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In memoriam Professor Charles E. Chidume (1947- 2021)

Solutions of Split Equality Hammerstein Type Equation Problems in Reflexive Real Banach Spaces

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ABSTRACT. The purpose of this study is to introduce an inertial algorithm for approximating a solution of the split equality Hammerstein type equation problem in general reflexive real Banach spaces. Strong convergence results are established under the assumption that the associated mappings are monotone and uniformly continuous. The results in this paper generalize and improve many of the existing results in the literature in the sense that the underlying mappings are relaxed from Lipschitz continuous to uniformly continuous and the spaces under consideration are extended from Hilbert spaces to reflexive real Banach spaces with a more general problem which includes the Hammerstein type equation problems.

1. INTRODUCTION

Let X be a real Banach space with dual X^*. Let $\langle \cdot, \cdot \rangle$ be the generalized duality pairing between X and X^*, and let $||\cdot||$ be the induced norm. Let C be a nonempty subset of X. A mapping $A : C \to X^*$ is said to be

(a) *monotone* on C if

$$\langle Ax - Ay, x - y \rangle \geq 0, \ \ \forall x, y \in C;$$

(b) $\alpha-$ *inverse strongly monotone* on C if there exists $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha ||Ax - Ay||^2, \text{ for all } x, y \in C;$$

(c) $L-$ *Lipschitz continuous* on C if there exists a constant $L > 0$, called the Lipschitz constant, such that

$$||Ax - Ay|| \leq L||x - y||, \ \ \forall x, y \in C.$$

If $L < 1$, then A is called a *contraction* and if $L = 1$, then A is said to be *nonexpansive*.

Remark 1.1. Notice that every $\alpha-$ inverse strongly monotone mapping is $\dfrac{1}{\alpha}-$Lipschitz monotone.

Let $A : X \to X^*$ be a monotone mapping. Then, A is said to be *maximal monotone* if its graph, $G(A) = \{(x, Ax) : x \in X\}$, is not properly contained in $G(B)$, where $B : X \to X^*$ is any other monotone mapping. That is, a monotone mapping A is maximal if and only if $y = Ax$, whenever $(x, y) \in X \times X^*$ and $\langle x - u, y - v \rangle \geq 0$ for every $(u, v) \in G(A)$.

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Monotone mappings play very important role in solving nonlinear integral Hammerstein type equations (see, e.g., [40]). An equation of the form

$$u(x) + \int_{\Omega} k(x,y) f(y, u(y)) dy = 0, \tag{1.1}$$

where Ω is a measure space, dy is a $\sigma-$finite measure on $\Omega \times \Omega$, f is a function from $\Omega \times \mathbb{R}$ to $\mathbb{R}$ and u is a real valued function defined on Ω, is called nonlinear integral equation of Hammerstein type. Several problems that arise from differential equations, for instance, elliptic boundary value problems whose linear parts possess Green's functions can be transformed into the form (1.1). Consider, for instance, the following problem of a pendulum with finite amplitude:

$$\frac{d^2\theta(t)}{d\theta(t)^2} + c^2 \sin\theta(t) = z(t), \;\; t \in [0,1], \; \theta(0) = \theta(1) = 0, \tag{1.2}$$

where θ is the amplitude (angular displacement) of the bob from the equilibrium point, z is the driving force and c is a nonzero constant which depends on the length of the pendulum and gravitational acceleration. Since the Green's function of the problem

$$\theta^{''}(t) = 0, \quad \theta(0) = \theta(1) = 0,$$

is given by

$$k(t,s) = \begin{cases} t(1-s), & 0 \le t \le s, \\ s(1-t), & s \le t \le 1, \end{cases}$$

the solution of problem (1.2) is nonlinear integral equation of the form

$$\theta(t) = -\int_0^1 k(t,s) \left[z(s) - c^2 \sin\theta(s) \right] ds, \;\; t \in [0,1]. \tag{1.3}$$

Now, if we put

$$g(t) = \int_0^1 k(t,s) z(s) ds, \;\; u(t) = \theta(t) + g(t), \;\; t \in [0,1], \tag{1.4}$$

then $\theta = u - g$ and hence (1.3) can be rewritten as

$$u(t) + \int_0^1 k(t,s) c^2 \sin[g(s) - u(s)] ds = 0,$$

which is the same as the Hammerstein integral equation of the form

$$u(t) + \int_0^1 k(t,s) f(s, u(s)) ds = 0,$$

where $f(s,u(s)) = c^2 \sin\left[g(s) - u(s)\right], \; t, \; s \in [0,1]$.

A more general form of the integral equation (1.1) is the Hammerstein type equation which is given as

$$u + KFu = 0, \tag{1.5}$$

where $u \in X$, $F : X \to X^*$ and $K : X^* \to X$ are linear or nonlinear mappings. Different problems that emerge from network systems, automation and optimal control can be formulated as (1.5) (see, e.g., [28]).

Many authors have studied and proved several existence and uniqueness results of Hammerstein type equations (see, e.g., [5, 6, 7, 8, 19]). Generally, as Hammerstein type equations are nonlinear, there is no closed way method to solve such type of equations. So, different authors have introduced different approximation methods for solving Hammerstein type equations (see, for instance, [10, 12, 14, 15, 16, 18, 20, 21, 25, 35, 36, 39]). Chidume and Zegeye [12, 15, 16] were the first to propose and study iterative processes for approximating the solution of (1.5).

In 2005, Chidume and Zegeye [15] introduced the following iterative scheme in real Hilbert spaces. Let H be a real Hilbert space and let $F\colon D(F) \to H$ and $K\colon D(K) \to H$ be bounded monotone mappings with $R(F) \subset D(K)$, where $R(F)$ and $D(K)$ are closed convex subsets of H satisfying certain conditions. Let $\{u_n\}$ and $\{v_n\}$ be sequences generated from arbitrary elements $u_0 \in D(F)$ and $v_0 \in D(K)$, respectively, by

$$(1.6) \qquad \begin{cases} u_{n+1} = P_{D(F)}\left[u_n - \gamma_n\left(Fu_n - v_n + \theta_n(u_n - w_1)\right)\right], \\ v_{n+1} = P_{D(K)}\left[v_n - \gamma_n\left(Kv_n + u_n + \theta_n(v_n - w_2)\right)\right], n \geq 0, \end{cases}$$

where (w_1, w_2) is an arbitrary fixed element of $D(F) \times D(K)$ and γ_n, θ_n are sequences in $(0,1)$ satisfying appropriate conditions. They proved, under some conditions, that there exists $\gamma_0 > 0$ such that if $\gamma_n \leq \gamma_0$ and $\dfrac{\gamma_n}{\theta_n} \leq \gamma_0^2$ for all $n \geq 0$, then the sequence $\{(u_n, v_n)\}$ converges strongly to $\{(u^*, v^*)\}$, where u^* is a solution of (1.5) with $v^* = Fu^*$.

In 2015, Tufa *et al.* [35] pointed out that the convergence of the method in (1.6) depends on the existence of a constant γ_0, but it is not clear how to choose such γ_0 during implementation. Thus, they introduced an iterative algorithm which converges strongly to a solution of the general Hammerstein type equation, $u + KFu = 0$, where K and F are Lipschitz monotone mappings. Though, the convergence of their method does not require the existence of a constant γ_0, it only holds true in the Hilbert space settings.

In 2016, Uba *et al.* [36] proposed the following algorithm for solving (1.5) in a Banach space setting: Let X be a uniformly convex and uniformly smooth real Banach space and let $F : X \to X^*$, $K : X^* \to X$ be maximal monotone and bounded mappings. For $u_1 \in X$, $v_1 \in X^*$ define the sequences $\{u_n\}$ and $\{v_n\}$ in X and X^*, respectively, by

$$(1.7) \qquad \begin{cases} u_{n+1} = J^{-1}\left[Ju_n - \lambda_n(Fu_n - v_n) - \lambda_n\theta_n(Ju_n - Ju_1)\right], \\ v_{n+1} = J_*^{-1}\left[J_*v_n - \lambda_n(Kv_n + u_n) - \lambda_n\theta_n(J_*v_n - J_*v_1)\right], \end{cases}$$

where $\{\lambda_n\}, \{\theta_n\} \subset (0,1)$ satisfy the relation $\lambda_n \leq \gamma_0\theta_n$, where $\gamma_0 > 0$. Under the assumption that the equation $u + KFu = 0$ has a solution, they proved that the sequences $\{u_n\}$ and $\{v_n\}$ converge strongly to u^* and v^*, respectively, where u^* is the solution of $u + KFu = 0$ with $v^* = Fu^*$.

In 2019, Daman *et al.* [17] introduced an iterative algorithm for solving Hammerstein type equations in a $2-$uniformly convex and uniformly smooth real Banach space where the mappings under consideration are Lipschitz monotone mappings. They established strong convergence of the system which does not depend on the existence of a constant.

The need to speed up the convergence of iterative algorithms has always been of great importance. One of the recent methods of speeding up the convergence of an algorithm is the inertial method. An inertial algorithm, introduced by Polyak [30], is an iterative procedure in which subsequent terms of the sequence are obtained from the preceding two terms.

In 2021, Bello *et al.* [3] introduced an inertial type algorithm for solving Hammerstein type equations in real Hilbert spaces: Let H be a real Hilbert space and let $F, K : H \to H$ be maximal monotone and bounded mappings. For arbitrary $u_1, v_1, u_2, v_2 \in H$, define the sequences $\{h_n\}, \{p_n\}, \{u_n\}$, and $\{v_n\}$ by

$$(1.8)\qquad \begin{cases} h_n = u_n + c_n(u_{n-1} - u_n), \\ p_n = v_n + c_n(v_{n-1} - v_n), \\ u_{n+1} = h_n - \lambda_n(Fh_n - p_n) - \lambda_n\theta_n h_n, n \geq 2, \\ v_{n+1} = p_n - \lambda_n(Kp_n + h_n) - \lambda_n\theta_n p_n, n \geq 2, \end{cases}$$

where $\{\theta_n\}$, $\{\lambda_n\}$, and $\{c_n\}$ are sequences in $(0, 1)$ satisfying some conditions. Under the assumption that the inclusion $0 \in u + KFu$ has a solution in H, they proved that there exists a real constant γ_0 such that $\lambda_n \leq \gamma_0\theta_n$ for all $n \geq n_0$, for some $n_0 \geq 2$, and the sequence $\{u_n\}$ converges strongly to a solution u^* of $0 \in u + KFu$.

The aforementioned results are valid only in Hilbert spaces or uniformly smooth and uniformly convex Banach spaces. Besides, the convergence processes of some of the methods depend on the existence of a constant γ_0 and the conditions on the underlying mappings are very strong.

Based on these results, we raise the following important question:

Question 1.1. Can we obtain an inertial method for approximating a solution of the Hammerstein type equation problem in general reflexive real Banach spaces for uniformly continuous mappings whose convergence does not depend on the existence of a constant γ_0?

Motivated and inspired by the aforementioned results in the literature, it is our purpose in this paper to introduce and study an inertial algorithm for solving Hammerstein type equation problems in reflexive real Banach spaces. In fact, our aim is to introduce a method for solving a more general problem called split equality Hammerstein type equation problem whose convergence does not depend on the existence of the constant γ_0.

The split equality Hammerstein type equation problem (SEHTEP) is defined as finding a point $(u, p) \in X \times Y$ such that

$$(1.9)\qquad u + K_1F_1u = 0,\ p + K_2F_2p = 0 : T_1u = S_1p,$$

where X and Y are reflexive real Banach spaces with dual spaces X^* and Y^*, respectively, F_1: $X \to X^*, K_1$: $X^* \to X$, F_2: $Y \to Y^*$ and K_2: $Y^* \to Y$ are uniformly continuous monotone mappings, T_1: $X \to Z$ and S_1: $Y \to Z$ are bounded linear mappings with adjoints T_1^* and S_1^*, respectively and Z is another reflexive real Banach space. The SEHTEP includes Hammerstein Type Equation Problems Common Solutions of Hammerstein Type Equation Problems and Split Hammerstein Type Equation Problems as special cases.

2. Preliminaries

This section contains some basic definitions and important results that will be used in the sequel.
Let X be a reflexive real Banach space and let $\{x_n\}$ be a sequence in X. The strong and weak convergence of a sequence $\{x_n\}$ to a point $x \in X$ are denoted by $x_n \to x$ and $x_n \rightharpoonup x$, respectively.
Let $B_X = \{x \in X : ||x|| = 1\}$.

We say that X is *strictly convex* if $\dfrac{||x + y||}{2} < 1$ for all $x, y \in B_X$ with $x \neq y$. If the limit

$$(2.10)\qquad \lim_{t \to 0} \frac{||x + ty|| - ||x||}{t}$$

exists for $x, y \in B_X$, then we say that X is *smooth*.

The domain of a convex function $f\colon X \to \mathbb{R}$, denoted by $domf$, is defined as $domf = \{x \in X : f(x) < +\infty\}$. We say that f is *proper* if $domf \neq \emptyset$. The *Fenchel conjugate* of f, denoted by f^*, is the function $f^*\colon X^* \to \mathbb{R}$ defined by $f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in X\}$ for any $x^* \in X^*$. The directional derivative of f at $x \in$ *int(domf)* in the direction of y is defined as

$$f^o(x,y) = \lim_{t\downarrow 0} \frac{f(x+ty)-f(x)}{t}. \tag{2.11}$$

The function f is said to be *Gâteaux differentiable* at x if the limit in (2.11) exists for every $y \in X$. In this case, the gradient of f at x, denoted by ∇fx, is the linear function $\langle \nabla fx, y\rangle = f^o(x,y)$ for all $y \in X$. The function f is said to be Gâteaux differentiable if it is Gâteaux differentiable at each $x \in$ *int(domf)*. If the limit in (2.11) is attained uniformly for any $y \in B_X$, then f is said to be *uniformly Fréchet differentiable* at x.
A function $f\colon X \to \mathbb{R}$ is said to be a *Legendre function* if it satisfies the following conditions:
(A) f is Gâteaux differentiable, *int(domf)*$\neq \emptyset$, and *dom*$\nabla f =$*int(domf)*;
(B) f^* is Gâteaux differentiable, *int(dom*f^**)*$\neq \emptyset$, and *dom*$\nabla f^* =$*int(dom*f^**)*.

If E is a strictly convex and smooth Banach space, then the function $f(x) = \frac{1}{p}||x||^p$ $(1 < p < \infty)$ is a proper, lower semi-continuous and Legendre function with Fenchel conjugate $f^*(x^*) = \frac{1}{q}||x^*||^q$ $(1 < q < \infty)$, (see, for instance, [2]), where $\frac{1}{p} + \frac{1}{q} = 1$. In this case, the gradient of f is equal to the *generalized duality mapping*, J_p, of X. That is, $\nabla f = J_p$, where $J_p : X \to 2^{X^*}$ is defined as

$$J_p(x) = \left\{x^* \in X^* : \langle x^*, x\rangle = ||x||^p,\ ||x^*|| = ||x||^{p-1}\right\}.$$

If $p = 2$, then we write $J_p = J$ and we call it the *normalized duality mapping* and if in addition $X = H$, where H is a real Hilbert space, then $J = I$, where I is the identity mapping on H. If $f\colon X \to (-\infty, +\infty]$ is a Legendre function and X is a reflexive Banach space, then $\nabla f^* = (\nabla f)^{-1}$(see, [4]). Moreover, we have that f is a Legendre function if and only if f^* is a Legendre function (see, [2]).

Lemma 2.1. [33] *If X is a smooth real Banach space and J_X is the normalized duality mapping on X, then*

$$||x+y||^2 \leq ||x||^2 + 2\langle J_X(x+y), y\rangle$$

for all $x, y \in X$.

Definition 2.1. A function $f\colon X \to \mathbb{R}\cup\{+\infty\}$ is said to be strongly coercive if $\lim_{||x||\mapsto\infty}\left(\frac{f(x)}{||x||}\right) = \infty$.

Definition 2.2. Let $f\colon X \to \mathbb{R}\cup\{+\infty\}$ be a convex Gâteaux differentiable function, where X is a Banach space. The function $D_f\colon domf \times int(domf) \to [0, +\infty)$ defined by

$$D_f(y,x) = f(y) - f(x) - \langle \nabla f(x), y - x\rangle, \tag{2.12}$$

is called the *Bregman distance* with respect to f.

Though the Bregman distance does not posses the usual properties of a metric such as symmetric and triangle inequality properties, it satisfies the following important properties:

(i) *Three point identity*:

$$D_f(w,x) + D_f(x,y) - D_f(w,y) = \langle \nabla f(x) - \nabla f(y), x - w\rangle, \tag{2.13}$$

for any $w \in domf$ and $x, y \in int(domf)$;

(ii) *Four point identity*:

$$D_f(x,z)+D_f(w,y)-D_f(x,y)-D_f(w,z)=\langle\nabla f(y)-\nabla f(z),x-w\rangle, \tag{2.14}$$

for any $x,w\in domf$ and $y,z\in int(domf)$.

Definition 2.3. A Gâteaux differentiable function $f\colon X\to\mathbb{R}\cup\{+\infty\}$ defined on a reflexive real Banach space X is said to be *strongly convex* if there exists a constant $\beta>0$, called strong convexity constant, such that

$$\langle\nabla fx-\nabla fy,x-y\rangle\geq\beta||x-y||^2,$$

for all $x,y\in domf$, or equivalently [27]

$$f(y)\geq f(x)+\langle\nabla fx,y-x\rangle+\frac{\beta}{2}||x-y||^2.$$

If X is a smooth and strictly convex Banach space, then the function $f(x)=\frac{1}{2}||x||^2$ is strongly coercive, lower semi-continuous, bounded, uniformly Fr$\acute{e}$chet differentiable and strongly convex with strong convexity constant $\beta\in(0,1]$ and conjugate $f^*(x^*)=\frac{1}{2}||x^*||^2$. Note that for a $\beta-$strongly convex function f the following property holds:

$$D_f(y,x)\geq\frac{\beta}{2}||x-y||^2, \tag{2.15}$$

for all $x\in int(domf)$ and $y\in domf$ (see, [37]).

Definition 2.4. Let X be real Banach space and let $f\colon X\to\mathbb{R}\cup\{+\infty\}$ be a convex and Gâteaux differentiable function. Let $C\subseteq int(domf)$ be a nonempty, closed and convex subset of X. Then, the Bregman projection of $x\in int(domf)$ onto C is the unique vector $P_C^f(x)$ of C with the property

$$D_f(P_C^f(x),x)=inf\left\{D_f(y,x):y\in C\right\}.$$

The Bregman projection also satisfies the following properties

$$z=P_C^f(x)\ if\ and\ only\ if\ \langle\nabla fx-\nabla fz,y-z\rangle\leq 0,\quad for\ \ all\ \ y\in C,\ and \tag{2.16}$$

$$D_f(y,P_C^f(x))+D_f(P_C^f(x),x)\leq D_f(y,x),\quad for\ \ all\ \ x\in X,y\in C. \tag{2.17}$$

Lemma 2.2. [1] *If X_1 and X_2 are smooth reflexive real Banach spaces, then the Cartesian product $X=X_1\times X_2$ is also a smooth reflexive real Banach space with dual $X^*=X_1^*\times X_2^*$ and duality pairing*

$$\langle(x_2,y_2),(x_1,y_1)\rangle=\langle x_2,x_1\rangle+\langle y_2,y_1\rangle,$$

for all $(x_1,y_1)\in X$, $(x_2,y_2)\in X^*$, and $(x_n,y_n)\rightharpoonup(x,y)$ implies $x_n\rightharpoonup x$ and $y_n\rightharpoonup y$.

Let C be a nonempty, closed, and convex subset of X and let $f\colon X_1\to(-\infty,+\infty]$, $g\colon X_2\to(-\infty,+\infty]$ be convex Gâteaux differentiable functions. Now, for any $(x,y)\in X$, if $(x^*,y^*)=P_C^h(x,y)$, then

$$\langle\nabla h(x,y)-\nabla h(x^*,y^*),(u,v)-(x^*,y^*)\rangle\leq 0, \tag{2.18}$$

for all $(u,v)\in C$, where $h\colon X\to(-\infty,+\infty]$ is defined by $h(x,y)=f(x)+g(y)$ and $\nabla h(x,y)=(\nabla fx,\nabla gy)$.

Lemma 2.3. [29] *Let X be a reflexive real Banach space and let $f\colon X \to (-\infty, +\infty]$ be a proper, lower semi-continuous, convex and Gâteaux differentiable function. Then, $f^*\colon X^* \to (-\infty, +\infty]$ is a proper, weak* lower semi-continuous and convex function. Thus, for all $x \in X$, we have*

$$D_f\left(x, \nabla f^*\left(\sum_{i=1}^{N} \alpha_i \nabla f(z_i)\right)\right) \leq \sum_{i=1}^{N} \alpha_i D_f(x, z_i),$$

where $\{z_i\}_{i=1}^N \subseteq X$ and $\{\alpha_i\}_{i=1}^N \subseteq (0,1)$ with $\sum_{i=1}^N \alpha_i = 1$.

A function f is said to be *uniformly convex* if there exists an increasing nonnegative function ϕ with $\phi(0) = 0$ such that

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) - \lambda(1-\lambda)\phi(||x-y||),$$

for all $x, y \in dom f$ and $\lambda \in [0,1]$. The function ϕ is called the *modulus of uniform convexity* of f. The subdifferential of f at x, denoted by ∂f, is the set defined by

$$\partial f(x) = \{x^* \in X^* : \langle x^*, y-x\rangle \leq f(y) - f(x), \forall y \in X\}, \text{ (see, [22]).}$$

Lemma 2.4. *Let $f\colon X \to (-\infty, +\infty]$ be a convex and lower semi-continuous function on a Banach space X. Then, the following are equivalent (see,* [38]*):*

(i) f is uniformly convex;
(ii) for all $(x, x^*), (y, y^*) \in Gph(\partial f)$ there exists a modulus ϕ such that $f(y) \geq f(x) + \langle x^*, y-x\rangle + \phi(||x-y||)$;
(iii) dom $f^* = X^*$, f^* is Fréchet differentiable and ∇f^* is uniformly continuous.

Note that every $\beta-$strongly convex function is uniformly convex with modulus of uniform convexity $\phi(x) = \dfrac{\beta}{2}x^2$ and hence the class of strongly convex functions is contained in the class of uniformly convex functions.

Lemma 2.5. [26] *Let X be a Banach space and let $f\colon X \to (-\infty, +\infty]$ be a Gâteaux differentiable function which is uniformly convex on bounded subsets of X. Let the sequences $\{x_n\}$ and $\{w_n\}$ be bounded in X. Then, $\lim_{n\to\infty} D_f(x_n, w_n) = 0$ if and only if $\lim_{n\to\infty} ||x_n - w_n|| = 0$.*

Let $f\colon X \to \mathbb{R}$ be a Gâteaux differentiable Legendre function. The non-negative real-valued function $V_f\colon X \times X^* \to [0, +\infty)$ defined by

$$V_f(x, x^*) = f(x) - \langle x^*, x\rangle + f^*(x^*), \quad for \;\; all \;\; x \in X, \;\; x^* \in X^* \tag{2.19}$$

satisfies the following two properties

$$V_f(x, x^*) = D_f(x, \nabla f^*(x^*)), \tag{2.20}$$

and

$$V_f(x, x^*) + \langle y^*, \nabla f^*(x^*) - x\rangle \leq V_f(x, x^* + y^*), \quad for \;\; all \; x \in X, \;\; x^*, \;\; y^* \in X^*. \tag{2.21}$$

Lemma 2.6. [32] *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\alpha_n)\, a_n + \alpha_n d_n,$$

where $\{\alpha_n\} \subset (0,1)$ with $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\{d_n\}$ is a sequence of real numbers. If $\limsup_{k\to\infty} d_{n_k} \leq 0$ for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying the condition

$$\liminf_{k\to\infty} (a_{n_k+1} - a_{n_k}) \geq 0,$$

then $\lim_{n\to\infty} a_n = 0$.

The *modulus of total convexity* of a Gâteaux differentiable function f is the function v_f: $int(domf) \times [0, \infty) \to [0, \infty)$ defined by $v_f(x, t) = \inf \{D_f(y, x) : y \in domf, ||y - x|| = t\}$. We say that f is *totally convex* at a point $x \in int(domf)$ if $v_f(x, t) > 0$ whenever $t > 0$. The function f is said to be totally convex if it is totally convex at every point in the interior of its domain.
Notice that the concepts of total convexity and uniform convexity coincide on bounded subsets of X (see, [9]).

Lemma 2.7. [23] *Let X be a reflexive real Banach space and f: $X \to \mathbb{R}$ be a totally convex function. If $\{D_f(x_n, x_0)\}$ is bounded for any $x_0 \in X$, then $\{x_n\}$ is bounded.*

Lemma 2.8. [31] *If f: $X \to \mathbb{R}$ is uniformly Fréchet differentiable function which is bounded on bounded subsets of X, then ∇f is norm-to-norm uniformly continuous on bounded subsets of X and hence both f and ∇f are bounded on bounded subsets of X.*

Lemma 2.9. [34] *Let X be a reflexive real Banach space with dual X^* and let g: $X \to (-\infty, +\infty]$ be a strongly coercive and strongly convex function with strongly convex conjugate g^* and let $x = (u, v) \in X \times X^*$. Then, the function f: $X \times X^* \to (-\infty, +\infty]$ defined by*

$$f(x) = g(u) + g^*(v),$$

is strongly coercive and strongly convex.

Lemma 2.10. [24] *Let X be a reflexive real Banach space with dual X^* and let A: $X \to X^*$ be a hemicontinuous monotone mapping. Then, A is maximal monotone.*

Note that for any continuous monotone mapping A, the set $N(A) = \{x \in X : Ax = 0\}$ is closed and convex (see, e.g., [39]).

Lemma 2.11. [24] *Let X be a reflexive real Banach space with dual X^*. Let $E = X \times X^*$ with norm $||x||_E^2 = ||u||_X^2 + ||v||_{X^*}^2$, where $x = (u, v) \in E$. If F: $X \to X^*$ and K: $X^* \to X$ are hemicontinuous monotone mappings, then the mapping A: $E \to E^*$ defined by $Ax = (Fu - v, Kv + u)$ is maximal monotone.*

Let X be a reflexive real Banach space with dual X^* and let g: $X \to (-\infty, +\infty]$ be a proper, convex and lower semi-continuous function. If g is strongly coercive, bounded on bounded subsets of X and Legendre function, then its Fenchel conjugate g^*: $X^* \to (-\infty, +\infty]$ is also strongly coercive, bounded on bounded subsets of X^* and Legendre function [34]. If we define f: $X \times X^* \to (-\infty, +\infty]$ by

$$f(x) = g(u) + g^*(v),\ x = (u, v) \in X \times X^*,$$

then it can be easily verified that f is a bounded Legendre function on bounded subsets of $X \times X^*$. Moreover, if g is uniformly convex and uniformly Fréchet differentiable function, then f is Fréchet differentiable and $\nabla fx = (\nabla gu, \nabla g^* v)$ [34]. By Lemma 2.4 and Lemma 2.8, we have that ∇g and ∇g^* are uniformly continuous on bounded subsets of their domains and hence ∇f and ∇f^* are uniformly continuous.

3. Main result

In this section, we present precise statement of our algorithm and discuss its convergence. We will make use of the following assumptions for the convergence of the proposed algorithm:

Conditions

(C1) Let X and Y be reflexive real Banach spaces with dual spaces X^* and Y^*, respectively;

(C2) Let F_1: $X \to X^*$, K_1: $X^* \to X$, F_2: $Y \to Y^*$ and K_2: $Y^* \to Y$ be uniformly continuous monotone mappings;

(C3) Let T_1: $X \to Z$ and S_1: $Y \to Z$ be bounded linear mappings with adjoints T_1^* and S_1^*, respectively, where Z is another reflexive real Banach space;

(C4) Let the set of solutions of (1.9), denoted by Λ, be nonempty. That is,
$$\Lambda = \{(u^*, p^*) \in X \times Y : u^* + K_1F_1u^* = 0, p^* + K_2F_2p^* = 0 \textit{ and } T_1u^* = S_1p^*\} \neq \emptyset;$$

(C5) Let g: $X, Y \to (-\infty, +\infty]$ be a strongly coercive, lower semi-continuous, strongly convex, bounded and uniformly Fréchet differentiable Legendre function on bounded subsets with strongly convex conjugate g^*. Let the strong convexity constants of g and g^* be β_1 and β_2, respectively, and let $\beta = \min\{\beta_1, \beta_2\}$;

(C6) Let $\{\alpha_n\} \subset (0, 1)$ be such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C7) Let $\{\zeta_n\}$ be a positive sequence such that $\zeta_n \in \left(0, \dfrac{\beta}{2}\right)$ and $\dfrac{\zeta_n}{\alpha_n} \to 0$ as $n \to \infty$.

Remark 3.2. We note that the conditions (C6) and (C7) ara easily satisfied by, for example, taking $\alpha_n = \dfrac{1}{n+1}$ and $\zeta_n = \dfrac{1}{n^2+1}$.

Lemma 3.12. *Let X be a real normed space with dual X^*. Let F: $X \to X^*$ and K: $X^* \to X$ be uniformly continuous mappings. Let $E = X \times X^*$ be the Cartesian product space with norm $||x||_E^2 = ||u||_X^2 + ||v||_{X^*}^2$, where $x = (u, v) \in E$. Then, the mapping A: $E \to E^*$ defined by $Ax = A(u, v) = (Fu - v, Kv + u)$ is uniformly continuous.*

Proof. Let $x_n = (u_n, v_n), y_n = (w_n, z_n) \in E$ be such that $x_n - y_n \to 0$ as $n \to \infty$. That is, $u_n - w_n \to 0$ and $v_n - z_n \to 0$ as $n \to \infty$. Now,

(3.22)
$$\begin{aligned} ||Ax_n - Ay_n|| &= ||\,(Fu_n - v_n, Kv_n + u_n) - (Fw_n - z_n, Kz_n + w_n)\,|| \\ &= ||\,(Fu_n - Fw_n + z_n - v_n, Kv_n - Kz_n + u_n - w_n)\,|| \\ &= \left\{||Fu_n - Fw_n + z_n - v_n||^2 + ||Kv_n - Kz_n + u_n - w_n||^2\right\}^{\frac{1}{2}} \\ &\leq \left\{\,[||Fu_n - Fw_n|| + ||z_n - v_n||]^2 + [||Kv_n - Kz_n|| + ||u_n - w_n||]^2\,\right\}^{\frac{1}{2}}. \end{aligned}$$

Since F and K are uniformly continuous and the norm is a continuous function, we have that $||Fu_n - Fw_n|| \to 0$ and $||Kv_n - Kz_n|| \to 0$ as $n \to \infty$. Thus, taking the limit of both sides of (3.22) yields that $||Ax_n - Ay_n|| \to 0$ as $n \to \infty$ and hence A is uniformly continuous. □

The following notations will be used in the undermentioned algorithm.
$$\Theta_n = \left\{||\nabla gu_n - \nabla gu_{n-1}||^2 + ||\nabla g^*v_n - \nabla g^*v_{n-1}||^2\right\}^{\frac{1}{2}},$$
$$\Phi_n = \left\{||\nabla gp_n - \nabla gp_{n-1}||^2 + ||\nabla g^*q_n - \nabla g^*q_{n-1}||^2\right\}^{\frac{1}{2}}.$$

Algorithm 3.1

Initialization: Let $u_0, u_1, \in X$, $v_0, v_1, \in X^*$, $p_0, p_1, \in Y$, $q_0, q_1, \in Y^*$, $\theta > 0$, $\mu \in (0, \beta)$, $l, \gamma, \in (0, 1)$. For $u \in X$, $v \in X^*$, $p \in Y$ and $q \in Y^*$, calculate $\{u_n\}$, $\{v_n\}$, $\{p_n\}$ and $\{q_n\}$ as follows:

Step 1: Let the current iterates be $u_{n-1}, u_n \in X$, $v_{n-1}, v_n \in X^*$, $p_{n-1}, p_n \in Y$ and $q_{n-1}, q_n \in Y^*$. Choose θ_n such that $0 \leq \theta_n \leq \bar{\theta_n}$, where

$$\bar{\theta_n} = \begin{cases} min\left\{\theta, \dfrac{\zeta_n}{\Theta_n + \Phi_n}\right\}, & if \;\; \Theta_n + \Phi_n \neq 0 \\ \theta & otherwise. \end{cases}$$

Step 2: Compute

$$\begin{aligned} e_{1n} &= \nabla g^* \left[\nabla g u_n + \theta_n \left(\nabla g u_n - \nabla g u_{n-1}\right)\right], \\ e_{2n} &= \nabla g \left[\nabla g^* v_n + \theta_n \left(\nabla g^* v_n - \nabla g^* v_{n-1}\right)\right], \\ h_{1n} &= \nabla g^* \left[\nabla g p_n + \theta_n \left(\nabla g p_n - \nabla g p_{n-1}\right)\right], \\ h_{2n} &= \nabla g \left[\nabla g^* q_n + \theta_n \left(\nabla g^* q_n - \nabla g^* q_{n-1}\right)\right]. \end{aligned}$$

Step 3: Compute

$$\begin{aligned} z_{1n} &= \nabla g^* \left[\nabla g e_{1n} - \gamma_n T_1^* J_Z \left(T_1 e_{1n} - S_1 h_{1n}\right)\right], \\ r_{1n} &= \nabla g^* \left[\nabla g h_{1n} - \gamma_n S_1^* J_Z \left(S_1 h_{1n} - T_1 e_{1n}\right)\right], \end{aligned}$$

where $0 < \rho \leq \gamma_n \leq \rho_1$ with

$$\rho_1 = min\left\{\rho + 1, \frac{\beta||T_1 e_{1n} - S_1 h_{1n}||^2}{2[||T_1^* J_Z(T_1 e_{1n} - S_1 h_{1n})||^2 + ||S_1^* J_Z(S_1 h_{1n} - T_1 e_{1n})||^2]}\right\}$$

for $n \in \Omega = \{m \in \mathbb{N} : T_1 e_{1m} - S_1 h_{1m} \neq 0\}$, otherwise $\gamma_n = \rho$.

Step 4: Compute

$$y_{1n} = \nabla g^* \left[\nabla g z_{1n} - \lambda_n(F_1 z_{1n} - e_{2n})\right], \tag{3.23}$$

$$y_{2n} = \nabla g \left[\nabla g^* e_{2n} - \lambda_n(K_1 e_{2n} + z_{1n})\right], \tag{3.24}$$

$$\begin{aligned} t_{1n} &= \nabla g^* \left[\nabla g r_{1n} - \eta_n(F_2 r_{1n} - h_{2n})\right], \\ t_{2n} &= \nabla g \left[\nabla g^* h_{2n} - \eta_n(K_2 h_{2n} + r_{1n})\right], \end{aligned}$$

$\lambda_n = \gamma l^{j_n}$ and $\eta_n = \gamma l^{m_n}$, where j_n is the smallest nonnegative integer j satisfying

$$\begin{aligned} \gamma l^j &\left[||F_1 y_{1n} - F_1 z_{1n} + e_{2n} - y_{2n}||^2 + ||K_1 y_{2n} - K_1 e_{2n} + y_{1n} - z_{1n}||^2\right] \\ &\leq \mu \left[||y_{1n} - z_{1n}||^2 + ||y_{2n} - e_{2n}||^2\right], \end{aligned} \tag{3.25}$$

and m_n is the smallest nonnegative integer m satisfying

$$\begin{aligned} \gamma l^m &\left[||F_2 t_{1n} - F_2 r_{1n} + h_{2n} - t_{2n}||^2 + ||K_2 t_{2n} - K_2 h_{2n} + t_{1n} - r_{1n}||^2\right] \\ &\leq \mu \left[||t_{1n} - r_{1n}||^2 + ||t_{2n} - h_{2n}||^2\right]. \end{aligned} \tag{3.26}$$

Step 5: Compute

$$\begin{aligned} a_{1n} &= \nabla g^* \left[\nabla g y_{1n} - \lambda_n(F_1 y_{1n} - F_1 z_{1n} + e_{2n} - y_{2n})\right], \\ a_{2n} &= \nabla g \left[\nabla g^* y_{2n} - \lambda_n(K_1 y_{2n} - K_1 e_{2n} + y_{1n} - z_{1n})\right], \\ k_{1n} &= \nabla g^* \left[\nabla g t_{1n} - \eta_n(F_2 t_{1n} - F_2 r_{1n} + h_{2n} - t_{2n})\right], \\ k_{2n} &= \nabla g \left[\nabla g^* t_{2n} - \eta_n(K_2 t_{2n} - K_2 h_{2n} + t_{1n} - r_{1n})\right]. \end{aligned}$$

Step 6: Compute

$$\begin{aligned} u_{n+1} &= \nabla g^* \left[\alpha_n \nabla g u + (1 - \alpha_n) \nabla g a_{1n}\right], \\ v_{n+1} &= \nabla g \left[\alpha_n \nabla g^* v + (1 - \alpha_n) \nabla g^* a_{2n}\right], \\ p_{n+1} &= \nabla g^* \left[\alpha_n \nabla g p + (1 - \alpha_n) \nabla g k_{1n}\right], \\ q_{n+1} &= \nabla g \left[\alpha_n \nabla g^* q + (1 - \alpha_n) \nabla g^* k_{2n}\right]. \end{aligned}$$

Set $n = n + 1$ and go to **Step 1**.

Lemma 3.13. *Assume that the conditions $(C1) - (C7)$ hold. Then, the Armijo-line search rules (3.25) and (3.26) are well defined.*

Proof. We consider two cases on z_{1n} and r_{1n}:

Case I. Assume that z_{1n} is a solution of the Hammerstein type equation $u + K_1F_1u = 0$. That is, $z_{1n} + K_1F_1z_{1n} = 0$ with $e_{2n} = F_1z_{1n}$. Then, we have

$$(F_1z_{1n} - e_{2n}, K_1e_{2n} + z_{1n}) = (0, 0),$$

which implies that

$$F_1z_{1n} - e_{2n} = 0, \tag{3.27}$$

and

$$K_1e_{2n} + z_{1n} = 0. \tag{3.28}$$

Substituting (3.27) and (3.28) into (3.23) and (3.24), respectively, we obtain that $y_{1n} = z_{1n}$ and $y_{2n} = e_{2n}$. Thus, we have

$$||y_{1n} - z_{1n}||^2 + ||y_{2n} - e_{2n}||^2 = 0. \tag{3.29}$$

On the other hand, we have

$$\gamma l^j \left[||F_1y_{1n} - F_1z_{1n} + e_{2n} - y_{2n}||^2 + ||K_1y_{2n} - K_1e_{2n} + y_{1n} - z_{1n}||^2\right] = 0. \tag{3.30}$$

for every nonnegative integer j. From (3.29) and (3.30), we conclude that inequality (3.25) holds for $j = 0$.

Case II. Assume that z_{1n} is not a solution of the Hammerstein type equation $u+K_1F_1u = 0$ and assume on the contrary that for all j we have

$$\begin{aligned}&\left[||F_1p_{nj} - F_1z_{1n} + e_{2n} - q_{nj}||^2 + ||K_1q_{nj} - K_1e_{2n} + p_{nj} - z_{1n}||^2\right]\\ &\qquad > \frac{\mu}{\gamma l^j}\left[||p_{nj} - z_{1n}||^2 + ||q_{nj} - e_{2n}||^2\right],\end{aligned} \tag{3.31}$$

where $p_{nj} = \nabla g^* \left[\nabla g z_{1n} - \gamma l^j(F_1z_{1n} - e_{2n})\right]$ and $q_{nj} = \nabla g \left[\nabla g^* e_{2n} - \gamma l^j(K_1e_{2n} + z_{1n})\right]$. Since ∇g^* and ∇g are continuous, we have

$$\lim_{j\to\infty} ||p_{nj} - z_{1n}|| = \lim_{j\to\infty} ||\nabla g^* \left[\nabla g z_{1n} - \gamma l^j(F_1z_{1n} - e_{2n})\right] - z_{1n}|| = 0. \tag{3.32}$$

Similarly,

$$\lim_{j\to\infty} ||q_{nj} - e_{2n}|| = 0. \tag{3.33}$$

Since F_1 and K_1 are uniformly continuous, we obtain using (3.32) and (3.33) that

$$\lim_{j\to\infty} \left(||F_1p_{nj} - F_1z_{1n} + e_{2n} - q_{nj}||^2 + ||K_1q_{nj} - K_1e_{2n} + p_{nj} - z_{1n}||^2\right) = 0. \tag{3.34}$$

Combining (3.31) and (3.34), we obtain

$$\lim_{j\to\infty} \left(\frac{\mu}{\gamma l^j}\left[||p_{nj} - z_{1n}||^2 + ||q_{nj} - e_{2n}||^2\right]\right) = 0,$$

from which we obtain

$$\lim_{j\to\infty} \left(\frac{p_{nj} - z_{1n}}{\gamma l^j}\right) = \lim_{j\to\infty} \left(\frac{q_{nj} - e_{2n}}{\gamma l^j}\right) = 0. \tag{3.35}$$

The Lipschitz continuity of ∇g together with (3.35) gives

$$\lim_{j\to\infty} \left(\frac{\nabla g p_{nj} - \nabla g z_{1n}}{\gamma l^j}\right) = 0. \tag{3.36}$$

Since $p_{nj} \in X$, one can write $p_{nj} = P_X^g \nabla g^* \left[\nabla g z_{1n} - \gamma l^j \left(F_1 z_{1n} - e_{2n}\right)\right]$ and thus we have by (2.16) that

$$\langle \nabla g z_{1n} - \gamma l^j (F_1 z_{1n} - e_{2n}) - \nabla g p_{nj}, y - p_{nj} \rangle \leq 0 \ \ for \ \ all \ \ y \in X, \tag{3.37}$$

which implies that

$$\left\langle \frac{\nabla g z_{1n} - \nabla g p_{nj}}{\gamma l^j}, y - p_{nj} \right\rangle - \langle F_1 z_{1n} - e_{2n}, y - p_{nj} \rangle \leq 0 \ \ for \ \ all \ \ y \in X. \tag{3.38}$$

Taking the limit as $j \to \infty$ in (3.38) and using (3.36), we obtain

$$- \lim_{j \to \infty} \langle F_1 z_{1n} - e_{2n}, y - p_{nj} \rangle \leq 0 \ \ for \ \ all \ \ y \in X. \tag{3.39}$$

Taking $y = -J^{-1}(F_1 z_{1n} - e_{2n}) + p_{nj}$ in (3.39), we obtain $\langle J^{-1}(F_1 z_{1n} - e_{2n}), F_1 z_{1n} - e_{2n} \rangle = ||F_1 z_{1n} - e_{2n}||^2 \leq 0$ which implies that

$$F_1 z_{1n} - e_{2n} = 0. \tag{3.40}$$

Similarly, we can show that

$$K_1 e_{2n} + z_{1n} = 0. \tag{3.41}$$

Combining (3.40) and (3.41), we get

$$(F_1 z_{1n} - e_{2n}, K_1 e_{2n} + z_{1n}) = (0, 0), \tag{3.42}$$

and this implies that z_{1n} is a solution of the Hammerstein type equation $u + K_1 F_1 u = 0$ which is a contradiction. Thus, (3.25) holds. Considering similar cases on r_{1n}, it can be shown that (3.26) holds and hence the proof is complete. ☐

Theorem 3.1. *Assume that conditions $(C1) - (C7)$ hold. Then, the sequences $\{u_n\} \subset X$, $\{v_n\} \subset X^*$, $\{p_n\} \subset Y$ and $\{q_n\} \subset Y^*$ generated by Algorithm 3.1 are bounded.*

Proof. We have from Lemma 2.10 that F_1 and K_1 are maximal monotone mappings. If we define a norm on $E_1 = X \times X^*$ by $||x||^2_{E_1} = ||u||^2_X + ||v||^2_{X^*}$, then we have by Lemma 2.11 that the mapping A_1: $E_1 \to E_1^*$ given by $A_1 x = (F_1 u - v, K_1 v + u)$, where $x = (u, v) \in E_1$, is a maximal monotone. Similarly, the mapping A_2: $E_2 \to E_2^*$ given by $A_2 w = (F_2 p - q, K_2 q + p)$, where $w = (p, q) \in E_2 = Y \times Y^*$, is a maximal monotone. If we let $E_3 = Z \times Z^*$ and $T : E_1 \to E_3,\ S : E_2 \to E_3$ are mappings defined by $T = (T_1, 0)$ and $S = (S_1, 0)$, then Algorithm 3.1 can be rewritten as follows:

Initialization: Let $x_0, x_1 \in E_1$, $w_0, w_1 \in E_2$, $\theta > 0$, $\mu \in (0, \beta)$, $l, \gamma, \in (0, 1)$. For $x = (u, v) \in E_1$ and $w = (p, q) \in E_2$, calculate (x_n, w_n) as follows:

Step 1: Given the current iterates $x_{n-1}, x_n \in E_1$ and $w_{n-1}, w_n \in E_2$, choose θ_n such that $0 \leq \theta_n \leq \bar{\theta_n}$, where

$$\bar{\theta_n} = \begin{cases} min\left\{\theta, \dfrac{\zeta_n}{\Theta_n + \Phi_n}\right\}, & if \ \ \Theta_n + \Phi_n \neq 0 \\ \theta & otherwise, \end{cases} \tag{3.43}$$

Step 2: Compute

$$\begin{aligned} e_n &= \nabla f^* \left[\nabla f x_n + \theta_n \left(\nabla f x_n - \nabla f x_{n-1}\right)\right], \\ h_n &= \nabla f^* \left[\nabla f w_n + \theta_n \left(\nabla f w_n - \nabla f w_{n-1}\right)\right]. \end{aligned} \tag{3.44}$$

Step 3: Compute

$$z_n = \nabla f^* \left[\nabla f e_n - \gamma_n T^* J_{E_3} \left(T e_n - S h_n\right)\right], \tag{3.45}$$

$$r_n = \nabla f^* \left[\nabla f h_n - \gamma_n S^* J_{E_3} \left(S h_n - T e_n\right)\right].$$

Step 4: Compute

$$y_n = \nabla f^* \left[\nabla f z_n - \lambda_n A_1 z_n\right], \tag{3.46}$$
$$t_n = \nabla f^* \left[\nabla f r_n - \eta_n A_2 r_n\right],$$

for $\lambda_n = \gamma l^{j_n}$ and $\eta_n = \gamma l^{m_n}$, where j_n and m_n are the smallest nonnegative integers j and m satisfying the relations

$$\gamma l^j ||A_1 y_n - A_1 z_n|| \leq \mu ||y_n - z_n|| \; and$$
$$\gamma l^m ||A_2 t_n - A_2 r_n|| \leq \mu ||t_n - r_n||,$$

respectively.
Step 5: Compute

$$a_n = \nabla f^* \Big[\nabla f y_n - \lambda_n (A_1 y_n - A_1 z_n)\Big], \tag{3.47}$$
$$k_n = \nabla f^* \Big[\nabla f t_n - \eta_n (A_2 t_n - A_2 r_n)\Big].$$

Step 6: Compute

$$x_{n+1} = \nabla f^* \Big[\alpha_n \nabla f x + (1 - \alpha_n) \nabla f a_n\Big], \tag{3.48}$$
$$w_{n+1} = \nabla f^* \Big[\alpha_n \nabla f w + (1 - \alpha_n) \nabla f k_n\Big].$$

Set $n = n + 1$ and go to **Step 1**.

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where f: $E_1, E_2 \to (-\infty, +\infty]$ is defined by $f(u, v) = g(u) + g^*(v)$. Note that $x_n = (u_n, v_n)$, $w_n = (p_n, q_n)$ and f is uniformly Fréchet differentiable Legendre function that is bounded on bounded subsets of E_1 and E_2. By Lemma 2.9, we have that f is strongly coercive and strongly convex with constant β.

Now, let $(u^*, p^*) \in \Lambda$. We observe that $x^* = (u^*, v^*)$ solves $A_1 x = 0$ if and only if u^* is a solution of the equation $u + K_1 F_1 u = 0$, where $v^* = F_1 u^*$. So, the set of null points of A_1 is nonempty. Similarly, it can be shown that the set of null points of A_2 is nonempty. In addition to this, we have $T_1 u^* = S_1 p^*$. Thus, $\Pi = \{(x^*, w^*) \in N(A_1) \times N(A_2) : T x^* = S w^*\}$ is nonempty.

Now, let $(\hat{x}, \hat{w}) \in \Pi$. From Lemma 2.3 and (3.48), we have

$$\begin{aligned} D_f(\hat{x}, x_{n+1}) &= D_f(\hat{x}, \nabla f^* \left(\alpha_n \nabla f x + (1 - \alpha_n) \nabla f a_n\right)) \\ &\leq \alpha_n D_f(\hat{x}, x) + (1 - \alpha_n) D_f(\hat{x}, a_n). \end{aligned} \tag{3.49}$$

From (2.12) and (3.47), we obtain

(3.50)
$$\begin{aligned} D_f\left(\hat{x}, a_n\right) &= D_f\left(\hat{x}, \nabla f^*\left(\nabla f y_n - \lambda_n\left(A_1 y_n - A_1 z_n\right)\right)\right) \\ &= f(\hat{x}) - \langle \nabla f y_n - \lambda_n (A_1 y_n - A_1 z_n), \hat{x} - a_n \rangle - f(a_n) \\ &= f(\hat{x}) + \langle \nabla f y_n, a_n - \hat{x} \rangle + \langle \lambda_n (A_1 y_n - A_1 z_n), \hat{x} - a_n \rangle - f(a_n) \\ &= f(\hat{x}) - \langle \nabla f y_n, \hat{x} - y_n \rangle - f(y_n) + \langle \nabla f y_n, \hat{x} - y_n \rangle + f(y_n) \\ &+ \langle \nabla f y_n, a_n - \hat{x} \rangle + \langle \lambda_n (A_1 y_n - A_1 z_n), \hat{x} - a_n \rangle - f(a_n) \\ &= D_f(\hat{x}, y_n) + \langle \nabla f y_n, a_n - y_n \rangle + f(y_n) - f(a_n) + \langle \lambda_n (A_1 y_n - A_1 z_n), \hat{x} - a_n \rangle \\ &= D_f(\hat{x}, y_n) - D_f(a_n, y_n) + \langle \lambda_n (A_1 y_n - A_1 z_n), \hat{x} - a_n \rangle. \end{aligned}$$

Using (2.14), we have

$$D_f(\hat{x}, y_n) - D_f(a_n, y_n) = D_f(\hat{x}, z_n) - D_f(a_n, z_n) + \langle \nabla f z_n - \nabla f y_n, \hat{x} - a_n \rangle. \tag{3.51}$$

Combining (3.50) and (3.51), we obtain

$$D_f(\hat{x}, a_n) = D_f(\hat{x}, z_n) - D_f(a_n, z_n) + \langle \lambda_n(A_1 y_n - A_1 z_n), \hat{x} - a_n \rangle + \langle \nabla f z_n - \nabla f y_n, \hat{x} - a_n \rangle. \tag{3.52}$$

From (3.45), (2.20) and (2.21), we have

$$\begin{aligned} D_f(\hat{x}, z_n) &= D_f(\hat{x}, \nabla f^*(\nabla f e_n - \gamma_n T^* J_{E_3}(Te_n - Sh_n))) \\ &= V_f(\hat{x}, \nabla f e_n - \gamma_n T^* J_{E_3}(Te_n - Sh_n)) \\ &\leq V_f(\hat{x}, \nabla f e_n) - \langle \gamma_n T^* J_{E_3}(Te_n - Sh_n), z_n - \hat{x} \rangle \\ &= D_f(\hat{x}, e_n) - \langle \gamma_n T^* J_{E_3}(Te_n - Sh_n), z_n - \hat{x} \rangle. \end{aligned} \tag{3.53}$$

Substituting (3.53) into (3.52), we get

$$\begin{aligned} D_f(\hat{x}, a_n) \leq{}& D_f(\hat{x}, e_n) - D_f(a_n, z_n) + \langle \lambda_n(A_1 y_n - A_1 z_n), \hat{x} - a_n \rangle \\ &+ \langle \nabla f z_n - \nabla f y_n, \hat{x} - a_n \rangle - \langle \gamma_n T^* J_{E_3}(Te_n - Sh_n), z_n - \hat{x} \rangle. \end{aligned} \tag{3.54}$$

From (2.13), we obtain

$$D_f(a_n, z_n) = D_f(a_n, y_n) + D_f(y_n, z_n) - \langle \nabla f y_n - \nabla f z_n, y_n - a_n \rangle. \tag{3.55}$$

Combining (3.54) and (3.55), we obtain

$$\begin{aligned} D_f(\hat{x}, a_n) \leq{}& D_f(\hat{x}, e_n) - D_f(a_n, y_n) - D_f(y_n, z_n) \\ &+ \langle \nabla f y_n - \nabla f z_n, y_n - a_n \rangle + \langle \lambda_n(A_1 y_n - A_1 z_n), \hat{x} - a_n \rangle \\ &+ \langle \nabla f z_n - \nabla f y_n, \hat{x} - a_n \rangle - \langle \gamma_n T^* J_{E_3}(Te_n - Sh_n), z_n - \hat{x} \rangle. \\ ={}& D_f(\hat{x}, e_n) - D_f(a_n, y_n) - D_f(y_n, z_n) + \langle \nabla f y_n - \nabla f z_n, y_n - \hat{x} \rangle \\ &+ \langle \lambda_n(A_1 y_n - A_1 z_n), \hat{x} - a_n \rangle - \langle \gamma_n T^* J_{E_3}(Te_n - Sh_n), z_n - \hat{x} \rangle. \end{aligned} \tag{3.56}$$

From (3.44) and (2.12), we have

$$\begin{aligned} D_f(\hat{x}, e_n) &= D_f(\hat{x}, \nabla f^*(\nabla f x_n + \theta_n(\nabla f x_n - \nabla f x_{n-1}))) \\ &= f(\hat{x}) - \langle \nabla f x_n + \theta_n(\nabla f x_n - \nabla f x_{n-1}), \hat{x} - e_n \rangle - f(e_n) \\ &= f(\hat{x}) - \langle \nabla f x_n, \hat{x} - e_n \rangle - \langle \theta_n(\nabla f x_n - \nabla f x_{n-1}), \hat{x} - e_n \rangle - f(e_n) \\ &= f(\hat{x}) - \langle \nabla f x_n, \hat{x} - x_n \rangle - f(x_n) + \langle \nabla f x_n, \hat{x} - x_n \rangle + f(x_n) \\ &\quad - \langle \nabla f x_n, \hat{x} - e_n \rangle - \langle \theta_n(\nabla f x_n - \nabla f x_{n-1}), \hat{x} - e_n \rangle - f(e_n) \\ &= D_f(\hat{x}, x_n) + \langle \nabla f x_n, e_n - x_n \rangle + f(x_n) - f(e_n) - \langle \theta_n(\nabla f x_n - \nabla f x_{n-1}), \hat{x} - e_n \rangle \\ &= D_f(\hat{x}, x_n) - D_f(e_n, x_n) - \langle \theta_n(\nabla f x_n - \nabla f x_{n-1}), \hat{x} - e_n \rangle. \end{aligned} \tag{3.57}$$

Substituting (3.57) into (3.56), we obtain

$$\begin{aligned}
D_f(\hat{x}, a_n) &\leq D_f(\hat{x}, x_n) - D_f(a_n, y_n) - D_f(y_n, z_n) - D_f(e_n, x_n)\\
&+ \langle \nabla f y_n - \nabla f z_n, y_n - \hat{x}\rangle + \langle \lambda_n(A_1 y_n - A_1 z_n), \hat{x} - a_n\rangle\\
&- \langle \theta_n(\nabla f x_n - \nabla f x_{n-1}), \hat{x} - e_n\rangle - \langle \gamma_n T^* J_{E_3}(Te_n - Sh_n), z_n - \hat{x}\rangle\\
&= D_f(\hat{x}, x_n) - D_f(a_n, y_n) - D_f(y_n, z_n) - D_f(e_n, x_n)\\
&+ \langle \nabla f y_n - \nabla f z_n, y_n - \hat{x}\rangle + \langle \lambda_n(A_1 y_n - A_1 z_n), \hat{x} - y_n + y_n - a_n\rangle\\
&- \langle \theta_n(\nabla f x_n - \nabla f x_{n-1}), \hat{x} - e_n\rangle - \langle \gamma_n T^* J_{E_3}(Te_n - Sh_n), z_n - \hat{x}\rangle\\
&= D_f(\hat{x}, x_n) - D_f(a_n, y_n) - D_f(y_n, z_n) - D_f(e_n, x_n)\\
&+ \langle \nabla f y_n - \nabla f z_n, y_n - \hat{x}\rangle + \langle \lambda_n(A_1 y_n - A_1 z_n), \hat{x} - y_n\rangle\\
&+ \langle \lambda_n(A_1 y_n - A_1 z_n), y_n - a_n\rangle - \langle \theta_n(\nabla f x_n - \nabla f x_{n-1}), \hat{x} - e_n\rangle\\
&- \langle \gamma_n T^* J_{E_3}(Te_n - Sh_n), z_n - \hat{x}\rangle\\
&= D_f(\hat{x}, x_n) - D_f(a_n, y_n) - D_f(y_n, z_n) - D_f(e_n, x_n)\\
&+ \langle \lambda_n(A_1 y_n - A_1 z_n), y_n - a_n\rangle - \langle \gamma_n T^* J_{E_3}(Te_n - Sh_n), z_n - \hat{x}\rangle\\
&- \langle \theta_n(\nabla f x_n - \nabla f x_{n-1}), \hat{x} - e_n\rangle\\
&- \langle \lambda_n(A_1 y_n - A_1 z_n) - (\nabla f y_n - \nabla f z_n), y_n - \hat{x}\rangle.
\end{aligned} \tag{3.58}$$

Since $\hat{x} \in N(A_1)$, we have from (3.46) that

$$\begin{aligned}
&\langle \lambda_n(A_1 y_n - A_1 z_n) - (\nabla f y_n - \nabla f z_n), y_n - \hat{x}\rangle\\
&\qquad = \langle \lambda_n A_1 y_n - (\nabla f z_n - \nabla f y_n) - (\nabla f y_n - \nabla f z_n), y_n - \hat{x}\rangle\\
&\qquad = \langle \lambda_n A_1 y_n, y_n - \hat{x}\rangle \geq 0,
\end{aligned}$$

where the last inequality follows by the virtue of monotonicity of A_1. Thus, the relation (3.58) can be simplified to

$$\begin{aligned}
D_f(\hat{x}, a_n) &\leq D_f(\hat{x}, x_n) - D_f(a_n, y_n) - D_f(y_n, z_n) - D_f(e_n, x_n)\\
&+ \langle \lambda_n(A_1 y_n - A_1 z_n), y_n - a_n\rangle - \langle \theta_n(\nabla f x_n - \nabla f x_{n-1}), \hat{x} - e_n\rangle\\
&- \langle \gamma_n T^* J_{E_3}(Te_n - Sh_n), z_n - \hat{x}\rangle.
\end{aligned} \tag{3.59}$$

Moreover, by the Cauchy Schwarz inequality and (2.15) we have

(3.60)

$$\begin{aligned}
\langle \theta_n(\nabla f x_n - \nabla f x_{n-1}), \hat{x} - e_n\rangle &\leq \theta_n ||\nabla f x_n - \nabla f x_{n-1}||\, ||\hat{x} - e_n||\\
&= \theta_n ||\nabla f x_n - \nabla f x_{n-1}||\, ||(\hat{x} - e_n) \times 1||\\
&\leq \frac{\theta_n}{2} ||\nabla f x_n - \nabla f x_{n-1}||\, \left[||\hat{x} - e_n||^2 + 1\right]\\
&= \frac{\theta_n}{2} ||\nabla f x_n - \nabla f x_{n-1}||\, \left[||\hat{x} - x_n + x_n - e_n||^2 + 1\right]\\
&\leq \frac{\theta_n}{2} ||\nabla f x_n - \nabla f x_{n-1}||\, \left[2||\hat{x} - x_n||^2 + 2||x_n - e_n||^2 + 1\right]\\
&\leq \frac{\theta_n}{2} ||\nabla f x_n - \nabla f x_{n-1}||\, \left[\frac{4}{\beta} D_f(\hat{x}, x_n) + \frac{4}{\beta} D_f(e_n, x_n) + 1\right]\\
&= \frac{2\theta_n}{\beta} ||\nabla f x_n - \nabla f x_{n-1}|| D_f(\hat{x}, x_n)\\
&+ \frac{2\theta_n}{\beta} ||\nabla f x_n - \nabla f x_{n-1}|| D_f(e_n, x_n) + \frac{\theta_n}{2} ||\nabla f x_n - \nabla f x_{n-1}||.
\end{aligned}$$

Substituting (3.60) into (3.59) and using (3.25) and the Cauchy Schwarz inequality, we obtain

(3.61)

$$\begin{aligned}
D_f(\hat{x},a_n) &\le D_f(\hat{x},x_n)-D_f(a_n,y_n)-D_f(y_n,z_n)-D_f(e_n,x_n)+\lambda_n\|A_1y_n-A_1z_n\|\,\|y_n-a_n\|\\
&+\frac{2\theta_n}{\beta}\|\nabla fx_n-\nabla fx_{n-1}\|D_f(\hat{x},x_n)+\frac{2\theta_n}{\beta}\|\nabla fx_n-\nabla fx_{n-1}\|D_f(e_n,x_n)\\
&+\frac{\theta_n}{2}\|\nabla fx_n-\nabla fx_{n-1}\|-\langle\gamma_nT^*J_{E_3}(Te_n-Sh_n),z_n-\hat{x}\rangle\\
&\le D_f(\hat{x},x_n)-D_f(a_n,y_n)-D_f(y_n,z_n)-D_f(e_n,x_n)+\mu\|y_n-z_n\|\,\|y_n-a_n\|\\
&+\frac{2\theta_n}{\beta}\|\nabla fx_n-\nabla fx_{n-1}\|D_f(\hat{x},x_n)+\frac{2\theta_n}{\beta}\|\nabla fx_n-\nabla fx_{n-1}\|D_f(e_n,x_n)\\
&+\frac{\theta_n}{2}\|\nabla fx_n-\nabla fx_{n-1}\|-\langle\gamma_nT^*J_{E_3}(Te_n-Sh_n),z_n-\hat{x}\rangle\\
&\le D_f(\hat{x},x_n)-D_f(a_n,y_n)-D_f(y_n,z_n)-D_f(e_n,x_n)+\mu\left[\frac{\|y_n-z_n\|^2+\|y_n-a_n\|^2}{2}\right]\\
&+\frac{2\theta_n}{\beta}\|\nabla fx_n-\nabla fx_{n-1}\|D_f(\hat{x},x_n)+\frac{2\theta_n}{\beta}\|\nabla fx_n-\nabla fx_{n-1}\|D_f(e_n,x_n)\\
&+\frac{\theta_n}{2}\|\nabla fx_n-\nabla fx_{n-1}\|-\langle\gamma_nT^*J_{E_3}(Te_n-Sh_n),z_n-\hat{x}\rangle.
\end{aligned}$$

From (3.61), (2.15) and (3.43), we obtain

(3.62)

$$\begin{aligned}
D_f(\hat{x},a_n) &\le D_f(\hat{x},x_n)-D_f(a_n,y_n)-D_f(y_n,z_n)-D_f(e_n,x_n)+\frac{\mu}{\beta}D_f(y_n,z_n)+\frac{\mu}{\beta}D_f(a_n,y_n)\\
&+\frac{2\theta_n}{\beta}\|\nabla fx_n-\nabla fx_{n-1}\|D_f(\hat{x},x_n)+\frac{2\theta_n}{\beta}\|\nabla fx_n-\nabla fx_{n-1}\|D_f(e_n,x_n)\\
&+\frac{\theta_n}{2}\|\nabla fx_n-\nabla fx_{n-1}\|-\langle\gamma_nT^*J_{E_3}(Te_n-Sh_n),z_n-\hat{x}\rangle\\
&\le D_f(\hat{x},x_n)-D_f(a_n,y_n)-D_f(y_n,z_n)-D_f(e_n,x_n)+\frac{\mu}{\beta}D_f(y_n,z_n)+\frac{\mu}{\beta}D_f(a_n,y_n)\\
&+\frac{2\zeta_n}{\beta}D_f(\hat{x},x_n)+\frac{2\zeta_n}{\beta}D_f(e_n,x_n)+\frac{\zeta_n}{2}-\langle\gamma_nT^*J_{E_3}(Te_n-Sh_n),z_n-\hat{x}\rangle\\
&=\left(1+\frac{2\zeta_n}{\beta}\right)D_f(\hat{x},x_n)-\left(1-\frac{2\zeta_n}{\beta}\right)D_f(e_n,x_n)-\left(1-\frac{\mu}{\beta}\right)D_f(a_n,y_n)\\
&-\left(1-\frac{\mu}{\beta}\right)D_f(y_n,z_n)+\frac{\zeta_n}{2}-\langle\gamma_nT^*J_{E_3}(Te_n-Sh_n),z_n-\hat{x}\rangle.
\end{aligned}$$

Since $\zeta_n\in\left(0,\frac{\beta}{2}\right)$ and $\mu\in(0,\beta)$, we obtain from (3.62) that

(3.63) $$D_f(\hat{x},a_n)\le\left(1+\frac{2\zeta_n}{\beta}\right)D_f(\hat{x},x_n)+\frac{\zeta_n}{2}-\langle\gamma_nT^*J_{E_3}(Te_n-Sh_n),z_n-\hat{x}\rangle.$$

Substituting (3.63) into (3.49), we obtain

(3.64)
$$\begin{aligned} D_f(\hat{x}, x_{n+1}) &\leq \alpha_n D_f(\hat{x}, x) + (1-\alpha_n)\left[\left(1+\frac{2\zeta_n}{\beta}\right) D_f(\hat{x}, x_n) - \left(1-\frac{2\zeta_n}{\beta}\right) D_f(e_n, x_n)\right] \\ &- (1-\alpha_n)\left[\left(1-\frac{\mu}{\beta}\right) D_f(a_n, y_n) + \left(1-\frac{\mu}{\beta}\right) D_f(y_n, z_n) - \frac{\zeta_n}{2}\right] \\ &- (1-\alpha_n)\langle \gamma_n T^* J_{E_3}(Te_n - Sh_n), z_n - \hat{x}\rangle. \end{aligned}$$

Similarly,

(3.65)
$$\begin{aligned} D_f(\hat{w}, w_{n+1}) &\leq \alpha_n D_f(\hat{w}, w) + (1-\alpha_n)\left[\left(1+\frac{2\zeta_n}{\beta}\right) D_f(\hat{w}, w_n) - \left(1-\frac{2\zeta_n}{\beta}\right) D_f(h_n, w_n)\right] \\ &- (1-\alpha_n)\left[\left(1-\frac{\mu}{\beta}\right) D_f(k_n, t_n) + \left(1-\frac{\mu}{\beta}\right) D_f(t_n, r_n) - \frac{\zeta_n}{2}\right] \\ &- (1-\alpha_n)\langle \gamma_n S^* J_{E_3}(Sh_n - Te_n), r_n - \hat{w}\rangle. \end{aligned}$$

Let $\Omega_n = D_f(\hat{x}, x_n) + D_f(\hat{w}, w_n)$ and $\Sigma = D_f(\hat{x}, x) + D_f(\hat{w}, w)$. Then, combining (3.64) and (3.65), we obtain

$$\begin{aligned} \Omega_{n+1} &\leq \alpha_n \Sigma + (1-\alpha_n)\left(1+\frac{2\zeta_n}{\beta}\right)\Omega_n + (1-\alpha_n)\zeta_n \\ &- (1-\alpha_n)\gamma_n\langle J_{E_3}(Te_n - Sh_n), Tz_n - Sr_n\rangle. \end{aligned} \tag{3.66}$$

But,

(3.67)
$$\begin{aligned} -\langle J_{E_3}(Te_n - Sh_n), Tz_n - Sr_n\rangle &= -\langle J_{E_3}(Te_n - Sh_n), Te_n - Sh_n\rangle \\ &- \langle J_{E_3}(Te_n - Sh_n), Tz_n - Te_n\rangle \\ &- \langle J_{E_3}(Te_n - Sh_n), Sh_n - Sr_n\rangle \\ &= -||Te_n - Sh_n||^2 - \langle T^* J_{E_3}(Te_n - Sh_n), z_n - e_n\rangle \\ &- \langle h_n - r_n, S^* J_{E_3}(Te_n - Sh_n)\rangle \\ &\leq -||Te_n - Sh_n||^2 + ||z_n - e_n||\,||T^* J_{E_3}(Te_n - Sh_n)|| \\ &+ ||h_n - r_n||\,||S^* J_{E_3}(Te_n - Sh_n)||. \end{aligned}$$

From the strong convexity of f and the definition of z_n, we have

$$\begin{aligned} ||z_n - e_n|| &= ||\nabla f^*\left(\nabla f(e_n) - \gamma_n T^* J_{E_3}(Te_n - Sh_n)\right) - \nabla f^*\left(\nabla f(e_n)\right)|| \\ &\leq \frac{\gamma_n}{\beta}||T^* J_{E_3}(Te_n - Sh_n)||. \end{aligned} \tag{3.68}$$

Similarly, the strong convexity of f and the definition of r_n imply that

$$\begin{aligned} ||r_n - h_n|| &= ||\nabla f^*\left(\nabla f h_n - \gamma_n S^* J_{E_3}\left(Sh_n - Te_n\right)\right) - \nabla f^*(\nabla f h_n)|| \\ &\leq \frac{\gamma_n}{\beta}||S^* J_{E_3}(Sh_n - Te_n)||. \end{aligned} \tag{3.69}$$

Substituting (3.69) and (3.68) into (3.67), we get

(3.70)

$$\begin{aligned}
-\gamma_n\langle J_{E_3}(Te_n - Sh_n), Tz_n - Sr_n\rangle &\leq -\gamma_n||Te_n - Sh_n||^2 + \frac{\gamma_n^2}{\beta}||T^*J_{E_3}(Te_n - Sh_n)||^2 \\
&+ \frac{\gamma_n^2}{\beta}||S^*J_{E_3}(Sh_n - Te_n)||^2 \\
&\leq -\frac{\rho}{2}||Te_n - Sh_n||^2 - \frac{\gamma_n}{2}||Te_n - Sh_n||^2 \\
&+ \frac{\gamma_n}{2}\left\{\frac{2\gamma_n}{\beta}||T^*J_{E_3}(Te_n - Sh_n)||^2\right\} \\
&+ \frac{\gamma_n}{2}\left\{\frac{2\gamma_n}{\beta}||S^*J_{E_3}(Sh_n - Te_n)||^2\right\} \\
&\leq -\frac{\rho}{2}||Te_n - Sh_n||^2.
\end{aligned}$$

Let $\varepsilon \in \left(0, \frac{\beta}{2}\right)$. Since $\frac{\zeta_n}{\alpha_n} \to 0$ as $n \to \infty$, there exists a natural number n_0 such that $\zeta_n < \alpha_n\varepsilon$ for all $n \geq n_0$. So, combining (3.70) and (3.66), we obtain

$$\begin{aligned}
\Omega_{n+1} &\leq \alpha_n\Sigma + (1-\alpha_n)\left(1 + \frac{2\zeta_n}{\beta}\right)\Omega_n + (1-\alpha_n)\zeta_n - (1-\alpha_n)\frac{\rho}{2}||Te_n - Sh_n||^2 \\
&\leq \alpha_n\Sigma + (1-\alpha_n)\Omega_n + \frac{2\varepsilon\alpha_n}{\beta}\Omega_n + \varepsilon\alpha_n \\
&= \left[1 - \alpha_n\left(1 - \frac{2\varepsilon}{\beta}\right)\right]\Omega_n + \alpha_n\left[\Sigma + \varepsilon\right] \qquad (3.71) \\
&= \left[1 - \alpha_n\left(1 - \frac{2\varepsilon}{\beta}\right)\right]\Omega_n + \alpha_n\left(1 - \frac{2\varepsilon}{\beta}\right)\left[\frac{\beta\Sigma}{\beta - 2\varepsilon} + \frac{\beta\varepsilon}{\beta - 2\varepsilon}\right] \\
&\leq \max\left\{\Omega_n, \frac{\beta\Sigma}{\beta - 2\varepsilon} + \frac{\beta\varepsilon}{\beta - 2\varepsilon}\right\}.
\end{aligned}$$

By the Principle of Mathematical induction, we have

$$\Omega_n \leq \max\left\{\Omega_0, \frac{\beta\Sigma}{\beta - 2\varepsilon} + \frac{\beta\varepsilon}{\beta - 2\varepsilon}\right\}.$$

Thus, $\{\Omega_n\}$ is a bounded sequence. This in turn implies that the sequences $\{D_f(\hat{x}, x_n)\}$ and $\{D_f(\hat{w}, w_n)\}$ are bounded. So, by Lemma 2.7 we have that $\{x_n\}$ and $\{w_n\}$ are bounded and hence $\{u_n\}$, $\{v_n\}$, $\{p_n\}$ and $\{q_n\}$ are bounded sequences. □

Theorem 3.2. *Assume that conditions $(C1) - (C7)$ hold. Then, the sequences $\{u_n\}$, $\{v_n\}$, $\{p_n\}$ and $\{q_n\}$ generated by Algorithm 3.1 converge strongly to $\mathring{u}$, $\mathring{v}$, $\mathring{p}$ and $\mathring{q}$, respectively, where $(\mathring{u}, \mathring{p}) \in \Lambda$, $\mathring{v} = F_1\mathring{u}$ and $\mathring{q} = F_2\mathring{p}$.*

Proof. Let $\Pi = \{(x^*, w^*) \in N(A_1) \times N(A_2) : Tx^* = Sw^*\}$, where A_1, A_2, T and S are defined as in the proof of Theorem 3.1. Now, let $(\mathring{x}, \mathring{w}) = P^f_\Pi(x, w)$, where $x = (u, v)$, $w = (p, q)$ $\mathring{x} = (\mathring{u}, \mathring{v})$ and $\mathring{w} = (\mathring{p}, \mathring{q})$. Then, by (2.18), we have

$$\langle(\nabla fx, \nabla fw) - (\nabla f\mathring{x}, \nabla f\mathring{w}),\ (z, r) - (\mathring{x}, \mathring{w})\rangle \leq 0, \qquad (3.72)$$

for all $(z, r) \in \Pi$. From (3.48), (2.20), (2.21) and Lemma 2.3, we get

(3.73)
$$\begin{aligned} D_f(\mathring{x}, x_{n+1}) &= D_f(\mathring{x}, \nabla f^*(\alpha_n \nabla f x + (1-\alpha_n)\nabla f a_n)) \\ &= V_f(\mathring{x}, \alpha_n \nabla f x + (1-\alpha_n)\nabla f a_n) \\ &\leq V_f(\mathring{x}, \alpha_n \nabla f x + (1-\alpha_n)\nabla f a_n - \alpha_n(\nabla f x - \nabla f \mathring{x})) \\ &+ \langle \alpha_n(\nabla f x - \nabla f \mathring{x}), x_{n+1} - \mathring{x}\rangle \\ &= V_f(\mathring{x}, \alpha_n \nabla f \mathring{x} + (1-\alpha_n)\nabla f a_n) + \langle \alpha_n(\nabla f x - \nabla f \mathring{x}), x_{n+1} - \mathring{x}\rangle \\ &= D_f(\mathring{x}, \nabla f^*(\alpha_n \nabla f \mathring{x} + (1-\alpha_n)\nabla f a_n)) + \langle \alpha_n(\nabla f x - \nabla f \mathring{x}), x_{n+1} - \mathring{x}\rangle. \end{aligned}$$

Since $D_f(\mathring{x}, \mathring{x}) = 0$, we obtain from (3.73) and (3.63) that

(3.74)
$$\begin{aligned} D_f(\mathring{x}, x_{n+1}) &\leq (1-\alpha_n) D_f(\mathring{x}, a_n) + \langle \alpha_n(\nabla f x - \nabla f \mathring{x}), x_{n+1} - \mathring{x}\rangle \\ &\leq (1-\alpha_n)\left[\left(1 + \frac{2\zeta_n}{\beta}\right) D_f(\mathring{x}, x_n) + \frac{\zeta_n}{2} - \langle \gamma_n T^* J_{E_3}(Te_n - Sh_n), z_n - \mathring{x}\rangle\right] \\ &+ \langle \alpha_n(\nabla f x - \nabla f \mathring{x}), x_{n+1} - \mathring{x}\rangle \\ &= (1-\alpha_n)\left(1 + \frac{2\zeta_n}{\beta}\right) D_f(\mathring{x}, x_n) + (1-\alpha_n)\frac{\zeta_n}{2} \\ &- (1-\alpha_n)\langle \gamma_n T^* J_{E_3}(Te_n - Sh_n), z_n - \mathring{x}\rangle + \langle \alpha_n(\nabla f x - \nabla f \mathring{x}), x_{n+1} - \mathring{x}\rangle \\ &\leq (1-\alpha_n) D_f(\mathring{x}, x_n) + \frac{2\zeta_n}{\beta} D_f(\mathring{x}, x_n) + \frac{\zeta_n}{2} \\ &- (1-\alpha_n)\langle \gamma_n T^* J_{E_3}(Te_n - Sh_n), z_n - \mathring{x}\rangle + \langle \alpha_n(\nabla f x - \nabla f \mathring{x}), x_{n+1} - \mathring{x}\rangle. \end{aligned}$$

Since $\frac{\zeta_n}{\alpha_n} \to 0$ as $n \to \infty$, for any $\varepsilon \in \left(0, \frac{\beta}{2}\right)$, there exists a natural number n_0 such that $\zeta_n < \alpha_n \varepsilon$ for all $n \geq n_0$. Thus, we obtain

(3.75)
$$\begin{aligned} D_f(\mathring{x}, x_{n+1}) &\leq (1-\alpha_n) D_f(\mathring{x}, x_n) + \frac{2\varepsilon\alpha_n}{\beta} D_f(\mathring{x}, x_n) + \frac{\zeta_n}{2} \\ &- (1-\alpha_n)\langle \gamma_n T^* J_{E_3}(Te_n - Sh_n), z_n - \mathring{x}\rangle + \langle \alpha_n(\nabla f x - \nabla f \mathring{x}), x_{n+1} - \mathring{x}\rangle \\ &= \left[1 - \alpha_n\left(1 - \frac{2\varepsilon}{\beta}\right)\right] D_f(\mathring{x}, x_n) + \frac{\zeta_n}{2} + \langle \alpha_n(\nabla f x - \nabla f \mathring{x}), x_{n+1} - \mathring{x}\rangle \\ &- (1-\alpha_n)\langle \gamma_n T^* J_{E_3}(Te_n - Sh_n), z_n - \mathring{x}\rangle \\ &\leq \left[1 - \alpha_n\left(1 - \frac{2\varepsilon}{\beta}\right)\right] D_f(\mathring{x}, x_n) + \alpha_n ||\nabla f x - \nabla f \mathring{x}|| \, ||x_{n+1} - x_n|| + \frac{\alpha_n \zeta_n}{2\alpha_n} \\ &+ \langle \alpha_n(\nabla f x - \nabla f \mathring{x}), x_n - \mathring{x}\rangle - (1-\alpha_n)\langle \gamma_n T^* J_{E_3}(Te_n - Sh_n), z_n - \mathring{x}\rangle \\ &\leq \left[1 - \alpha_n\left(1 - \frac{2\varepsilon}{\beta}\right)\right] D_f(\mathring{x}, x_n) - (1-\alpha_n)\langle \gamma_n T^* J_{E_3}(Te_n - Sh_n), z_n - \mathring{x}\rangle \\ &+ \alpha_n\left(1 - \frac{2\varepsilon}{\beta}\right)\left[\frac{\beta ||\nabla f x - \nabla f \mathring{x}|| \, ||x_{n+1} - x_n||}{\beta - 2\varepsilon}\right] \\ &+ \alpha_n\left(1 - \frac{2\varepsilon}{\beta}\right)\left[+\frac{\beta\langle \nabla f x - \nabla f \mathring{x}, x_n - \mathring{x}\rangle}{\beta - 2\varepsilon} + \frac{\beta\zeta_n}{(2\beta - 4\varepsilon)\alpha_n}\right] \\ &= \left[1 - \alpha_n\left(1 - \frac{2\varepsilon}{\beta}\right)\right] D_f(\mathring{x}, x_n) + \alpha_n\left(1 - \frac{2\varepsilon}{\beta}\right)\Delta_n \\ &- (1-\alpha_n)\langle \gamma_n T^* J_{E_3}(Te_n - Sh_n), z_n - \mathring{x}\rangle, \end{aligned}$$

where

$$\Delta_n = \left[\frac{\beta||\nabla fx - \nabla f\mathring{x}||\,||x_{n+1} - x_n||}{\beta - 2\varepsilon} + \frac{\beta\langle\nabla fx - \nabla f\mathring{x}, x_n - \mathring{x}\rangle}{\beta - 2\varepsilon} + \frac{\beta\zeta_n}{(2\beta - 4\varepsilon)\,\alpha_n}\right].$$

Similarly,

$$\begin{aligned} D_f(\mathring{w}, w_{n+1}) \le \left[1 - \alpha_n\left(1 - \frac{2\varepsilon}{\beta}\right)\right] D_f(\mathring{w}, w_n) + \alpha_n\left(1 - \frac{2\varepsilon}{\beta}\right)\Upsilon_n \\ - (1 - \alpha_n)\langle\gamma_n S^* J_{E_3}(Sh_n - Te_n), r_n - \mathring{w}\rangle, \end{aligned} \tag{3.76}$$

where

$$\Upsilon_n = \left[\frac{\beta||\nabla fw - \nabla f\mathring{w}||\,||w_{n+1} - w_n||}{\beta - 2\varepsilon} + \frac{\beta\langle\nabla fw - \nabla f\mathring{w}, w_n - \mathring{w}\rangle}{\beta - 2\varepsilon} + \frac{\beta\zeta_n}{(2\beta - 4\varepsilon)\,\alpha_n}\right].$$

Denote $\Sigma^* = D_f(\mathring{x}, x) + D_f(\mathring{w}, w)$ and $\Omega_n^* = D_f(\mathring{x}, x_n) + D_f(\mathring{w}, w_n)$. Now, combining (3.75) and (3.76) and using the relation (3.70), we obtain

(3.77)

$$\begin{aligned} \Omega_{n+1}^* &\le \left[1 - \alpha_n\left(1 - \frac{2\varepsilon}{\beta}\right)\right]\Omega_n^* + \alpha_n\left(1 - \frac{2\varepsilon}{\beta}\right)(\Delta_n + \Upsilon_n) - (1 - \alpha_n)\frac{\rho}{2}||Te_n - Sh_n||^2 \\ &\le \left[1 - \alpha_n\left(1 - \frac{2\varepsilon}{\beta}\right)\right]\Omega_n^* + \alpha_n\left(1 - \frac{2\varepsilon}{\beta}\right)(\Delta_n + \Upsilon_n). \end{aligned}$$

Adding the relations (3.64) and (3.65) with $\hat{x} = \mathring{x}$ and $\hat{w} = \mathring{w}$ and using (3.70) gives

$$\begin{aligned} &(1 - \alpha_n)\left(1 - \frac{2\zeta_n}{\beta}\right) D_f(e_n, x_n) + (1 - \alpha_n)\left(1 - \frac{\mu}{\beta}\right) D_f(a_n, y_n) \\ &+ (1 - \alpha_n)\left(1 - \frac{\mu}{\beta}\right) D_f(y_n, z_n) + (1 - \alpha_n)\left(1 - \frac{2\zeta_n}{\beta}\right) D_f(h_n, w_n) \\ &+ (1 - \alpha_n)\left(1 - \frac{\mu}{\beta}\right) D_f(k_n, t_n) + (1 - \alpha_n)\left(1 - \frac{\mu}{\beta}\right) D_f(t_n, r_n) \\ &+ \frac{\rho}{2}||Te_n - Sh_n||^2 \le \Omega_n^* - \Omega_{n+1}^* + \alpha_n\left[\Sigma^* + \left(\frac{2\varepsilon}{\beta} - 1\right)\Omega_n^* + \frac{\zeta_n}{\alpha_n}\right]. \end{aligned} \tag{3.78}$$

Now, suppose $\{\Omega_{n_k}^*\}$ is a subsequence of $\{\Omega_n^*\}$ with the property

$$\liminf_{k\to\infty}\left(\Omega_{n_k+1}^* - \Omega_{n_k}^*\right) \ge 0. \tag{3.79}$$

Taking the limit on both sides of (3.78) we obtain

$$\lim_{k\to\infty}||Te_{n_k} - Sh_{n_k}|| = 0. \tag{3.80}$$

From (3.78), (3.79) and Lemma 2.5, we obtain

$$\lim_{k\to\infty}||e_{n_k} - x_{n_k}|| = \lim_{k\to\infty}||a_{n_k} - y_{n_k}|| = \lim_{k\to\infty}||y_{n_k} - z_{n_k}|| = 0, \tag{3.81}$$

and

$$\lim_{k\to\infty}||h_{n_k} - w_{n_k}|| = \lim_{k\to\infty}||k_{n_k} - t_{n_k}|| = \lim_{k\to\infty}||t_{n_k} - r_{n_k}|| = 0. \tag{3.82}$$

From (3.45) and (3.80), we obtain

$$\begin{aligned} ||\nabla fz_{n_k} - \nabla fe_{n_k}|| &= \gamma_{n_k}||T^* J_{E_3}(Te_{n_k} - Sh_{n_k})|| \\ &\le (\rho + 1)||T^* J_{E_3}(Te_{n_k} - Sh_{n_k})|| \\ &\le (\rho + 1)||T^*||\,||J_{E_3}(Te_{n_k} - Sh_{n_k})|| \to 0 \; as \; k \to \infty, \end{aligned}$$

which implies, together with the uniform continuity of ∇f^*, that

$$\lim_{k\to\infty} ||z_{n_k} - e_{n_k}|| = 0. \tag{3.83}$$

From (3.48) and the condition on α_{n_k}, we have

$$\begin{aligned} \lim_{n\to\infty} ||\nabla f x_{n_k+1} - \nabla f a_{n_k}|| &= \lim_{k\to\infty} ||\alpha_{n_k} \nabla f x + (1-\alpha_{n_k})\nabla f a_{n_k} - \nabla f a_{n_k}|| \\ &= \lim_{k\to\infty} \alpha_{n_k} ||\nabla f x - \nabla f a_{n_k}|| = 0. \end{aligned} \tag{3.84}$$

From (3.84) and the uniform continuity of ∇f^*, we obtain

$$\lim_{k\to\infty} ||x_{n_k+1} - a_{n_k}|| = 0. \tag{3.85}$$

Consequently, from (3.81), (3.83) and (3.85), we obtain

$$\begin{aligned} ||x_{n_k+1} - x_{n_k}|| &\le ||x_{n_k+1} - a_{n_k}|| + ||a_{n_k} - y_{n_k}|| + ||y_{n_k} - z_{n_k}|| \\ &+ ||z_{n_k} - e_{n_k}|| + ||e_{n_k} - x_{n_k}|| \to 0 \ \ as \ k \to \infty. \end{aligned} \tag{3.86}$$

Similarly,

$$\lim_{k\to\infty} ||w_{n_k+1} - w_{n_k}|| = 0. \tag{3.87}$$

Since $\{(x_{n_k}, w_{n_k})\}$ is bounded in $E_1 \times E_2$ and E_1 and E_2 are reflexive, there exist a subsequence $\left\{(x_{n_{k_j}}, w_{n_{k_j}})\right\}$ of $\{(x_{n_k}, w_{n_k})\}$ and an element $(\tilde{x}, \tilde{w})$ of $E_1 \times E_2$ such that $(x_{n_{k_j}}, w_{n_{k_j}}) \rightharpoonup (\tilde{x}, \tilde{w})$ and

$$\begin{aligned} &\limsup_{k\to\infty} \langle (\nabla f x, \nabla f w) - (\nabla f \mathring{x}, \nabla f \mathring{w}), (x_{n_k}, w_{n_k}) - (\mathring{x}, \mathring{w}) \rangle \\ &= \lim_{j\to\infty} \langle (\nabla f x, \nabla f w) - (\nabla f \mathring{x}, \nabla f \mathring{w}), (x_{n_{k_j}}, w_{n_{k_j}}) - (\mathring{x}, \mathring{w}) \rangle. \end{aligned}$$

Moreover, we have $x_{n_{k_j}} \rightharpoonup \tilde{x}$ and $w_{n_{k_j}} \rightharpoonup \tilde{w}$. Now, we show that $(\tilde{x}, \tilde{w}) \in \Pi$. Let $(\bar{y}, \bar{z}) \in G(A_1)$, where $G(A_1)$ is the graph of A_1. We have from (3.46) that

$$\nabla f y_{n_{k_j}} = \nabla f e_{n_{k_j}} - \lambda_{n_{k_j}} A_1 e_{n_{k_j}},$$

that is,

$$\frac{1}{\lambda_{n_{k_j}}} \left(\nabla f e_{n_{k_j}} - \nabla f y_{n_{k_j}} \right) - A_1 e_{n_{k_j}} = 0.$$

Thus, we have

$$\left\langle \bar{z} - A_1 \bar{y} - \frac{1}{\lambda_{n_{k_j}}} \left(\nabla f e_{n_{k_j}} - \nabla f y_{n_{k_j}} \right) + A_1 e_{n_{k_j}}, \bar{y} - y_{n_{k_j}} \right\rangle = 0,$$

which implies that

(3.88)

$$\begin{aligned}\langle \bar{z}, \bar{y} - y_{n_{k_j}} \rangle &= \Big\langle A_1\bar{y} + \frac{1}{\lambda_{n_{k_j}}}\Big(\nabla f e_{n_{k_j}} - \nabla f y_{n_{k_j}}\Big) - A_1 e_{n_{k_j}}, \bar{y} - y_{n_{k_j}} \Big\rangle \\ &= \langle A_1\bar{y} - A_1 e_{n_{k_j}}, \bar{y} - y_{n_{k_j}} \rangle + \Big\langle \frac{1}{\lambda_{n_{k_j}}}\Big(\nabla f e_{n_{k_j}} - \nabla f y_{n_{k_j}}\Big), \bar{y} - y_{n_{k_j}} \Big\rangle \\ &= \langle A_1\bar{y} - A_1 y_{n_{k_j}} + A_1 y_{n_{k_j}} - A_1 e_{n_{k_j}}, \bar{y} - y_{n_{k_j}} \rangle \\ &+ \Big\langle \frac{1}{\lambda_{n_{k_j}}}\Big(\nabla f e_{n_{k_j}} - \nabla f y_{n_{k_j}}\Big), \bar{y} - y_{n_{k_j}} \Big\rangle \\ &= \langle A_1\bar{y} - A_1 y_{n_{k_j}}, \bar{y} - y_{n_{k_j}} \rangle + \langle A_1 y_{n_{k_j}} - A_1 e_{n_{k_j}}, \bar{y} - y_{n_{k_j}} \rangle \\ &+ \Big\langle \frac{1}{\lambda_{n_{k_j}}}\Big(\nabla f e_{n_{k_j}} - \nabla f y_{n_{k_j}}\Big), \bar{y} - y_{n_{k_j}} \Big\rangle \\ &\geq \langle A_1 y_{n_{k_j}} - A_1 e_{n_{k_j}}, \bar{y} - y_{n_{k_j}} \rangle + \Big\langle \frac{1}{\lambda_{n_{k_j}}}\Big(\nabla f e_{n_{k_j}} - \nabla f y_{n_{k_j}}\Big), \bar{y} - y_{n_{k_j}} \Big\rangle,\end{aligned}$$

where the last inequality holds by the virtue of the monotonicity of A_1. Since A_1 and ∇f are uniformly continuous, we have from (3.81) and (3.83) that

$$\lim_{j\to\infty} ||A_1 e_{n_{k_j}} - A_1 y_{n_{k_j}}|| = \lim_{j\to\infty} ||\nabla f e_{n_{k_j}} - \nabla f y_{n_{k_j}}|| = 0.$$

Thus, from (3.88) we conclude that $\langle \bar{z}, \bar{y} - \tilde{x} \rangle \geq 0$. By the maximality of A_1, we have that $\tilde{x} \in N(A_1)$. Similarly, one can show that $\tilde{w} \in N(A_2)$.

Moreover, by Lemma 2.1, we have

(3.89)

$$\begin{aligned}||T\tilde{x} - S\tilde{w}||^2 &= ||Te_{n_{k_j}} - Sh_{n_{k_j}} + T\tilde{x} - Te_{n_{k_j}} + Sh_{n_{k_j}} - S\tilde{w}||^2 \\ &\leq ||Te_{n_{k_j}} - Sh_{n_{k_j}}||^2 + 2\langle J_{E_3}(T\tilde{x} - S\tilde{w}),\ T\tilde{x} - Te_{n_{k_j}} + Sh_{n_{k_j}} - S\tilde{w} \rangle.\end{aligned}$$

From (3.81) and (3.82), we have that $e_{n_{k_j}} \rightharpoonup \tilde{x}$ and $h_{n_{k_j}} \rightharpoonup \tilde{w}$. Since T and S bounded linear mappings, they are sequentially weakly continuous and hence we have $Te_{n_{k_j}} \rightharpoonup T\tilde{x}$ and $Sh_{n_{k_j}} \rightharpoonup S\tilde{w}$. Thus, we obtain from (3.80) and (3.89) that $T\tilde{x} = S\tilde{w}$. Therefore, $(\tilde{x}, \tilde{w}) \in \Pi$.

By (2.18), we have

$$\begin{aligned}&\limsup_{k\to\infty}\langle(\nabla f x, \nabla f w) - (\nabla f \mathring{x}, \nabla f \mathring{w}), (x_{n_k}, w_{n_k}) - (\mathring{x}, \mathring{w})\rangle \\ &= \lim_{j\to\infty}\langle(\nabla f x, \nabla f w) - (\nabla f \mathring{x}, \nabla f \mathring{w}), (x_{n_{k_j}}, w_{n_{k_j}}) - (\mathring{x}, \mathring{w})\rangle \\ &= \langle(\nabla f x, \nabla f w) - (\nabla f \mathring{x}, \nabla f \mathring{w}), (\tilde{x}, \tilde{w}) - (\mathring{x}, \mathring{w})\rangle \leq 0.\end{aligned} \tag{3.90}$$

From (3.77), (3.86), (3.87), (3.90) and Lemma 2.6, we obtain

$$\lim_{n\to\infty} \Omega_n^* = 0,$$

which implies that $\lim_{n\to\infty} D_f(\mathring{x}, x_n) = \lim_{n\to\infty} D_f(\mathring{w}, w_n) = 0$ and hence we obtain, by Lemma 2.5, that $||x_n - \mathring{x}|| \to 0$ as $n \to \infty$. Therefore, $\lim_{n\to\infty} x_n = \lim_{n\to\infty}(u_n, v_n) = (\mathring{u}, \mathring{v}) = \mathring{x}$, where $\mathring{u}$ is a solution of $u + K_1F_1u = 0$ with $\mathring{v} = F_1\mathring{u}$. Similarly, $\lim_{n\to\infty} w_n = \lim_{n\to\infty}(p_n, q_n) = (\mathring{p}, \mathring{q}) = \mathring{w}$, where $\mathring{p}$ is a solution of $p + K_2F_2p = 0$ with $\mathring{q} = F_2\mathring{p}$. The proof is complete. □

If, in Theorem 3.2, we assume that F_1, F_2, K_1 and K_2 are Lipschitz monotone mappings, then we obtain the following corollary.

Corollary 3.1. *Assume that the conditions $(C1), (C3) - (C7)$ hold. If $F_1 : X \to X^*$, $K_1 : X^* \to X$, $F_2 : Y \to Y^*$, $K_2 : Y^* \to Y$ are Lipschitz monotone mappings, then the sequences $\{u_n\}$, $\{v_n\}$, $\{p_n\}$ and $\{q_n\}$ generated by Algorithm 3.1 converge strongly to u^*, v^*, p^* and q^*, respectively, where $(u^*, p^*) \in \Lambda$ with $v^* = F_1u^*$ and $q^* = F_2p^*$.*

If we assume $g(x) = \dfrac{1}{2}||x||^2$ in Theorem 3.2, then $\nabla g = J$ and $\nabla g^* = J^{-1}$. Thus, we get the following corollary.

Corollary 3.2. *Assume that the conditions $(C1) - (C4), (C6)$, $(C7)$ are satisfied. Then, the sequences $\{u_n\}$, $\{v_n\}$, $\{p_n\}$ and $\{q_n\}$ generated by Algorithm 3.1, with $\nabla g = J$ and $\nabla g^* = J^{-1}$, converge strongly to u^*, v^*, p^* and q^*, respectively, where $(u^*, p^*) \in \Lambda$ with $v^* = F_1u^*$ and $q^* = F_2p^*$.*

If we assume that X, Y and Z are real Hilbert spaces and $g(x) = \dfrac{1}{2}||x||^2$ in Theorem 3.2, then $\nabla g = J = I$ and $\nabla g^* = J^{-1} = I$ and hence we obtain the following corollary.

Corollary 3.3. *Let X, Y and Z be real Hilbert spaces and assume that the Conditions $(C2) - (C4), (C6)$, $(C7)$ are satisfied. Then, the sequences $\{u_n\}$, $\{v_n\}$, $\{p_n\}$ and $\{q_n\}$ generated by Algorithm 3.1, with $\nabla g = I_X$ and $\nabla g^* = I_{X^*}$, converge strongly to u^*, v^*, p^* and q^*, respectively, where $(u^*, p^*) \in \Lambda$ with $v^* = F_1u^*$ and $q^* = F_2p^*$.*

4. Application

This section deals with applications of the main result to specific cases.

4.1. **Common Solutions of Hammerstein Type Equation Problems.** If we consider $X = Y = Z$ and $T_1 = S_1 = I_X$ in Algorithm 3.1, then SEHTEP reduces to *common solutions of the Hammerstein type equation problem*, which is defined as finding two points $u, p \in X$ such that $u + K_1F_1u = 0$, $p + K_2F_2p = 0$ and $u = p$.

Denote $\Gamma = \{(u, p) \in X \times X : u + K_1F_1u = 0, p + K_2F_2p = 0 \text{ and } u = p\}$.

Corollary 4.4. *Assume that conditions $(C1)$, $(C2)$ and $(C5) - (C7)$, with $X = Y = Z$ hold. If $\Gamma \neq \emptyset$, then the sequences $\{u_n\}$, $\{v_n\}$, $\{p_n\}$ and $\{q_n\}$ generated by Algorithm 3.1 with $T_1 = S_1 = I_X$ converge strongly to u^*, v^*, p^* and q^*, respectively, where $(u^*, p^*) \in \Gamma$ with $v^* = F_1u^*$ and $q^* = F_2p^*$.*

Corollary 4.5. *Assume that conditions $(C1)$ and $(C5) - (C7)$, with $X = Y = Z$, hold. Let F_1: $X \to X^*$, K_1: $X^* \to X$, F_2: $X \to X^*$, K_2: $X^* \to X$ be Lipschitz monotone mappings. Assume that $\Gamma \neq \emptyset$. Then, the sequences $\{u_n\}$, $\{v_n\}$, $\{p_n\}$ and $\{q_n\}$ generated by Algorithm 3.1 with $T_1 = S_1 = I_X$ converge strongly to u^*, v^*, p^* and q^*, respectively, where $(u^*, p^*) \in \Gamma$ with $v^* = F_1u^*$ and $q^* = F_2p^*$.*

4.2. **Split Hammerstein Type Equation Problems.** If we take $Y = Z$ and $S_1 = I_Y$ in Algorithm 3.1, then the SEHTEP reduces to *split Hammerstein type equation problem* which seeks to find two points $u \in X$ and $p \in Y$ such that $u + K_1F_1u = 0$, $p + K_2F_2p = 0$ and $T_1u = p$.

Denote $\Gamma^* = \{(u, p) \in X \times Y : u + K_1F_1u = 0, \ p + K_2F_2p = 0 \text{ and } T_1u = p\}$.

Corollary 4.6. *Assume that conditions $(C1) - (C3)$ and $(C5) - (C7)$ hold with $Y = Z$, $S_1 = I_Y$. Let $\Gamma^* \neq \emptyset$. Then, the sequences $\{u_n\}$, $\{v_n\}$, $\{p_n\}$ and $\{q_n\}$ generated by Algorithm 3.1 converge strongly to u^*, v^*, p^* and q^*, respectively, where $(u^*, p^*) \in \Gamma^*$ with $v^* = F_1u^*$ and $q^* = F_2p^*$.*

Corollary 4.7. *Assume that conditions $(C1),(C3)$ and $(C5)-(C7)$ with $Y = Z$ and $S_1 = I_Y$ hold. Let $F_1\colon X \to X^*$, $K_1\colon X^* \to X$, $F_2\colon Y \to Y^*$, $K_2\colon Y^* \to Y$ be Lipschitz monotone mappings and $\Gamma^* \neq \emptyset$. Then, the sequences $\{u_n\}$, $\{v_n\}$, $\{p_n\}$ and $\{q_n\}$ generated by Algorithm 3.1 converge strongly to u^*, v^*, p^* and q^*, respectively, where $(u^*, p^*) \in \Gamma^*$ with $v^* = F_1u^*$ and $q^* = F_2p^*$.*

4.3. **Hammerstein Type Equation Problems.** If we take $X = Y = Z$, $F_2 = \nabla f$, $K_2 = -\nabla f^*$, $T_1 = S_1 = 0$ in Algorithm 3.1, then the SEHTEP reduces to *Hammerstein type equation problem* which seeks to find a point $u \in X$ such that $u + K_1F_1u = 0$. Denote $\Psi = \{u \in X : u + K_1F_1u = 0\}$.

Corollary 4.8. *Let X be a reflexive real Banach space with its dual X^*. Let $F_1\colon X \to X^*$ and $K_1\colon X^* \to X$ be uniformly continuous monotone mappings. Assume that $\Psi \neq \emptyset$. Let $g\colon X \to (-\infty, +\infty]$ be a strongly coercive, lower semi-continuous, strongly convex, bounded and uniformly Fréchet differentiable Legendre function on bounded subsets of X with strongly convex conjugate g^*. Let the strong convexity constants of g and g^* be β_1 and β_2, respectively, and let $\beta = \min\{\beta_1, \beta_2\}$. If conditions $(C6)-(C7)$, hold, then the sequence $\{(u_n, v_n)\}$ generated by Algorithm 3.1, with $X = Z$, $F_2 = \nabla f$, $K_2 = -\nabla f^*$ and $T_1 = S_1 = 0$, converges strongly to (u^*, v^*), where $u^* \in \Psi$ and $v^* = F_1u^*$.*

Corollary 4.9. *Let X be a reflexive real Banach space with its dual X^*. Let $F_1\colon X \to X^*$ and $K_1\colon X^* \to X$ be Lipschitz monotone mappings. Assume that $\Psi \neq \emptyset$. Let $g\colon X \to (-\infty, +\infty]$ be a strongly coercive, lower semi-continuous, strongly convex, bounded and uniformly Fréchet differentiable Legendre function on bounded subsets of X with strongly convex conjugate g^*. Let the strong convexity constants of g and g^* be β_1 and β_2, respectively, and let $\beta = \min\{\beta_1, \beta_2\}$. If conditions $(C6)-(C7)$, hold, then the sequence $\{(u_n, v_n)\}$ generated by Algorithm 3.1, with $X = Z$, $F_2 = \nabla f$, $K_2 = -\nabla f^*$ and $T_1 = S_1 = 0$, converges strongly to (u^*, v^*), where $u^* \in \Psi$ and $v^* = F_1u^*$.*

5. Numerical example

In this section, we give examples of uniformly continuous monotone mappings which satisfy the conditions of Theorem 3.2. A numerical experiment is also provided to illustrate the applicability of the algorithm.

Example 5.1. Let $X = Y = Z = L_p^{\mathbb{R}}([0,1])$, where $1 < p < \infty$ with the norm $||x||_{L_p} = \left(\int_0^1 |x(t)|^p dt\right)^{\frac{1}{p}}$ and $g\colon X \to (-\infty, +\infty]$ be defined by $g(x) = \frac{1}{p}||x||^p$. Then, $X^* = Y^* = Z^* = L_q^{\mathbb{R}}([0,1])$, where $\frac{1}{p} + \frac{1}{q} = 1$. Let F_1, F_2: $X \to X^*$ and K_1, K_2: $X^* \to X$ be defined by $(F_1u)(t) = 2\nabla gu(t) = 2J_pu(t)$, $(F_2p)(t) = 3\nabla gp(t) = 3J_pp(t)$, $(K_1v)(t) = \nabla g^*v(t) - 2 = J_p^{-1}v(t) - 2$ and $(K_2q)(t) = \nabla g^*q(t) + 1 = J_p^{-1}q(t) + 1$. One can show that F_1, K_1, F_2 and K_2 are uniformly continuous monotone mappings and the functions $u^*(t) = \frac{2}{3}$ and $p^*(t) = \frac{-1}{4}$ are solutions of the equations $u(t) + K_1F_1u(t) = 0$ and $p(t) + K_2F_2p(t) = 0$, respectively. Now, define T_1: $X \to Z$ and S_1: $Y \to Z$ by

$$(T_1u)(t) = \frac{1}{2}u(t) \quad \text{and} \quad (S_1p)(t) = \left(\frac{-4}{3}\right)p(t).$$

Clearly, T_1 and S_1 are bounded linear mappings with

$$T_1\left(\frac{2}{3}\right) = \frac{1}{3} = S_1\left(\frac{-1}{4}\right).$$

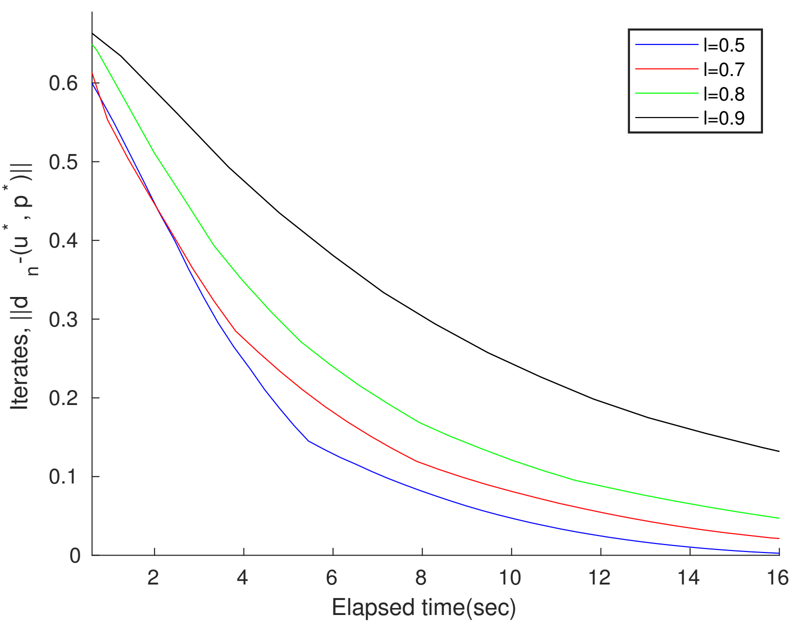
Therefore,

$$(u^*(t), p^*(t)) = \left(\frac{2}{3}, \frac{-1}{4}\right) \in \Lambda =$$
$$\{(u^*, p^*) \in X \times Y : u^* + K_1F_1u^* = 0, p^* + K_2F_2p^* = 0 \; and \; T_1u^* = S_1p^*\}.$$

For simplicity of the computation, we take $p = 2$ so that $g(x) = g^*(x) = \frac{1}{2}||x||^2$ and $\nabla gx = \nabla g^*x = Jx = Ix = x$, where I is the identity mapping on $L_2^{\mathbb{R}}([0,1])$. Then, the mappings F_1, K_1, F_2 and K_2 reduce to $(F_1u)(t) = 2u(t)$, $(K_1v)(t) = v(t) - 2$, $(F_2p)(t) = 3p(t)$ and $(K_2q)(t) = q(t) + 1$. If we consider $\zeta_n = \frac{1}{n^2 + 1000}$, $\alpha_n = \frac{1}{n + 100000}$, $\mu = 0.3$, $\gamma = 0.7$ and

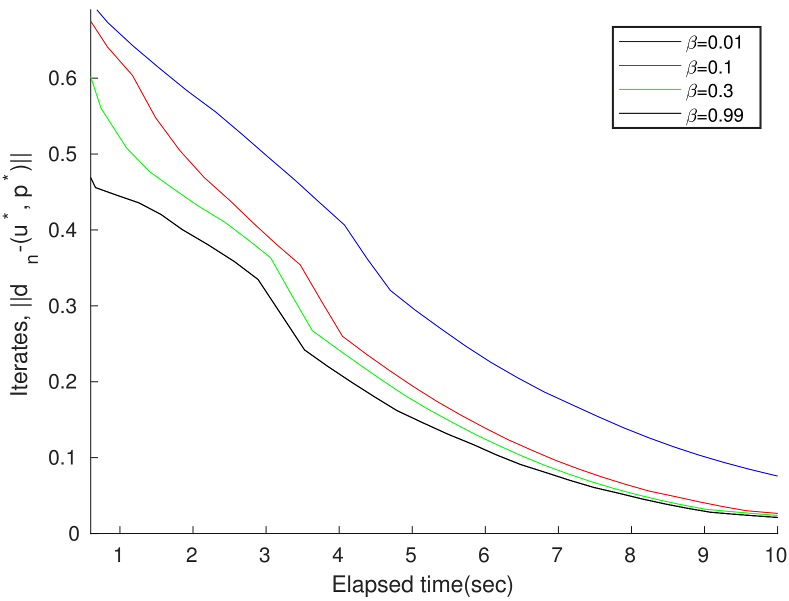
$$\gamma_n = \begin{cases} \left(\frac{\beta}{2}\right) \dfrac{\left\|\frac{1}{2}e_{1n} + \frac{4}{3}h_{1n}\right\|^2}{\left\|\frac{1}{4}e_{1n} + \frac{2}{3}h_{1n}\right\|^2 + \left\|\frac{16}{9}h_{1n} + \frac{2}{3}e_{1n}\right\|^2} & if \;\; n \in \Omega, \\ \frac{1}{1000000} \; if \;\; n \notin \Omega, \end{cases}$$

then the conditions $(C1) - (C7)$ are satisfied and the numerical MATLAB experiments shown below indicate that the sequence $\{d_n\} = \{(u_n, p_n)\}$ generated by Algorithm 3.1 converges strongly to a solution (u^*, p^*) of Problem 5.1 for different choices of l, β and θ as it is shown in the figures below.



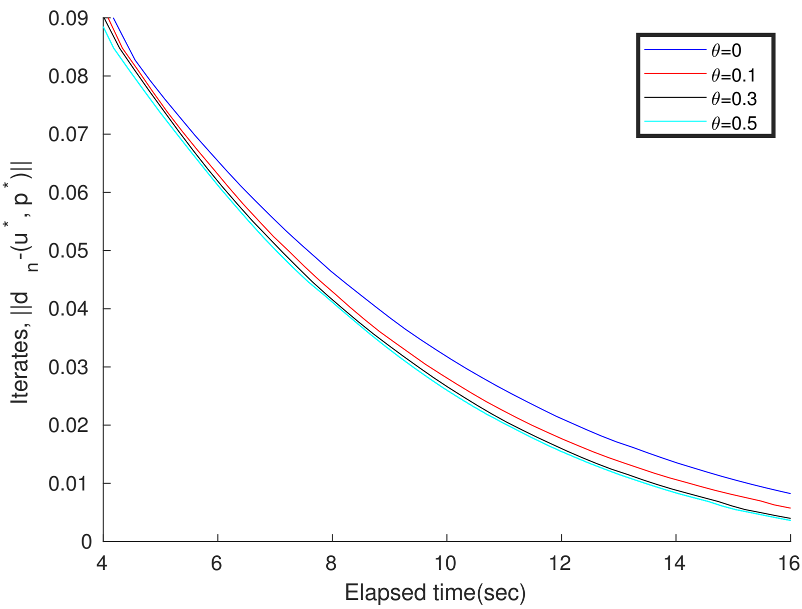
$(u_0, v_0, p_0, q_0) = (0, 0.5, 0, 0.5), \;\; \theta = 0.5, \;\; \beta = 0.5.$

FIGURE 1



$(u_0, v_0, p_0, q_0) = (0.5, 0, 0.5, 0),\ \theta = 0.5,\ l = 0.5.$

FIGURE 2



$(u_0, v_0, p_0, q_0) = (0.5, 0.5, 0.5, 0.5),\ l = 0.2,\ \beta = 0.5.$

FIGURE 3

Remark 5.3. FIGURE 1 reveals that the convergence of the method gets faster as l gets closer to zero while all other parameters and initial points are kept fixed. We also observe, from FIGURE 2, that the convergence of the method gets faster as the strong convexity constant, β, of f gets closer to 1 keeping all other parameters and the initial point fixed. From FIGURE 3, we observe that: (i) the rate of convergence of the method with inertial algorithm is faster than that of the non-inertial version, and (ii) the rate of convergence is different for different values of the inertial parameter θ and it also seems that the larger θ has better convergence rate.

6. CONCLUSIONS

In this paper, we have proposed an inertial iterative algorithm which solves the split equality Hammerstein type equation problems in reflexive real Banach spaces. A strong convergence theorem is established under the assumption that the associated mappings are uniformly continuous and monotone. The convergence of the method does not require the existence of a constant γ_0, unlike the results in Chidume and Zegeye [15], Uba *et al.* [36] and Bello *et al.* [3]. A numerical example is also provided to clearly exhibit the behavior of the convergence of the proposed method. Generally, the main result in this paper extends many of the results in the literature in the sense that the space considered is a more general reflexive real Banach space with a more general split equality Hammerstein type equation problems.

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