$, its effective domain and graph are defined as

$$\mathrm{D}(M) := \{x \in X : \exists y \in X : y \in M(x)\} = \{x \in X : M(x) \neq \emptyset\}$$

and

$$\mathrm{Graph}(M) := \{(x, y) \in X \times X : x \in \mathrm{D}(M),\, y \in M(x)\},$$

respectively. The range of M is defined by

$$\mathrm{R}(M) := \{y \in X : \exists x \in X : (x, y) \in \mathrm{Graph}(M)\}.$$

The inverse M^{-1} of M is $\{(y,x) : (x,y) \in \text{Graph}(M)\}$. For an arbitrary real constant ρ and multi-valued operators $M, N : X \to 2^X$, we define ρM and $M + N$ by

$$\rho M = \{(x, \rho y) : (x,y) \in \text{Graph}(M)\}$$

and

$$M + N = \{(x, y+z) : (x,y) \in \text{Graph}(M),\ (x,z) \in \text{Graph}(N)\},$$

respectively.

Definition 2.1. A multi-valued operator $M : X \to 2^X$ is said to be

(a) monotone if $\langle u - v, x - y\rangle \geq 0$ for all$(x,u),(y,v) \in \text{Graph}(M)$;
(b) strictly monotone if M is monotone and equality holds if and only if $x = y$;
(c) r-strongly monotone if there exists a constant $r > 0$ such that $\langle u - v, x - y\rangle \geq r\|x-y\|^2$ for all $(x,u),(y,v) \in \text{Graph}(M)$;
(d) m-relaxed monotone if there exists a constant $m > 0$ such that $\langle u - v, x - y\rangle \geq -m\|x-y\|^2$ for all $(x,u),(y,v) \in \text{Graph}(M)$.

We derive the following characterization of m-relaxed monotone operators which plays a key role in the sequel.

Proposition 2.1. *A multi-valued operator $M : X \to 2^X$ is m-relaxed monotone if and only if for every $\rho \in \left(0, \frac{1}{2m}\right)$,*

$$\|x - y\| \leq \frac{1}{\sqrt{1 - 2\rho m}}\|x - y + \rho(u - v)\|, \quad \forall (x,u),\ (y,v) \in \text{Graph}(M). \tag{2.1}$$

Proof. Let $\rho \in \left(0, \frac{1}{2m}\right)$ be an arbitrary real constant. Since M is m-relaxed monotone, for all $(x,u),\ (y,v) \in \text{Graph}(M)$, we compute

$$\begin{aligned}\|x - y + \rho(u-v)\|^2 &= \|x-y\|^2 + \rho^2\|u-v\|^2 + 2\rho\langle u-v, x-y\rangle \\ &\geq \|x-y\|^2 + \rho^2\|u-v\|^2 - 2\rho m\|x-y\|^2 \\ &\geq (1 - 2\rho m)\|x-y\|^2,\end{aligned}$$

which implies that

$$\|x - y\| \leq \frac{1}{\sqrt{1-2\rho m}}\|x - y + \rho(u-v)\|.$$

Conversely, assume that (2.1) holds for every $\rho \in \left(0, \frac{1}{2m}\right)$. Then, for all $(x,u),(y,v) \in \text{Graph}(M)$, we have

$$\begin{aligned}(1-2\rho m)\|x-y\|^2 &\leq \|x-y+\rho(u-v)\|^2 \\ &= \|x-y\|^2 + \rho^2\|u-v\|^2 + 2\rho\langle u-v, x-y\rangle,\end{aligned}$$

from which we conclude that

$$-2\rho m\|x-y\|^2 \leq \rho^2\|u-v\|^2 + 2\rho\langle u-v, x-y\rangle.$$

Clearly, m-relaxed monotonicity of M obtains by dividing both the sides by $2\rho > 0$ and letting $\rho \to 0$. □

In the light of the above-mentioned characterization, we obtain the following result.

Theorem 2.1. *Let $M : X \to 2^X$ be an m-relaxed monotone operator. Then, for any $\rho \in \left(0, \frac{1}{2m}\right)$, the operator $(I + \rho M)^{-1}$ from $\mathrm{R}(I + \rho M)$ to X is single-valued.*

Proof. Let $\rho \in \left(0, \frac{1}{2m}\right)$ be an arbitrary real constant, $u, v \in \mathrm{R}(I+\rho M)$ be given points and let $x \in (I+\rho M)^{-1}(u)$ and $y \in (I+\rho M)^{-1}(v)$. Then, $u \in x+\rho M(x)$ and $v \in y+\rho M(y)$. By (2.1), we get

$$\|x-y\| \leq \frac{1}{\sqrt{1-2\rho m}}\left\|x-y+\rho\left(\frac{u-x}{\rho}-\frac{v-y}{\rho}\right)\right\| = \frac{1}{\sqrt{1-2\rho m}}\|u-v\|.$$

Taking into account of this fact and picking $u=v$, it follows that $x=y$, that is, the operator $(I+\rho M)^{-1}: \mathrm{R}(I+\rho M) \to X$ is single-valued. □

Recall that a multi-valued operator $M: X \to 2^X$ is said to be maximal monotone [20] if M is monotone and $\mathrm{R}(I+\rho M)=X$ for all $\rho>0$, where I denotes the identity mapping on X.

We now turn our attention to a more general class of monotone operators which provides a unifying framework for maximal monotone operators and the classical monotone operators.

Definition 2.2. A multi-valued operator $M: X \to 2^X$ is said to be maximal m-relaxed monotone if M is m-relaxed monotone and $\mathrm{R}(I+\rho M)=X$ for every $\rho>0$.

We point out that the notion of a maximal m-relaxed monotone operator is a special case of the concept of A-maximal m-relaxed monotone operator (also known as A-monotone operator [19]). Indeed, Definition 2.2 is obtained by taking $A \equiv I$, the identity mapping on X, in Definition 2.2 in [19].

Since every monotone operator is m-relaxed monotone for any real constant $m>0$, it follows that every maximal monotone operator is maximal m-relaxed monotone for any real constant $m>0$. In other words, for any real constant $m>0$, the class of maximal monotone operators is contained within the class of maximal m-relaxed monotone operators.

The following characterization of maximal m-relaxed monotone operators provides a useful and manageable way for recognizing that an element u belongs to $M(x)$.

Proposition 2.2. *A multi-valued operator $M: X \to 2^X$ is maximal m-relaxed monotone if and only if for given points $x, u \in X$, the property*

$$\langle u-v, x-y\rangle + m\|x-y\|^2 \geq 0, \quad \forall (y,v) \in \mathrm{Graph}(M), \tag{2.2}$$

implies that $(x,u) \in \mathrm{Graph}(M)$.

Proof. Suppose first that M is a maximal m-relaxed monotone operator and $x, u \in X$ are two given points such that (2.2) holds. Assume contrary that $(x,u) \notin \mathrm{Graph}(M)$. Since M is a maximal m-relaxed monotone operator, we have $\mathrm{R}(I+\rho M)=X$ for every $\rho>0$. Let $\rho \in \left(0, \frac{1}{2m}\right)$ be an arbitrary real constant. Then, there exists $(x_0, u_0) \in \mathrm{Graph}(M)$ such that

$$x_0+\rho u_0 = x+\rho u. \tag{2.3}$$

Picking $(y,v)=(x_0,u_0)$ and taking into account the assumption, yields

$$\langle u-u_0, x-x_0\rangle + m\|x-x_0\|^2 \geq 0. \tag{2.4}$$

By (2.3) and (2.4), we obtain

$$\begin{aligned} 2\rho m\|x-x_0\|^2 < -\rho m\|x-x_0\|^2 &\leq -\rho\langle u-u_0, x-x_0\rangle \\ &= -\langle x-x_0, x-x_0\rangle = -\|x-x_0\|^2. \end{aligned}$$

The preceding relation implies that $2\rho m < -1$ which contradicts the choice of $\rho \in \left(0, \frac{1}{2m}\right)$.

Conversely, assume that for any given points $x, u \in X$, (2.2) implies that $(x, u) \in \mathrm{Graph}(M)$. This fact ensures that the operator M is m-relaxed monotone and its graph is not properly contained in the graph of any other m-relaxed monotone operator. In other words, M is an m-relaxed monotone operator which has no proper m-relaxed monotone extension and so M is a maximal m-relaxed monotone operator. □

Thanks to the arguments mentioned above, we note that M is a maximal m-relaxed monotone operator if and only if M is m-relaxed monotone and there is no other m-relaxed monotone operator whose graph contains strictly $\mathrm{Graph}(M)$. The maximal m-relaxed monotonicity is to be understood in terms of inclusion of graphs. If $M : X \to 2^X$ is a maximal m-relaxed monotone operator, then adding anything to its graph so as to obtain the graph of a new multi-valued operator, destroys the m-relaxed monotonicity. In fact, the extended operator is no longer m-relaxed monotone. In other words, for every pair $(x, u) \in X \times X \setminus \mathrm{Graph}(M)$, there exists $(y, v) \in \mathrm{Graph}(M)$ such that

$$\langle u - v, x - y\rangle + m\|x - y\|^2 < 0.$$

As an immediate consequence of Theorem 2.2, we have the following result.

Corollary 2.1. *Let $M : X \to 2^X$ be a maximal m-relaxed monotone operator. Then, for any $\rho \in \left(0, \frac{1}{2m}\right)$, the operator $(I + \rho M)^{-1} : X \to X$ is single-valued.*

Based on Corollary 2.1, associated with a maximal m-relaxed monotone operator M and an arbitrary real constant $\rho \in \left(0, \frac{1}{2m}\right)$, one can define the resolvent operator J_M^ρ as follows.

Definition 2.3. For any maximal m-relaxed monotone operator $M : X \to 2^X$ and arbitrary real constant $\rho \in \left(0, \frac{1}{2m}\right)$, the resolvent operator $J_M^\rho : X \to X$ associated with M and ρ is defined by

$$J_M^\rho(u) := (I + \rho M)^{-1}(u), \quad \forall u \in X.$$

We now prove the Lipschitz continuity of the resolvent operator J_M^ρ and calculate an estimate of its Lipschitz constant under some suitable conditions.

Theorem 2.2. *Let $M : X \to 2^X$ be a maximal m-relaxed monotone operator. Then, for any $\rho \in \left(0, \frac{1}{2m}\right)$, the resolvent operator $J_M^\rho : X \to X$ is $\frac{1}{\sqrt{1-2\rho m}}$-Lipschitz continuous, i.e.,*

$$\|J_M^\rho(u) - J_M^\rho(v)\| \le \frac{1}{\sqrt{1 - 2\rho m}}\|u - v\|, \quad \forall u, v \in X.$$

Proof. Let $\rho \in \left(0, \frac{1}{2m}\right)$ be an arbitrary real constant and $u, v \in X$ be any given points. Since the operator M is maximal m-relaxed monotone, invoking Corollary 2.1, the resolvent operator $J_M^\rho : X \to X$ is single-valued. Let $J_M^\rho(u) = \{x\}$ and $J_M^\rho(v) = \{y\}$ for some $x, y \in X$. In the light of the definition of J_M^ρ, it follows that $u \in x + \rho M(x)$ and $v \in y + \rho M(y)$. Then, by (2.1), we yield

$$\begin{aligned}\|J_M^\rho(u) - J_M^\rho(v)\| = \|x - y\| &\le \frac{1}{\sqrt{1 - 2\rho m}}\left\|x - y + \rho\left(\frac{u - x}{\rho} - \frac{v - y}{\rho}\right)\right\| \\ &= \frac{1}{\sqrt{1 - 2\rho m}}\|u - v\|.\end{aligned}$$

This completes the proof. □

It is worthwhile to stress that the conclusions derived in this section can be viewed as extensions of the corresponding results relating to monotone and maximal monotone operators, see, for example, [3].

3. Formulation of the Problem and Existence Results

Given an operator $T : X \to X$ and a maximal m-relaxed monotone operator $M : X \to 2^X$, we consider the problem of finding $x \in X$ such that

$$\mathbf{0} \in T(x) + M(x), \tag{3.5}$$

which is known as a variational inclusion problem (VIP) involving maximal m-relaxed monotone operator and has been studied by many authors in the setting of Hilbert / Banach spaces under suitable conditions imposed on the operators T and M, see, for example, [11] and the references therein. We denote by $\mathrm{VIP}(X, M, T)$ the set of solutions of the VIP (3.5).

The following lemma, which follows directly from Definition 2.3 and some simple arguments, gives a characterization of a solution of the VIP (3.5).

Lemma 3.1. *Let X, M and T be the same as in the* VIP (3.5)*. Then, $x \in X$ is a solution of the* VIP (3.5) *if and only if $x = J_M^\rho[x - \rho T(x)]$, where $\rho \in \left(0, \frac{1}{2m}\right)$ is a constant.*

Before to deal with the existence theorem for a solution of the VIP (3.5), we need to recall the following definitions.

Definition 3.4. A mapping $T : X \to X$ is said to be

(a) monotone if $\langle T(x) - T(y), x - y\rangle \geq 0$ for all $x, y \in X$;
(b) r-strongly monotone if there exists a constant $r > 0$ such that $\langle T(x) - T(y), x - y\rangle \geq r\|x - y\|^2$ for all $x, y \in X$;
(c) ϱ-Lipschitz continuous if there exists a constant $\varrho > 0$ such that $\|T(x) - T(y)\| \leq \varrho\|x - y\|$ for all $x, y \in X$.

Theorem 3.3. *Let $T : X \to X$ be an r-strongly monotone and ϱ-Lipschitz continuous operator, and $M : X \to 2^X$ be a maximal m-relaxed monotone operator. Suppose further that there exists a real constant $\rho \in \left(0, \frac{1}{2m}\right)$ such that*

$$\sqrt{\frac{1 - 2\rho r + \rho\varrho^2}{1 - 2\rho m}} < 1 \quad \textit{and} \quad 2\rho r < 1 + \rho^2\varrho^2. \tag{3.6}$$

Then, the VIP (3.5) *has a unique solution.*

Proof. Define a mapping $F : X \to X$ by

$$F(x) = J_M^\rho[x - \rho T(x)], \quad \forall x \in X. \tag{3.7}$$

We prove that F is a contraction mapping. For this end, assume that $x, x' \in X$ be chosen arbitrarily but fixed. By (3.7) and Theorem 2.2, we deduce that

$$\begin{aligned} \|F(x) - F(x')\| &= \|J_M^\rho[x - \rho T(x)] - J_M^\rho[x' - \rho T(x')]\| \\ &\leq \frac{1}{\sqrt{1 - 2\rho m}}\|x - x' - \rho(T(x) - T(x'))\|. \end{aligned} \tag{3.8}$$

Since T is an r-strongly monotone and ϱ-Lipschitz continuous operator, we yield

$$\begin{aligned} \|x - x' - \rho(T(x) - T(x'))\|^2 &= \|x - x'\|^2 - 2\rho\langle T(x) - T(x'), x - x'\rangle + \rho^2\|T(x) - T(x')\|^2 \\ &\leq (1 - 2\rho r + \rho^2\varrho^2)\|x - x'\|^2, \end{aligned}$$

which implies that

$$\|x - x' - \rho(T(x) - T(x'))\| \leq \sqrt{1 - 2\rho r + \rho^2\varrho^2}\,\|x - x'\|. \tag{3.9}$$

Substituting (3.9) into (3.8), we obtain

$$\|F(x) - F(x')\| \leq \sqrt{\frac{1 - 2\rho r + \rho^2 \varrho^2}{1 - 2\rho m}} \|x - x'\| = \vartheta \|x - x'\|, \tag{3.10}$$

where $\vartheta = \sqrt{\frac{1-2\rho r+\rho\varrho^2}{1-2\rho m}} < 1$. Clearly, (3.6) ensures that $\vartheta \in (0,1)$ and so (3.10) implies that F is a contraction mapping. By Banach fixed point theorem, there exists a unique point $x^* \in X$ such that $F(x^*) = x^*$. Thereby, recalling (3.7), it follows that $x^* = J_M^\rho[x^* - \rho T(x^*)]$. Thanks to Lemma 3.1, we conclude that $x^* \in X$ is a unique solution of the VIP (3.5). □

Recall that a mapping $T : X \to X$ is said to be nonexpansive if $\|T(x) - T(y)\| \leq \|x - y\|$ for all $x, y \in X$. During the past few decades, considerable effort has been aimed to introduce various generalizations of the notion of a nonexpansive mapping in different contexts and to study the approximate conditions for the existence of fixed points of such mappings. Goebel and Kirk [13] introduced the following notion of asymptotically non-expansive mappings which is an extension of a nonexpansive mapping.

Definition 3.5. [13]A nonlinear mapping $T : X \to X$ is said to be asymptotically non-expansive if there exists a sequence $\{a_n\} \subset (0, +\infty)$ with $\lim_{n\to\infty} a_n = 0$ such that for each $n \in \mathbb{N}$,

$$\|T^n(x) - T^n(y)\| \leq (1 + a_n)\|x - y\|, \quad \forall x, y \in X.$$

Equivalently, we say that T is asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, +\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that for each $n \in \mathbb{N}$,

$$\|T^n(x) - T^n(y)\| \leq k_n \|x - y\|, \quad \forall x, y \in X.$$

It is significant to emphasize that every nonexpansive mapping is asymptotically non-expansive with $k_n = 1$ for all $n \in \mathbb{N}$, but the converse is not true necessarily. A further generalization, known as total uniformly L-Lipschitzian, of a nonexpansive mapping is given by Kiziltunc and Purtas [14].

Definition 3.6. *A nonlinear mapping $T : X \to X$ is said to be*

(a) *total asymptotically nonexpansive (also referred to as $(\{a_n\}, \{b_n\}, \phi)$-total asymptotically nonexpansive in the literature)* [1] *if there exist nonnegative real sequences $\{a_n\}$ and $\{b_n\}$ with $a_n, b_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(0) = 0$ such that for all $x, y \in X$,*

$$\|T^n(x) - T^n(y)\| \leq \|x - y\| + a_n \phi(\|x - y\|) + b_n, \quad \forall n \in \mathbb{N}.$$

(b) *total uniformly L-Lipschitzian (or $(\{a_n\}, \{b_n\}, \phi)$-total uniformly L-Lipschitzian)* [14] *if there exist a constant $L > 0$, nonnegative real sequences $\{a_n\}$ and $\{b_n\}$ with $a_n, b_n \to 0$ as $n \to \infty$ and strictly increasing continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(0) = 0$ such that for each $n \in \mathbb{N}$,*

$$\|T^n(x) - T^n(y)\| \leq L[\|x - y\| + a_n \phi(\|x - y\|) + b_n], \quad \forall x, y \in X.$$

We note that for given nonnegative real sequences $\{a_n\}$ and $\{b_n\}$ and a strictly increasing continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$, an $(\{a_n\}, \{b_n\}, \phi)$-total asymptotically non-expansive mapping is $(\{a_n\}, \{b_n\}, \phi)$-total uniformly L-Lipschitzian with $L = 1$, but the converse is not true necessarily. The following example shows that the class of total uniformly L-Lipschitzian mappings properly includes the class of total asymptotically non-expansive mappings.

Example 3.1. Consider $X = \mathbb{R}$ with the Euclidean norm $\|\cdot\| = |\cdot|$ and let the self-mapping T of X be defined by

$$T(x) = \begin{cases} \frac{1}{\beta}, & \text{if } x \in [0,\alpha) \cup \{\beta\}, \\ \beta, & \text{if } x = \alpha, \\ 0, & \text{if } x \in (-\infty, 0) \cup (\alpha,\beta) \cup (\beta, +\infty), \end{cases}$$

where $\alpha > 0$ and $\beta > \frac{\alpha+\sqrt{\alpha^2+4}}{2}$ are arbitrary real constants such that $\alpha\beta > 1$. Since the mapping T is discontinuous at the points $x = 0, \alpha, \beta$, it follows that T is not Lipschitzian and so it is not an asymptotically nonexpansive mapping. Take $a_n = \frac{\sigma}{n}$ and $b_n = \frac{\alpha}{k^n}$ for each $n \in \mathbb{N}$, where $\sigma > 0$ and $k > 1$ are arbitrary constants. Moreover, let the function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by $\phi(t) = \gamma t^p$ for all $t \in \mathbb{R}^+$, where $p \in \mathbb{N}$ and $\gamma \in \left(0, \frac{k^p(\beta^2-\alpha\beta-1)}{\alpha^p\beta\sigma(k-1)^p}\right)$ are arbitrary constants. Picking $x = \alpha$ and $y = \frac{\alpha}{k}$, we have $T(x) = \beta$ and $T(y) = \frac{1}{\beta}$. Thanks to the fact that $0 < \gamma < \frac{k^p(\beta^2-\alpha\beta-1)}{\alpha^p\beta\sigma(k-1)^p}$, we conclude that

$$\begin{aligned} |T(x) - T(y)| = \beta - \frac{1}{\beta} &> \alpha + \frac{\sigma\gamma(k-1)^p\alpha^p}{k^p} \\ &= \frac{(k-1)\alpha}{k} + \frac{\sigma\gamma(k-1)^p\alpha^p}{k^p} + \frac{\alpha}{k} \\ &= |x-y| + \sigma\gamma|x-y|^p + \frac{\alpha}{k} \\ &= |x-y| + a_1\phi(|x-y|) + b_1, \end{aligned}$$

which ensures that T is not an $(\{a_n\},\{b_n\},\phi)$-total asymptotically nonexpansive mapping.

However, for all $x, y \in X$, we obtain

$$\begin{aligned} |T(x) - T(y)| \le \beta &\le \frac{k\beta}{\alpha}\left(|x-y| + \sigma\gamma|x-y|^p + \frac{\alpha}{k}\right) \\ &= \frac{k\beta}{\alpha}(|x-y| + a_1\phi(|x-y|) + b_1), \end{aligned} \tag{3.11}$$

and for all $n \ge 2$, taking into account that $T^n(z) = \frac{1}{\beta}$ for all $z \in X$, yields

$$\begin{aligned} |T^n(x) - T^n(y)| &< \frac{k\beta}{\alpha}\left(|x-y| + \frac{\sigma\gamma}{n}|x-y|^p + \frac{\alpha}{k^n}\right) \\ &= \frac{k\beta}{\alpha}(|x-y| + a_n\phi(|x-y|) + b_n). \end{aligned} \tag{3.12}$$

Therefore, (3.11) and (3.12) imply that T is a $(\{\frac{\sigma}{n}\},\{\frac{\alpha}{k^n}\},\phi)$-total uniformly L-Lipschitzian mapping for each $L \ge \frac{k\beta}{\alpha}$.

4. Iterative Algorithms and graph convergence

Let $S : X \to X$ be an $(\{a_n\},\{b_n\},\phi)$-total uniformly L-Lipschitzian mapping and let M and T be the same as in Lemma 3.1. Denote by $\mathrm{Fix}(S)$ the set of all fixed points of S. If $x^* \in \mathrm{Fix}(S) \cap \mathrm{VIP}(X, M, T)$, then invoking Lemma 3.1, we conclude that

$$x^* = S^n x^* = J_M^\rho[x^* - \rho T(x^*)] = S^n J_M^\rho[x^* - \rho T(x^*)]. \tag{4.13}$$

The fixed point formulation (4.13) enables us to construct the following iterative algorithm.

Algorithm 1. Suppose that X and T are the same as in the VIP (3.5). For each $n \ge 0$, let $M_n : X \to 2^X$ be a maximal m_n-relaxed monotone operator and let $S : X \to X$ be an $(\{a_n\}, \{b_n\}, \phi)$-total uniformly L-Lipschitzian mapping. For an arbitrary chosen initial point $x_0 \in X$, compute the iterative sequence $\{x_n\}_{n=0}^{\infty}$ in X by the iterative scheme

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) S^n J_{M_n}^{\rho_n}[x_n - \rho_n T(x_n)], \tag{4.14}$$

where $n \geq 0$; $\rho_n \in \left(0, \frac{1}{2m_n}\right)$ are real constants, and $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in the interval $[0,1)$ such that $\limsup_{n\to\infty} \alpha_n < 1$.

If $S \equiv I$, the identity mapping on X, $\rho_n = \rho$ and $M_n = M$, then Algorithm 4 reduces to the following iterative algorithm.

Algorithm 2. Let X, M and T be the same as in the VIP (3.5). For any given $x_0 \in X$, define the iterative sequence $\{x_n\}_{n=0}^{\infty}$ in X in the following way:

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) J_M^{\rho}[x_n - \rho T(x_n)],$$

where $n \geq 0$; $\rho \in \left(0, \frac{1}{2m}\right)$ is a real constant and the sequence $\{\alpha_n\}_{n=0}^{\infty}$ is the same as in Algorithm 4.

Definition 4.7. [4] *Let X be a real Hilbert space and $M_n, M : X \to 2^X$ $(n \geq 0)$ be* multi-valued mappings. We say that the sequence $\{M_n\}_{n=0}^{\infty}$ is graph-convergent to M, denoted by $M_n \xrightarrow{G} M$, if for every point $(x,u) \in \mathrm{Graph}(M)$, there exists a sequence of points $(x_n, u_n) \in \mathrm{Graph}(M_n)$ such that $x_n \to x$ and $u_n \to u$, as $n \to \infty$.

We now derive an equivalence relation between the graph convergence of a sequence of maximal m-relaxed monotone operators and their associated resolvent operators, respectively, to a given maximal m-relaxed monotone operator and its associated resolvent operator.

Theorem 4.4. *Let X be a real Hilbert space, and $M, M_n : X \to 2^X$ $(n \geq 0)$ be maximal m-relaxed monotone and maximal m_n-relaxed monotone operators, respectively. Suppose further that $\{\rho_n\}$ is a sequence of real constants such that $\rho_n \in \left(0, \frac{1}{2m_n}\right)$ for each $n \geq 0$, $\rho_n \to \rho \in \left(0, \frac{1}{2m}\right)$ as $n \to \infty$, and the sequence $\left\{\frac{1}{\sqrt{1-2\rho_n m_n}}\right\}_{n=0}^{\infty}$ is bounded. Then, $M_n \xrightarrow{G} M$ if and only if $J_{M_n}^{\rho_n}(z) \to J_M^{\rho}(z)$, for all $z \in X$, as $n \to \infty$, where $J_M^{\rho} = (I + \rho M)^{-1}$ and for each $n \geq 0$, $J_{M_n}^{\rho_n} = (I + \rho_n M_n)^{-1}$.*

Proof. Assume that for all $z \in X$, $\lim_{n\to\infty} J_{M_n}^{\rho_n}(z) = J_M^{\rho}(z)$. Then, for any $(x,u) \in \mathrm{Graph}(M)$, we have $x = J_M^{\rho}[x + \rho u]$, and so $J_{M_n}^{\rho_n}[x+\rho u] \to x$ as $n \to \infty$. Letting $x_n = J_{M_n}^{\rho_n}[x + \rho u]$ for each $n \geq 0$, we deduce that $x + \rho u \in (I + \rho_n M_n)(x_n)$. Therefore, for each $n \geq 0$, one can choose $u_n \in M(x_n)$ such that $x + \rho u = x_n + \rho_n u_n$. By virtue of the fact that $x_n \to x$ as $n \to \infty$, it follows that $\rho_n u_n \to \rho u$ as $n \to \infty$. Furthermore, for all $n \geq 0$, we yield

$$\begin{aligned}\rho\|u_n - u\| = \|\rho u_n - \rho u\| &\leq \|\rho_n u_n - \rho u_n\| + \|\rho_n u_n - \rho u\| \\ &= |\rho_n - \rho|\|u_n\| + \|\rho_n u_n - \rho u\|.\end{aligned}$$

Taking into account that $\rho_n \to \rho$ and $\rho_n u_n \to \rho u$, as $n \to \infty$, we conclude that the right-hand side of the preceding inequality approaches zero, as $n \to \infty$, which implies that $u_n \to u$ as $n \to \infty$. In the light of Definition 4.7, we deduce that $M_n \xrightarrow{G} M$.

Conversely, suppose that $M_n \xrightarrow{G} M$, and let $z \in X$ be chosen arbitrarily but fixed. Since M is a maximal m-relaxed monotone operator, it follows that the range of $I + \rho M$ is precisely X and so there exists $(x,u) \in \mathrm{Graph}(M)$ such that $z = x + \rho u$. Definition 4.7 ensures the existence of a sequence $\{(x_n, u_n)\}_{n=0}^{\infty} \subset \mathrm{Graph}(M)$ such that $x_n \to x$ and

$u_n \to u$ as $n \to \infty$. In view of the facts that $(x, u) \in \text{Graph}(M)$ and $(x_n, u_n) \in \text{Graph}(M_n)$, we obtain

$$x = J_M^{\rho}[x + \rho u] \quad \text{and} \quad x_n = J_{M_n}^{\rho_n}[x_n + \rho_n u_n]. \tag{4.15}$$

Assuming $z_n = x_n + \rho_n u_n$ for all $n \geq 0$, utilizing Theorem 2.2, (4.15) and the assumptions, we derive that for all $n \geq 0$,

$$\begin{aligned}
\left\|J_{M_n}^{\rho_n}(z) - J_M^{\rho}(z)\right\| &\leq \left\|J_{M_n}^{\rho_n}(z) - J_{M_n}^{\rho_n}(z_n)\right\| + \left\|J_{M_n}^{\rho_n}(z_n) - J_M^{\rho}(z)\right\| \\
&\leq \frac{1}{\sqrt{1-2\rho_n m_n}}\|z_n - z\| + \left\|J_{M_n}^{\rho_n}[x_n + \rho_n u_n] - J_M^{\rho}[x + \rho u]\right\| \\
&\leq \frac{1}{\sqrt{1-2\rho_n m_n}}\|z_n - z\| + \|x_n - x\| \\
&= \frac{1}{\sqrt{1-2\rho_n m_n}}\|x_n + \rho_n u_n - x - \rho u\| + \|x_n - x\| \\
&\leq \frac{1}{\sqrt{1-2\rho_n m_n}}(\|x_n - x\| + \|\rho_n u_n - \rho u\|) + \|x_n - x\| \\
&\leq \frac{1}{\sqrt{1-2\rho_n m_n}}(\|x_n - x\| + \|\rho_n u_n - \rho_n u\| + \|\rho_n u - \rho u\|) + \|x_n - x\| \\
&= \left(1 + \frac{1}{\sqrt{1-2\rho_n m_n}}\right)\|x_n - x\| + \frac{\rho_n}{\sqrt{1-2\rho_n m_n}}\|u_n - u\| \\
&\quad + \frac{|\rho_n - \rho|}{\sqrt{1-2\rho_n m_n}}\|u\|.
\end{aligned} \tag{4.16}$$

Since the sequence $\left\{\frac{1}{\sqrt{1-2\rho_n m_n}}\right\}_{n=0}^{\infty}$ is bounded and $\lim\limits_{n\to\infty} \rho_n = \rho$, it follows that the sequence $\left\{\frac{\rho_n}{\sqrt{1-2\rho_n m_n}}\right\}_{n=0}^{\infty}$ is also bounded. Since $x_n \to x$, $u_n \to u$ and $\rho_n \to \rho$ as $n \to \infty$, we have that the right-hand side of (4.16) tends to zero as $n \to \infty$, which guarantees $J_{M_n}^{\rho_n}(z) \to J_M^{\rho}(z)$ as $n \to \infty$. □

Before proceeding to the main result of this paper, we need to present a significant lemma which plays a critical role in our proof. We first recall the following result.

Lemma 4.2. [17] *If a sequence* $\{u_n\}_{n=0}^{\infty}$ *that satisfies*

$$u_{n+1} \leq q u_n + \alpha, \quad \forall n \geq 0,$$

for some $q \in [0, 1)$ *and* $\alpha > 0$*, then*

$$u_n \leq \frac{\alpha}{1-q} + \left(u_0 - \frac{\alpha}{1-q}\right) q^n.$$

Lemma 4.3. *Let* $\{c_n\}_{n=0}^{\infty}$*,* $\{d_n\}_{n=0}^{\infty}$ *and* $\{t_n\}_{n=0}^{\infty}$ *be three real sequences of nonnegative numbers which satisfy the following conditions:*

(i) $0 \leq t_n < 1$ *for all* $n \geq 0$ *and* $\limsup\limits_{n} t_n < 1$*;*

(ii) $c_{n+1} \leq t_n c_n + d_n$*, for all* $n \geq 0$*, and* $\lim\limits_{n\to\infty} d_n = 0$.

Then, $\lim\limits_{n\to\infty} c_n = 0$.

Proof. For any $\epsilon > 0$, take $n_0 \in \mathbb{N}$ such that $\limsup\limits_{n} t_n < 1 - \epsilon$, and $d_n < \epsilon^2$ for all $n \geq n_0$. Then by (ii), we have

$$c_{n+1} \leq (1-\epsilon)c_n + \epsilon^2.$$

Now, letting $q = 1 - \epsilon$ and $\alpha = \epsilon^2$, from Lemma 4.2, it follow that

$$c_n \leq \epsilon + (c_{n_0} - \epsilon)(1 - \epsilon)^n, \quad \forall n \geq n_0.$$

The preceding inequality implies that $\limsup_n c_n \leq \epsilon$. □

Remark 4.1.

(a) Taking $c_n = \kappa$, $d_n = \frac{\kappa}{n}$ and $t_n = 1 - \frac{1}{n}$ for all $n \in \mathbb{N}$, where $\kappa > 0$ is an arbitrary but fixed real number, we have $c_{n+1} \leq t_n c_n + d_n$ for all $n \in \mathbb{N}$, $\lim_{n\to\infty} d_n = 0$ and $\limsup_{n\to\infty} t_n = 1$, but $\lim_{n\to\infty} c_n = \kappa \neq 0$. Hence, it is important to emphasize that the condition $\limsup_{n\to\infty} t_n < 1$ imposed on the sequence $\{t_n\}$ in Lemma 4.3 is essential and cannot be dropped.

(b) We point out that Lemma 4.3 extends and unifies Lemma 5.1 in [9,12] and Lemma 2.2 in [16].

Theorem 4.5. *Let X, M and T be the same as in Theorem 3.3 and let all the conditions of Theorem 3.3 hold. Let $S : X \to X$ be an $(\{a_n\}, \{b_n\}, \phi)$-total uniformly L-Lipschitzian mapping such that $L\vartheta < 1$, where ϑ is the same as in* (3.10) *and* $\mathrm{Fix}(S) \cap \mathrm{VIP}(X, M, T) \neq \emptyset$. *Suppose that for each $n \geq 0$, $M_n : X \to 2^X$ is a maximal m_n-relaxed monotone operator such that $M_n \xrightarrow{G} M$ and $m_n \to m$ as $n \to \infty$. If there exist real constants $\rho_n > 0$ $(n \geq 0)$ satisfying* (4.14) *and $2\rho_n r < 1 + \rho_n^2 \varrho^2$ for each $n \geq 0$, and a real constant $\rho \in \left(0, \frac{1}{2m}\right)$ satisfying* (3.6) *such that $\rho_n \to \rho$ as $n \to \infty$, then the iterative sequence $\{x_n\}_{n=0}^{\infty}$ generated by Algorithm 4 converges strongly to the only element x^* of* $\mathrm{Fix}(S) \cap \mathrm{VIP}(X, M, T)$.

Proof. It follows from Theorem 3.3 that $\mathrm{VIP}(X, M, T)$ is a singleton set, that is, $\mathrm{VIP}(X, M, T) = \{x^*\}$. Since $\mathrm{Fix}(S) \cap \mathrm{VIP}(X, M, T) \neq \emptyset$, we have $x^* \in \mathrm{Fix}(S)$. Thus, by Lemma 3.1, for each $n \geq 0$, we have

$$x^* = \alpha_n x^* + (1 - \alpha_n) S^n J_M^{\rho}[x^* - \rho T(x^*)], \tag{4.17}$$

where the sequence $\{\alpha_n\}_{n=0}^{\infty}$ is the same as in Algorithm 4. Utilizing Theorem 2.2 and in the light of the hypothesis, we derive that for each $n \geq 0$,

$$\begin{aligned}
&\|J_{M_n}^{\rho_n}[x_n - \rho_n T(x_n)] - J_M^{\rho}[x^* - \rho T(x^*)]\| \\
&\leq \|J_{M_n}^{\rho_n}[x_n - \rho_n T(x_n)] - J_{M_n}^{\rho_n}[x^* - \rho T(x^*)]\| \\
&\quad + \|J_{M_n}^{\rho_n}[x^* - \rho T(x^*)] - J_M^{\rho}[x^* - \rho T(x^*)]\| \\
&\leq \frac{1}{\sqrt{1 - 2\rho_n m_n}} \|x_n - \rho_n T(x_n) - (x^* - \rho T(x^*))\| + \|\mu_n\| \\
&\leq \frac{1}{\sqrt{1 - 2\rho_n m_n}} \big(\|x_n - \rho_n T(x_n) - (x^* - \rho_n T(x^*))\| \\
&\quad + |\rho_n - \rho| \|T(x^*)\|\big) + \|\mu_n\| \\
&\leq \sqrt{\frac{1 - 2\rho_n r + \rho_n^2 \varrho^2}{1 - 2\rho_n m}} \|x_n - x^*\| + \frac{|\rho_n - \rho|}{\sqrt{1 - 2\rho_n m_n}} \|T(x^*)\| + \|\mu_n\|,
\end{aligned} \tag{4.18}$$

where for each $n \geq 0$, $\mu_n = J_{M_n}^{\rho_n}[x^* - \rho T(x^*)] - J_M^{\rho}[x^* - \rho T(x^*)]$.

Relying on the assumptions and employing (4.14), (4.17) and (4.18), for each $n \geq 0$, yields

$$
\begin{aligned}
\|x_{n+1}-x^*\| &\leq \alpha_n\|x_n-x^*\| + (1-\alpha_n)\|S^n J^{\rho_n}_{M_n}[x_n-\rho_n T(x_n)] - S^n J^{\rho}_{M}[x^*-\rho T(x^*)]\| \\
&\leq \alpha_n\|x_n-x^*\| + (1-\alpha_n)L\big(\|J^{\rho_n}_{M_n}[x_n-\rho_n T(x_n)] - J^{\rho}_{M}[x^*-\rho T(x^*)]\| \\
&\quad + a_n\phi(\|J^{\rho_n}_{M_n}[x_n-\rho_n T(x_n)] - J^{\rho}_{M}[x^*-\rho T(x^*)]\|) + b_n\big) \\
&\leq \alpha_n\|x_n-x^*\| + (1-\alpha_n)L\Bigg(\sqrt{\frac{1-2\rho_n r+\rho_n^2\varrho^2}{1-2\rho_n m_n}}\|x_n-x^*\| \\
&\quad + \frac{|\rho_n-\rho|}{\sqrt{1-2\rho_n m_n}}\|T(x^*)\| + \|\mu_n\| + a_n\phi\Big(\sqrt{\frac{1-2\rho_n r+\rho_n^2\varrho^2}{1-2\rho_n m_n}}\|x_n-x^*\| \\
&\quad + \frac{|\rho_n-\rho|}{\sqrt{1-2\rho_n m_n}}\|T(x^*)\| + \|\mu_n\|\Big) + b_n\Bigg) \\
&= \alpha_n\|x_n-x^*\| + (1-\alpha_n)L\Bigg(\vartheta_n\|x_n-x^*\| + \frac{|\rho_n-\rho|}{\sqrt{1-2\rho_n m_n}}\|T(x^*)\| \\
&\quad + \|\mu_n\| + a_n\phi(\vartheta_n\|x_n-x^*\| + \frac{|\rho_n-\rho|}{\sqrt{1-2\rho_n m_n}}\|T(x^*)\| + \|\mu_n\|) + b_n\Bigg),
\end{aligned}
\tag{4.19}
$$

where for each $n \geq 0$, $\vartheta_n = \sqrt{\frac{1-2\rho_n r+\rho_n^2\varrho^2}{1-2\rho_n m_n}}$. Since $\rho_n \to \rho$ and $m_n \to m$ as $n \to \infty$, it follows that $\vartheta_n \to \vartheta$ as $n \to \infty$. Taking into account that $\vartheta \in (0,1)$, there exist $n_0 \in \mathbb{N}$ and $\hat{\vartheta} \in (\vartheta, 1)$ such that $\vartheta_n \leq \hat{\vartheta}$ for all $n \geq n_0$. Thereby, for all $n > n_0$, by (4.19), we yield

$$
\begin{aligned}
\|x_{n+1}-x^*\| &\leq (\alpha_n + (1-\alpha_n)L\hat{\vartheta})\|x_n-x^*\| \\
&\quad + (1-\alpha_n)La_n\phi\left(\hat{\vartheta}\|x_n-x^*\| + \frac{|\rho_n-\rho|}{\sqrt{1-2\rho_n m_n}}\|T(x^*)\| + \|\mu_n\|\right) \\
&\quad + (1-\alpha_n)L\left(\frac{|\rho_n-\rho|}{\sqrt{1-2\rho_n m_n}}\|T(x^*)\| + \|\mu_n\| + b_n\right) \\
&= (L\hat{\vartheta} + (1-L\hat{\vartheta})\alpha_n)\|x_n-x^*\| \\
&\quad + (1-\alpha_n)La_n\phi\left(\hat{\vartheta}\|x_n-x^*\| + \frac{|\rho_n-\rho|}{\sqrt{1-2\rho_n m_n}}\|T(x^*)\| + \|\mu_n\|\right) \\
&\quad + (1-\alpha_n)L\left(\frac{|\rho_n-\rho|}{\sqrt{1-2\rho_n m_n}}\|T(x^*)\| + \|\mu_n\| + b_n\right).
\end{aligned}
\tag{4.20}
$$

Assuming that $t_n = L\hat{\vartheta} + (1-L\hat{\vartheta})\alpha_n$ for each $n \geq 0$ and in view of the facts that $L\hat{\vartheta} < 1$ and $\limsup\limits_{n\to\infty} \alpha_n < 1$, we get

$$
\limsup_{n\to\infty} t_n = \limsup_{n\to\infty}(L\hat{\vartheta} + (1-L\hat{\vartheta})\alpha_n) \leq L\hat{\vartheta} + (1-L\hat{\vartheta})\limsup_{n\to\infty}\alpha_n < 1.
$$

Considering the fact that $M_n \xrightarrow{G} M$, Theorem 4.4 guarantees that $\|\mu_n\| \to 0$ as $n \to \infty$. At the same time, since $\lim\limits_{n\to\infty} \rho_n = \rho$, it follows that $\lim\limits_{n\to\infty} \frac{|\rho_n-\rho|}{\sqrt{1-2\rho_n m_n}}\|T(x^*)\| = 0$. In virtue of the fact that S is an $(\{a_n\}, \{b_n\}, \phi_i)$-total uniformly L-Lipschitzian mapping, by Definition

3.6 (b), we have $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$. Let us now take for all $n \geq 0$, $c_n = \|x_n - x^*\|_*$ and

$$d_n = (1-\alpha_n)La_n\phi\left(\hat{\vartheta}\|x_n - x^*\| + \frac{|\rho_n - \rho|}{\sqrt{1-2\rho_n m_n}}\|T(x^*)\| + \|\mu_n\|\right) + (1-\alpha_n)L\left(\frac{|\rho_n - \rho|}{\sqrt{1-2\rho_n m_n}}\|T(x^*)\| + \|\mu_n\| + b_n\right).$$

Then, in the light of the above-mentioned arguments and the fact that $\limsup_{n\to\infty} \alpha_n < 1$, we conclude that $\lim_{n\to\infty} d_n = 0$. Since $L\hat{\vartheta} < 1$, we note that all the conditions of Lemma 4.3 are satisfied and so making use of (4.20) and Lemma 4.3, it follows that $\lim_{n\to\infty} c_n = 0$, i.e., $\lim_{n\to\infty} x_n = x^*$. Therefore, the sequence $\{x_n\}_{n=0}^{\infty}$ generated by Algorithm 4 converges strongly to the unique solution of the VIP (3.5), that is, the only element of $\mathrm{Fix}(S) \cap \mathrm{SGVI}(X, M, T)$. □

As a direct consequence of the above theorem, we obtain the following corollary.

Corollary 4.2. *Suppose that X, M and T are the same as in Theorem 3.3 and let all the conditions of Theorem 3.3 hold. Then the iterative sequence $\{x_n\}_{n=0}^{\infty}$ generated by Algorithm 4 converges strongly to the unique solution of the* VIP (3.5).

5. Conclusion

The history of monotone operators, and in particular maximal monotone ones which are rooted in the calculus of variations, and nonlinear operator equations with monotone and maximal monotone operators can be traced back to the early of nineteen sixties. Since then, it has a rapid development and a prolific growth of its applications. This is mainly due to the fact that monotone and maximal monotone operators are effective tools in the study of boundary value problems. Inspired by their wide applications, the introduction and study of a variety of generalizations of the concepts of maximal monotone operators in the setting of different spaces have been the focus of many researchers in the last two decades. The above description motivated us to study the class of maximal m-relaxed monotone operators as an extension of the class of maximal monotone operators and to present a characterization of such operators. By defining the resolvent operator associated with a maximal m-relaxed monotone operator, we have proved its Lipschitz continuity and computed an estimate of its Lipschitz constant under some appropriate conditions. By using the resolvent operator associated with a maximal m-relaxed monotone operator, we have constructed an iterative algorithm for approximating a common element of the set of fixed points of a total uniformly L-Lipschitzian mapping and the set of solutions of a variational inclusion problem involving maximal m-relaxed monotone operators. We have used the concept of graph convergence for maximal m-relaxed monotone operators and established a new equivalence relationship between the graph convergence of a sequence of maximal m-relaxed monotone operators and their associated resolvent operators, respectively, to a given maximal m-relaxed monotone operator and its associated resolvent operator. Finally, we have studied the strong convergence of the sequence generated by the proposed iterative algorithm to a common element of the above mentioned sets. The results derived in this paper can be extended to different classes of generalized monotone operators existing in the literature.

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