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In memoriam Professor Charles E. Chidume (1947- 2021)

Fixed points and coupled fixed points in b-metric spaces via graphical contractions

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ABSTRACT. In this paper some existence and stability results for cyclic graphical contractions in complete metric spaces are given. An application to a coupled fixed point problem is also derived.

1. INTRODUCTION AND PRELIMINARIES

In this paper, we will prove some fixed point and coupled fixed point theorems in complete b-metric spaces. Our results extend some recent theorems proved in classical metric spaces.

We recall first some notions and results.

Definition 1.1. Let M be a nonempty set and let $s \geq 1$ be a given real number. A functional $d : M \times M \to \mathbb{R}_+$ is said to be a b-metric (also called in some papers quasi-metric) with constant $s \geq 1$ if the Fréchet axioms of the metric are satisfied, except the so-called triangle inequality axiom, which has the following form:

$$(\star)\ \ d(x,z) \leq s[d(x,y) + d(y,z)], \text{ for all } x, y, z \in M.$$

A pair (M, d) with the above properties is called a b-metric space with constant $s \geq 1$.

Some interesting examples and a very recent work regarding the origins of the notion of b-metric space are given in [2], [3], [4], [5], [6], [9]. It is known that some topological properties in the setting of b-metric spaces are the same as in metric spaces.

Definition 1.2. Let (M, d) be a b-metric space. Then, a subset Y of M is called:

(1) compact if for every sequence of elements of Y there exists a subsequence that converges to an element of Y.

(2) closed if for each sequence $(x_n)_{n\in\mathbb{N}}$ in Y which converges to an element x, we have $x \in Y$.

The b-metric space (M, d) is complete if every Cauchy sequence from M converges in X.

Lemma 1.1. *Notice that in a b-metric space (M, d) the following assertions hold:*
(i) a convergent sequence has a unique limit;
(ii) each convergent sequence is Cauchy.

Although, there are some important distance-type differences: the b-metric on M need not be continuous, open balls in b-metric spaces need not be open sets, the closed ball is not necessary a closed set, to recall few.

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Definition 1.3. [8] Let (M,d) be a b-metric space. Let p be a positive integer with $p \geq 2$, let $K_1, K_2, ..., K_p$ be subsets of M, and $\tilde{K} := \bigcup_{i=1}^{p} K_i$. Then, $T : \tilde{K} \to \tilde{K}$ is called a cyclic operator if

(i) the sets $K_i \neq \emptyset$ for every $i \in \{1, 2, ...p\}$;

(ii) $\bigcup_{i=1}^{p} K_i$ is a cyclical representation of $\tilde{K}$ with respect to T, i.e.,

$$T(K_1) \subseteq K_2, T(K_2) \subseteq K_3, \cdots T(K_{p-1}) \subseteq K_p, T(K_p) \subseteq K_1.$$

Let X be a nonempty set and $T : X \to X$ be a single-valued operator. We denote by $Fix(T) := \{x \in X : x = T(x)\}$ the fixed point set of T.

Definition 1.4. [20] Let (M,d) a b-metric space. An operator $T : M \to M$ is called a weakly Picard operator (WPO) if the sequence $(T^n(x))_{n\in\mathbb{N}}$ converges for all $x \in M$ and its limit, denoted by $T^\infty(x)$, is a fixed point for T.

Definition 1.5. [20] In the above context, if T is a WPO and $Fix(T) = \{x^*\}$, then, by definition, T is a Picard operator.

If (M,d) is a b-metric space and $F : M \times M \to M$ is an operator, then, by definition, a coupled fixed point for F is a pair $(x^*, y^*) \in M \times M$ satisfying

$$\left\{ \begin{array}{l} x^* = F(x^*, y^*) \\ y^* = F(y^*, x^*). \end{array} \right. \tag{1.1}$$

Another generalization of the classical metric of Fréchet is the vector-valued metric. In this case, if M is a nonempty set, then a mapping $d : M \times M \to \mathbb{R}^m$ is a vector-valued metric (or a Perov type metric) if d satisfies all the axioms of the metric with respect to the componentwise inequality between vectors in $\mathbb{R}^m$. If the triangle inequality takes the form given in $(\star)$, then we say that (M,d) is a generalized b-metric space in the sense of Perov with constant $s \geq 1$. In particular, if $m = 1$ we obtain the above presented notion of b-metric.

We denote by $M_{mm}(\mathbb{R}_+)$ the set of all $m \times m$ matrices with positive elements, by I_m the identity $m \times m$ matrix and by O_m the null $m \times m$ matrix.

Definition 1.6. A square matrix $A \in M_{mm}(\mathbb{R}_+)$ is said to be convergent to zero if and only if its spectral radius $\rho(A)$ is strictly less than 1. In other words, this means that all the eigenvalues of A are in the open unit disc.

We have the following characterization theorem for a matrix convergent to zero.

Lemma 1.2. *(see e.g.* [16], [18]*) Let $A \in \mathcal{M}_{m,m}(\mathbb{R}_+)$. Then the following statements are equivalent:*

(1) A is a matrix convergent to zero;
(2) $A^n \longrightarrow O_m$ as $n \to \infty$;
(3) $I_m - A$ is non-singular and $(I_m - A)^{-1} = I_m + A + \ldots + A^n + \ldots$;
(4) $I_m - A$ is non-singular and $(I_m - A)^{-1}$ has nonnegative elements.

Definition 1.7. Let (M,d) be a generalized b-metric space in the sense of Perov and let $f : M \to M$ be an operator. Then, f is called an A-contraction if and only if $A \in \mathcal{M}_{m,m}(\mathbb{R}_+)$ is a matrix convergent to zero and

$$d(f(x), f(y)) \leq Ad(x,y), \text{ for any } (x,y) \in M \times M.$$

If the above condition holds for every $(x,y) \in Graph(f)$, i.e.,

$$d\left(f(x), f^2(x)\right) \leq Ad(x, f(x)), \text{ for any } x \in M,$$

then f is called a graphical (orbital) A-contraction.

Notice that any A-contraction $f : M \to M$ on a generalized b-metric space in the sense of Perov (M, d) is continuous, in the sense that for any convergent sequence $\{x_n\}_{n\in\mathbb{N}} \subset M$ to $\tilde{x} \in M$, the sequence $\{f(x_n)\}_{n\in\mathbb{N}}$ converges to $f(\tilde{x})$. Not the same is true for graphical (orbital) A-contraction.

In particular, if $m = 1$ we get the classical notions of (Banach) a-contraction and graphical (orbital) a-contraction in b-metric spaces, where $A := a \in]0, 1[$.

2. Main results

We recall first the following important result given by Miculescu and Mihail.

Lemma 2.3. [11] *Every sequence $(x_n)_{n\in\mathbb{N}}$ of elements from a b-metric space (M, d) with constant s having the property that there exists $\gamma \in [0, 1[$ such that $d(x_{n+1}, x_n) \leq \gamma d(x_n, x_{n-1})$, $n \in \mathbb{N}$ is a Cauchy sequence. Moreover, the following estimation holds*

$$d(x_{n+1}, x_{n+p}) \leq \frac{\gamma^n S}{1-\gamma} d(x_0, x_1), \text{ for all } n, p \in \mathbb{N},$$

where $S := \sum_{i=1}^{\infty} \gamma^{2i \log_\gamma s + 2^{i-1}}$.

Our first main result is the following theorem in b-metric spaces.

Theorem 2.1. *Let (M, d) be a complete b-metric space with constant $s \geq 1$, $p \in \mathbb{N}$ with $p \geq 2$ and let $K_1, K_2, ..., K_p$ be nonempty and closed subsets of M. Consider $\tilde{K} = \bigcup_{i=1}^{p} K_i$ and $T : \tilde{K} \to \tilde{K}$ be such that $\bigcup_{i=1}^{p} K_i$ is a cyclical representation of $\tilde{K}$ with respect to T. Suppose that T is a cyclic graphical (orbital) a-contraction, i.e., $a \in]0, 1[$ and*

$$d(T(x), T^2(x)) \leq a d(x, T(x)), \text{ for every } x \in \tilde{K}.$$

Then:

(i) $\bigcap_{i=1}^{p} K_i \neq \emptyset$ *and* $T : \bigcap_{i=1}^{p} K_i \to \bigcap_{i=1}^{p} K_i$;

(ii) if, additionally, T has closed graph, then:

(ii)-(a) T is a weakly Picard operator with the constant $\frac{1}{1-a}$ on $\bigcap_{i=1}^{p} K_i$, i.e., $Fix(T) \neq \emptyset$ and, for every element $x \in \bigcap_{i=1}^{p} K_i$, the sequence $\{T^n(x)\}_{n\in\mathbb{N}}$ converges to $T^\infty(x) \in Fix(T)$;

(ii)-(b) the following apriori estimation holds:

$$d(T^{n+1}(x), T^\infty(x)) \leq \frac{a^n s S}{1-a} d(x, T(x)), n \in \mathbb{N}, \text{ for all } x \in \bigcap_{i=1}^{p} K_i,$$

where $S := \sum_{i=1}^{\infty} a^{2i \log_a s + 2^{i-1}}$;

(iii)-(c) the following retraction-displacement condition holds

$$d(x, T^\infty(x)) \leq \frac{s(1-a+sS)}{1-a} d(x, T(x)), n \in \mathbb{N}, \text{ for all } x \in \bigcap_{i=1}^{p} K_i,$$

where $S := \sum_{i=1}^{\infty} a^{2i \log_a s + 2^{i-1}}$;

(iv)-(d) if $s < \sqrt{\frac{1-a}{2S}}$*, then* T *is a quasi-contraction, in the sense that*

$$d(T(x), T^{\infty}(x)) \leq \beta d(x, T^{\infty}(x)), \text{ for all } x \in \bigcap_{i=1}^{p} K_i,$$

where $\beta := \frac{s^2 S}{1-a-s^2 S} \in]0,1[$ *and* $S := \sum_{i=1}^{\infty} a^{2i \log_a s + 2^{i-1}}$.

Proof. (i) Let $x_0 \in \bigcup_{i=1}^{p} K_i$ be arbitrary. Then, there exists $i_0 \in \mathbb{N}$ such that $x_0 \in K_{i_0}$. Hence, $x_1 := T(x_0) \subset T(K_{i_0}) \subset K_{i_0+1}$. Then, for $x_1 \in K_{i_0+1}$ we have $x_2 := T(x_1) \in T(K_{i_0+1}) \subset K_{i_0+2}$. Inductively, we get a sequence $\{x_n\}_{n \in \mathbb{N}}$, with $x_{n+1} = T(x_n) = T^{n+1}(x_0) \in \bigcup_{i=1}^{p} K_i$, for each $n \in \mathbb{N}$.

If $x_n = x_{n+1}$, then x_n is a fixed point of T. We suppose that $x_n \neq x_{n+1}$ for all $n \geq 0$.

From the graphical contraction condition it follows that

$$d(x_n, x_{n+1}) = d(T(x_{n-1}), T(x_n)) = d(T(x_{n-1}), T^2(x_{n-1})) \leq ad(x_{n-1}, T(x_{n-1})) = ad(x_{n-1}, x_n).$$

Applying Lemma 2.3 for $\gamma = a$, we deduce that $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence. From the same lemma we also have that

$$d(x_{n+1}, x_{n+p}) \leq \frac{a^n S}{1-a} d(x_0, x_1), \text{ for all } n, p \in \mathbb{N}, \tag{2.2}$$

where $S := \sum_{i=1}^{\infty} a^{2i \log_a s + 2^{i-1}}$.

Since $(x_n)_{n \in \mathbb{N}}$ is Cauchy, by the completeness of the $b-$metric, we have that the sequence converges $x^* := x^*(x) \in \bigcup_{i=1}^{p} K_i$.

Moreover, we observe that infinitely many terms of $(x_n)_{n \in \mathbb{N}}$ lie in each K_i, $i \in \{1, 2, ..., p\}$. Thus $x^* \in \bigcap_{i=1}^{p} K_i$. By the cyclical representation of $\tilde{K}$ with respect to T, we get that $T : \bigcap_{i=1}^{p} K_i \to \bigcap_{i=1}^{p} K_i$.

(ii) Since $(T^n(x_0))_n$ converges to x^*, the closed graph condition of T implies that $x^* \in Fix(T)$.

In addition, from (2.2), we get

$$d(T^{n+1}(x_0), x^*) \leq s(d(x_{n+1}, x_{n+k}) + d(x_{n+k}, x^*) \leq$$
$$\frac{a^n sS}{1-a} d(x_0, T(x_0)) + sd(x_{n+k}, x^*), n, k \in \mathbb{N}.$$

By letting $k \to \infty$ we obtain that

$$d(T^{n+1}(x_0), x^*) \leq \frac{a^n sS}{1-a} d(x_0, T(x_0)), n \in \mathbb{N}.$$

(iii) By (ii), for $n = 0$ we get

$$d(T(x), T^{\infty}(x)) \leq \frac{sS}{1-a} d(x, T(x)), \text{ for all } x \in \bigcap_{i=1}^{p} K_i.$$

Thus, for all $x \in \bigcap_{i=1}^{p} K_i$, we have that

$$d(x, T^\infty(x)) \le s(d(x, T(x)) + d(T(x), T^\infty(x))) \le \frac{s(1-a+sS)}{1-a} d(x, T(x)).$$

(iv) As before, by (ii), for $n = 0$ we get

$$d(T(x), T^\infty(x)) \le \frac{sS}{1-a} d(x, T(x)), \text{ for all } x \in \bigcap_{i=1}^{p} K_i.$$

Then, we have:

$$d(T(x), T^\infty(x)) \le \frac{sS}{1-a} d(x, T(x)) \le \frac{s^2 S}{1-a} \left[d(x, T^\infty(x)) + d(T(x), T^\infty(x))\right].$$

Hence, we conclude that

$$d(T(x), T^\infty(x)) \le \frac{s^2 S}{1-a-s^2 S} d(x, T^\infty(x)), \text{ for all } x \in \bigcap_{i=1}^{p} K_i.$$

□

Example 2.1. Let $X = [0, +\infty[$ be equipped with $d : X \times X \to \mathbb{R}^+$, defined by $d = |x-y|^2$.

Let $A_1 = [0, \frac{1}{2}],\ \ A_2 = [\frac{1}{4}, 1]$ be subsets of $X = \mathbb{R}^+$. Define $T : \bigcup_{i=1}^{3} A_i \to \bigcup_{i=1}^{3} A_i$ by

$$T(x) := \begin{cases} \frac{2}{5}, & x \in [0, \frac{1}{2}[\\ 1 - x, & x \in [\frac{1}{2}, 1]. \end{cases}$$

Notice that (X, d) is a complete b-metric space with $b = \frac{1}{2}$. Moreover $T(A_1) \subseteq A_2, T(A_2) \subseteq A_1$. Then $\bigcup_{i=1}^{2} A_i$ is a cyclic representation with respect to T. Additionally, T satisfies all the assumptions (i-iv) in Theorem 2.1, i.e., T is a cyclic graphical $\frac{1}{4}$-contraction with respect to d.

We also observe that $Fix(T) = \{\frac{2}{5}, \frac{1}{2}\}$.

As a consequence of the first main result we can prove some stability results for cyclic graphical contractions in b-metric spaces.

Definition 2.8. Let (M, d) be a b-metric space with constant $s \ge 1$, $T : M \to M$ be an operator with $Fix(T) \ne \emptyset$ and let $r : M \to Fix(T)$ be a set retraction. Then:

(a) the fixed point equation $x = T(x), x \in M$ is said to be well-posed in the sense of Reich and Zaslavski if for each $x^* \in Fix(T)$ and for any sequence $(y_n)_{n\in\mathbb{N}}$ in $r^{-1}(x^*)$ for which

$$d(y_n, T(y_n)) \to 0 \text{ as } n \to \infty,$$

we have that

$$y_n \to x^* \text{ as } n \to \infty.$$

(b) the fixed point equation

$$x = T(x), x \in M, \tag{2.3}$$

is said to be Ulam-Hyers stable if there exists $c > 0$ such that for any $\varepsilon > 0$ and any ε-solution z of the fixed point equation (2.3), i.e.,

$$d(z, T(z)) \le \varepsilon$$

there exists $x^* \in Fix(T)$ such that $d(z, x^*) \le c\varepsilon$.

(c) The operator T has the Ostrowski stability property if for each $x^* \in Fix(T)$ and for any sequence $(z_n)_{n\in\mathbb{N}}$ in $r^{-1}(x^*)$ for which

$$d(z_{n+1}, T(z_n)) \to 0 \text{ as } n \to \infty,$$

we have that

$$z_n \to x^* \text{ as } n \to \infty.$$

We have the following stability results for a fixed point equation with cyclic graphical contractions in complete b-metric spaces.

Theorem 2.2. *Let (M, d) be a complete b-metric space with constant $s \geq 1$, let $p \in \mathbb{N}$ with $p \geq 2$ and $K_1, K_2, ..., K_p$ be nonempty and closed subsets of M. Let $\tilde{K} := \bigcup_{i=1}^{p} K_i$ and let $T : \tilde{K} \to \tilde{K}$ be such that $\bigcup_{i=1}^{p} K_i$ is a cyclical representation of $\tilde{K}$ with respect to T. Suppose that T is a cyclic graphical (orbital) a-contraction, i.e., $a \in]0, 1[$ and*

$$d(T(x), T^2(x)) \leq ad(x, T(x)), \text{ for every } x \in \tilde{K}.$$

Then, the fixed point equation $x = T(x), x \in \tilde{K}$ is well-posed in the sense of Reich and Zaslavski and it is Ulam-Hyers stable.

Proof. By Theorem 2.1 we know that T is a weakly $\frac{1}{1-a}$-Picard on $\bigcap_{i=1}^{p} K_i$ and the following retraction-displacement condition holds:

$$d(x, T^\infty(x)) \leq \frac{s(1 - a + sS)}{1 - a} d(x, T(x)), \text{ for all } x \in \bigcap_{i=1}^{p} K_i, \tag{2.4}$$

where, for each $x \in \bigcap_{i=1}^{p} K_i$, the value $T^\infty(x) \in Fix(T)$ is the limit of the sequence of Picard iterates $\{T^n(x)\}_{n\in\mathbb{N}}$ and $S := \sum_{i=1}^{\infty} a^{2i \log_a s + 2^{i-1}}$. Since $T : \bigcap_{i=1}^{p} K_i \to \bigcap_{i=1}^{p} K_i$ is a weakly Picard operator, the mapping $T^\infty : \bigcap_{i=1}^{p} K_i \to Fix(T)$ is a set retraction.

Consider first $x^* \in Fix(T)$ and $(y_n)_{n\in\mathbb{N}}$ a sequence such that $T^\infty(y_n) = x^*$ and

$$d(y_n, T(y_n)) \to 0 \text{ as } n \to \infty.$$

If we consider in (2.4) $x := y_n$, then we get that

$$d(y_n, x^*) = d(y_n, T^\infty(y_n)) \leq \frac{s(1 - a + sS)}{1 - a} d(y_n, T(y_n)) \to 0, \text{ as } n \to \infty.$$

Thus, the fixed point equation $x = T(x), x \in \bigcap_{i=1}^{p} K_i$ is well-posed in the sense of Reich and Zaslavski.

Consider now any $\varepsilon > 0$ and any ε-solution z of the fixed point equation $x = T(x), x \in \bigcap_{i=1}^{p} K_i$. Thus, $d(z, T(z)) \leq \varepsilon$. As before, since T is a weakly $\frac{1}{1-a}$-Picard on $\bigcap_{i=1}^{p} K_i$ we have that $Fix(T) \neq \emptyset$ and for each $x \in \bigcap_{i=1}^{p} K_i$, the sequence of Picard iterates $\{T^n(x)\}_{n\in\mathbb{N}}$

converges to $T^\infty(x) \in Fix(T)$. Using again the retraction-displacement condition (2.4) with $x := z$, we get that

$$d(z, T^\infty(z)) \leq \frac{s(1-a+sS)}{1-a} d(z, T(z)) \leq \frac{s(1-a+sS)}{1-a}\varepsilon.$$

hence, the fixed point equation $x = T(x), x \in \bigcap_{i=1}^{p} K_i$ is Ulam-Hyers stable. $\square$

The following result is know as Cauchy-Toeplitz Lemma.

Lemma 2.4. *(Cauchy-Toeplitz Lemma, see, for example,* [20]*) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}_+$, such that the series $\sum_{n\geq 0} a_n$ is convergent and $(b_n)_{n\in\mathbb{N}} \in \mathbb{R}_+$ be a sequence such that $\lim_{n\to\infty} b_n = 0$. Then*

$$\lim_{n\to\infty} (\sum_{k=0}^{n} a_{n-k} b_k) = 0.$$

Theorem 2.3. *Let (M, d) be a complete b-metric space with constant $s \geq 1$, let $p \in \mathbb{N}$ with $p \geq 2$ and $K_1, K_2, ..., K_p$ be nonempty and closed subsets of M. Let $\tilde{K} := \bigcup_{i=1}^{p} K_i$ and let $T : \tilde{K} \to \tilde{K}$ be such that $\bigcup_{i=1}^{p} K_i$ is a cyclical representation of $\tilde{K}$ with respect to T. Suppose that T is a cyclic graphical (orbital) a-contraction, i.e., $a \in]0, 1[$ and*

$$d(T(x), T^2(x)) \leq ad(x, T(x)), \textit{ for every } x \in \tilde{K}.$$

If $\frac{s^3 S}{1-a-s^2 S} < 1$, then the operator T has the Ostrowski property on $\bigcap_{i=1}^{p} K_i$.

Proof. Since $\frac{s^3 S}{1-a-s^2 S} < 1$ we get that $s < \sqrt{\frac{1-a}{2S}}$. Then, by Theorem 2.1 we know that T is a quasi-contraction, i.e.,

$$d(T(x), T^\infty(x)) \leq \beta d(x, T^\infty(x)), \text{ for all } x \in \bigcap_{i=1}^{p} K_i,$$

where $\beta := \frac{s^2 S}{1-a-s^2 S} \in]0, 1[$ and $S := \sum_{i=1}^{\infty} a^{2i \log_a s + 2^{i-1}}$. Moreover $s\beta < 1$. Then, T has the Ostrowski property on $\bigcap_{i=1}^{p} K_i$. For this conclusion, let $x^* \in Fix(T)$ and let $(z_n)_{n\in\mathbb{N}}$ a sequence in $\bigcap_{i=1}^{p} K_i$ such that $T^\infty(z_n) = x^*$ and

$$d(z_{n+1}, T(z_n)) \to 0 \text{ as } n \to \infty.$$

Then, we have

$$\begin{gathered} d(z_{n+1}, x^*) = d(z_{n+1}, T^\infty(z_n)) \leq s\left[d(z_{n+1}, T(z_n)) + d(T(z_n), x^*)\right] = \\ s\left[d(z_{n+1}, T(z_n)) + d(T(z_n), T^\infty(z_n))\right] \leq \\ s\left[d(z_{n+1}, T(z_n)) + \beta d(z_n, T^\infty(z_n))\right] = s\left[d(z_{n+1}, T(z_n)) + \beta d(z_n, x^*)\right] \leq \\ sd(z_{n+1}, T(z_n)) + s^2\beta\left[d(z_n, T(z_{n-1})) + d(T(z_{n-1}), x^*)\right] \leq \\ \cdots \end{gathered}$$

$$s\left[d(z_{n+1},T(z_n))+s\beta d(z_n,T(z_{n-1}))+\cdots+(s\beta)^n d(z_1,T(z_0))\right]+(s\beta)^n d(z_0,x^*).$$

Now, by the Cauchy-Toeplitz Lemma we get the conclusion. □

If we consider now the case of a generalized b-metric space in the sense of Perov, then the following lemma follows in a similar way to Lemma 2.1 given by Miculescu and Mihail in [11].

Lemma 2.5. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of elements from a generalized b-metric space in the sense of Perov (X,d) . Then, the inequality*

$$d(x_0,x_k)\le s^n\sum_{i=0}^{k-1}d(x_i,x_{i+1})$$

holds for each $n\in\mathbb{N}$ and each $k\in\{1,2,3,...,2^{n-1},2^n\}$.

Using the above lemma, it is an open question to prove a similar result with Lemma 2.2 given by Miculescu and Mihail in [11], for the case of vector-valued b-metric space.

Conjecture. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of elements from a vector-valued b-metric space (M,d) of constant $s>1$ having the property that there exists $A\in\mathfrak{M}_{m,m}(\mathbb{R}_+)$, such that:*

(i) *A is convergent to zero;*

(ii) *$d(x_{n+1},x_n)\le Ad(x_n,x_{n-1})$ for every $n\in\mathbb{N}$.*

Then $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (M,d).

As an application of the main result we can obtain a coupled fixed point theorem in complete b-metric spaces. We give first the following immediate consequence of Theorem 2.1.

Theorem 2.4. *Let (X,d), (Y,ρ) be two complete b-metric space, with $s\ge1$, $p\in\mathbb{N}$ with $p\ge2$ and $A_1,A_2,...,A_p$, $B_1,B_2,...,B_p$ be nonempty and closed subsets of X. Consider $Z=\bigcup_{i=1}^{p}(A_i\times B_i)$ and the operator $F:Z\to Z$ be such that $F(A_i\times B_i)\subset A_{i+1}\times B_{i+1}$, for every $i\in\{1,...,p\}$, where $A_{p+1}=A_1$ and $B_{p+1}=B_1$. Suppose that there exists $a\in]0,1[$ such that*

$$\bar{d}(F(x,y),F^2(x,y))\le a\bar{d}((x,y),F(x,y)),\ \text{for every } (x,y)\in Z,$$

where $\bar{d}$ is a scalar b-metric generated by d and ρ.
Then:

(i) *$\bigcap_{i=1}^{p}(A_i\times B_i)\ne\emptyset$ and $F:\bigcap_{i=1}^{p}(A_i\times B_i)\to\bigcap_{i=1}^{p}(A_i\times B_i)$;*

(ii) *if, additionally F has closed graph, then $Fix(F)\ne\emptyset$ and the following apriori estimation holds:*

$$\bar{d}(F^n(x_0),z^*)\le\frac{a^n sS}{1-a}\bar{d}(x_0,F(x_0)),n\in\mathbb{N},$$

where $S:=\sum_{i=1}^{\infty}a^{2i\log_a s+2^{i-1}}$.

Using the above result we can obtain the following extended coupled fixed point theorem.

Theorem 2.5. *Let (X,d), (Y,ρ) be two complete b-metric space with constant $s\ge1$, $p\in\mathbb{N}$ with $p\ge2$ and $A_1,A_2,...,A_p$, $B_1,B_2,...,B_p$ be nonempty and closed subsets of X. Consider $Z=\bigcup_{i=1}^{p}(A_i\times B_i)$ and $F_1:Z\to\bigcup_{i=1}^{p}A_i$ and $F_2:Z\to\bigcup_{i=1}^{p}B_i$ be such that the following assumptions hold:*

(i) $F_1(A_i \times B_i) \subset A_{i+1}$ *and* $F_2(A_i \times B_i) \subset B_{i+1}$, *for every* $i \in \{1, ..., p\}$, *where* $A_{p+1} = A_1$ *and* $B_{p+1} = B_1$.

(ii) Suppose that there exist $a_1, a_2 \in (0,1)$ *such that*

$$d(F_1(x,y), F_1^2(x,y)) \leq a_1 d(x, F_1(x,y)), \text{ for every } (x,y) \in Z,$$

$$\rho(F_2(x,y), F_2^2(x,y)) \leq a_2 \rho(y, F_2(x,y)), \text{ for every } (x,y) \in Z,$$

where $F_1^n(x,y) = F_1^{n-1}(F_1(x,y), F_2(x,y))$ *and* $F_2^n(x,y) = F_2^{n-1}(F_1(x,y), F_2(x,y))$ *for* $n \in \mathbb{N}, n \geq 2$.
Then, the following conclusions hold:

(i) $\bigcap\limits_{i=1}^{p} (A_i \times B_i) \neq \emptyset$;

(ii) *if, additionally,* F_1, F_2 *have closed graph, then, for each element* $z = (x,y) \in \bigcap\limits_{i=1}^{p} (A_i \times B_i)$, *the sequence* $(F_1^n(z), F_2^n(z))_{n\in\mathbb{N}}$ *converges to a solution* $(x^*, y^*) \in Fix(F) \cap \bigcap\limits_{i=1}^{p} (A_i \times B_i)$ *of the operator system*

$$\begin{cases} x = F_1(x,y) \\ y = F_2(x,y). \end{cases} \tag{2.5}$$

Moreover for every $(x,y) \in \bigcap\limits_{i=1}^{p} (A_i \times B_i)$ *the following apriori estimation holds:*

$$d(F_1^n(x_0), x^*) + \rho(F_2^n(x_0), x^*) \leq \frac{\max\{a_1, a_2\}^n sS}{1 - \max\{a_1, a_2\}} (d(x_0, F_1(x_0)) + d(x_0, F_2(x_0))), n \in \mathbb{N},$$

where $S := \sum\limits_{i=1}^{\infty} \max\{a_1, a_2\}^{2i \log_{\max\{a_1,a_2\}} s + 2^{i-1}}$.

Proof. Let us consider the following b-metric

$$\tilde{d}((x,y),(u,v)) := d(x,u) + \rho(y,v)$$

defined on $X \times Y$. By the hypothesis we have that $(X \times Y, \tilde{d})$ is a complete b-metric space.
Let us define the operator $T_{F_1,F_2} : Z \to Z$ by

$$T_{F_1,F_2}(x,y) := (F_1(x,y), F_2(x,y)). \tag{2.6}$$

Notice that the fixed point set of this operator coincides with the solution set of (2.5).

Let us notice that the operator T_{F_1,F_2} satisfies all the conditions of Theorem 2.4. We have that $T_{F_1,F_2}(A_i \times B_i) \subset A_{i+1} \times B_{i+1}$ and also

$$\tilde{d}(T_{F_1,F_2}(x,y), T^2_{F_1,F_2}(x,y)) \leq a\tilde{d}((x,y), T_{F_1,F_2}(x,y)), \text{ for every } (x,y) \in Z,$$

where $a = \max\{a_1, a_2\}$.

Applying the previous theorem we obtain the conclusion. □

Remark 2.1. In particular, if in the above theorem we consider $F_1(x,y) = F(x,y)$ and $F_2(x,y) = F(y,x)$, where $F : X \times X \to X$ is a given operator, then we obtain an existence and approximation result for the coupled fixed point problem (1.1).

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