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In memoriam Professor Charles E. Chidume (1947- 2021)

Graphical Ekeland's variational principle with a generalized w-distance and a new approach to quasi-equilibrium problems

PARIN CHAIPUNYA[1,2], NANTAPORN CHUENSUPANTHARAT[3] and PRINTAPORN SANGUANSUTTIGUL[4]

ABSTRACT. In this paper, we introduce the generalized Ekeland's variational principle in several forms. The general setting of our results includes a graphical metric structure and also employs a generalized w-distance. We then applied the proposed variational principles to obtain existence theorems for a class of quasi-equilibrium problems whose constraint maps are induced from the graphical structure. The conditions used in our existence results are based on a very general concept called a convergence class. Finally, we deduce the existence of a generalized Nash equilibrium via its quasi-equilibrium reformulation. A validating example is also presented.

1. INTRODUCTION

The Ekeland's Variational Principle (EVP) was first introduced in [6, 7] and since then became largely involved in nonconvex nonsmooth analysis and optimization. The EVP itself has several equivalent formulations, each of which are of importance in their own aspects. One of the most renowned equivalent form is the Caristi fixed point theorem [5]. Hence it is quite natural that the improvements done in metric fixed point theory would seamlessly make their ways into the study of EVP (see e.g. [3] for an example of a recent development).

Generalizing the metric conditions is one of the famous extensions in fixed point theory. Following this pipeline, several researchers have successfully replaced the distance function used in the original EVP with a generalized distance (see e.g. [11, 12, 14, 16]). This outlook enables us to relax the cone-shaped supports of a lsc function into some other shapes. Among these, we would like to emphasize the work of Lin and Du [11] which applied a generalized w-distance to the EVP and deduced several existence conditions for variational problems including minimax inequalities and equilibrium problems.

Recently, in 2019, Alfuraidan and Khamsi [1] introduced the graphical version of the EVP. The graphical approach used in [1] was originally developed in [8] also for metric fixed point theory. This method can be seen as an augmentation to the same theory in partially ordered metric spaces [13, 15], while some assumptions were ripped out. The authors of [1] also studied the graphical Caristi fixed point theorem in addition to their EVP, and applied their results to obtain an approximate Fermat's rule for Fréchet derivative. By looking closely at the graphical EVP, we can also regard the problem at hand as a quasi-optimization problem, which asks for a minimizer over a moving constraint. This

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Corresponding author: Nantaporn Chuensupantharat; nantaporn.ch@bsru.ac.th

is the perspective that we embrace so dearly in this paper and we show that the graphical approach has the capability to capture a quasi-equilibrium problem, while maintaining the simplicity to the same level with those of equilibrium problems. The study of equilibrium-type problems via Ekeland's variational principle has also recently been explored in [2, 18].

In this paper, we introduce the graphical EVP where a new concept of generalized w-distance function is used. Our EVP improves and refines the existing works of [11] and [1]. Viewing the graphical structure as a road to quasi-optimization problem, we continue further to deduce a variational principle and existence theorems for a quasi-equilibrium problem (QEP). To obtain the existence theorems for a QEP and at the same time avoid the restriction of metric compactness, we introduce the notion of a convergence class to unify several weaker convergence concepts. This will cover the weak convergence from the classical Banach space theory and also from the $\mathrm{CAT}(0)$ theory of Kirk and Panyanak [10]. Finally, we show the existence of a generalized Nash equilibrium to a generalized non-cooperative game, with a support of a validating example.

2. Preliminaries

In this section, we collect the necessary concepts, especially from metric graph theory and $\mathrm{CAT}(0)$ spaces, that will be used in the main results of this paper.

First, we recall the graphical metric structures. Let $G = (V(G), E(G))$ be a (directed) graph endowed with a metric d, that is, $(V(G), d)$ is a metric space where the vertex set $V(G)$ is possibly infinite and the edge set $E(G)$ is any subset of $V(G) \times V(G)$. We say that G is *reflexive* if $(x, x) \in E(G)$ for all $x \in V(G)$ and that it is *transitive* if for any $x, y, z \in V(G)$, $(x, y), (y, z) \in E(G)$ implies $(x, z) \in E(G)$. For any $x \in V(G)$, we write $K(x) := \{y \in V(G) \mid (y, x) \in E(G)\}$. This (set-valued) map K associated to the graph G will be referred to as the *inward adjacency map* of G. Let $C \subseteq V(G)$. If the limit point of a convergent sequence (x_n) in C which satisfies the condition $x_n \in K(x_{n-1})$ for all $n \in N$ is included in C, we say that C is *G-sequentially closed*. Moreover, if every sequence (x_n) in C which satisfies the condition $x_n \in K(x_{n-1})$ for all $n \in N$ has a convergent subsequence with a limit in C, we say that C is *G-compact*. The metric d is said to be *G-complete* if every Cauchy sequence (x_n) such that $x_n \in K(x_{n-1})$ for all $n \in N$ is convergent. We say that a function $\varphi : V(G) \to \mathbb{R}$ is *G-lsca* (or *G-lower semicontinuous from above*) on $C \subseteq V(G)$ if for any $x \in C$, the inequality $\varphi(x) \leq \lim_n \varphi(x_n)$ holds whenever a sequence (x_n) in C is convergent to x, $x_n \in K(x_{n-1})$ for all $n \in \mathbb{N}$, and $(\varphi(x_n))$ is decreasing. We say that G satisfies the (OSC) condition if any convergent sequence (x_n) in $V(G)$ such that $x_n \in K(x_{n-1})$ for all $n \in \mathbb{N}$ has the following properties, where x denotes the limit of (x_n):

(OSC1) $x \in K(x_m)$ for all $m \in \mathbb{N}$, and
(OSC2) $y \in K(x)$ provided that $y \in K(x_m)$ for all $m \in \mathbb{N}$.

Note that (OSC1) implies $K(u)$ is G-sequentially closed for all $u \in V(G)$.

Now, we make a brief recall of basics about a $\mathrm{CAT}(0)$ space. A metric space (M, d) is said to be *uniquely geodesic* if for any two points $x, y \in M$, there exists a unique curve $c : [0, 1] \to M$ such that $d(c(s), c(t)) = d(x, y)\,|s - t|$ for all $s, t \in [0, 1]$. The notation $x \#_t y := c(t)$, for $t \in [0, 1]$ is also widely adopted. A uniquely geodesic metric space (M, d) is said to be a $\mathrm{CAT}(0)$ *space* if

$$d(x \#_t y, z)^2 \leq (1 - t) d(x, z)^2 + t d(y, z)^2 - t(1 - t) d(x, y)^2$$

for every $x, y, z \in M$ and every $t \in [0, 1]$. Hilbert spaces, simply connected Riemannian manifolds with nonpositive sectional curvature and metric trees are typical examples of a $\mathrm{CAT}(0)$ space. A subset $C \subseteq M$ is said to be *convex* if $x, y \in C$ implies $x \#_t y \in C$ for all $t \in [0,1]$ and a function $\varphi : M \to \mathbb{R}$ is *convex* whenever $x, y \in C$ implies $\varphi(x\#_t y) \leq (1-t)\varphi(x) + t\varphi(y)$ for all $t \in [0,1]$.

Next, we recall the weak convergence in a $\mathrm{CAT}(0)$ space introduced in [10]. The idea here is very similar to weak convergence in Hilbert spaces or some particular Banach spaces. Suppose for now that (M,d) is a complete $\mathrm{CAT}(0)$ space. A bounded sequence (x_n) in M is said to be *weakly convergent* to $x^* \in M$ if $x^* = \arg\min_M \left[\limsup_n d^2(\cdot, x_{n_k})\right]$ for every subsequence (x_{n_k}) of (x_n). The point x^* here is called the *weak sequential limit* of (x_n). It is known [10] that a sequence can have at most one weak sequential limit. Moreover, we say that a subset $C \subseteq M$ is *weakly sequentially closed* if every weakly convergent sequence in C has its weak sequential limit in C. We say that C is *weakly compact* if any sequence in C has a weakly convergent subsequence whose weak sequential limit is also contained in C. The following properties of weak convergence are of great importance.

Proposition 2.1. *Let (M,d) be a complete $\mathrm{CAT}(0)$ space. Then the following properties hold:*

(i) *Every bounded sequence has a weakly convergent subsequence. [10]*
(ii) *If (x_n) is a sequence in a closed convex set C which is weakly convergent, then its weak sequential limit is in C. [4]*

In particular, the two statements (i) *and* (ii) *implies that every bounded closed convex set is weakly compact.*

3. Ekeland's variational principle

In this section, we present the graphical alternative of the renowned Ekeland's variational principles in both the strong and weak forms. Note the heavy usage of the inward adjacency map K in our formulations throughout the paper. Moreover, we replace the distance function with a new generalized w-distance given in the following.

Definition 3.1. Let (M,d) be a metric space. We say that $p : M \times M \to [0,\infty)$ is a *generalized w-distance on M* if the following conditions are satisfied:

($w1$) $p(x,z) \leq p(x,y) + p(y,z)$ for any $x, y, z \in M$.
($w2$) Let $x \in M$ and (y_n) be a sequence in M that converges to $y \in M$. If $\tau \geq 0$ satisfies $p(x, y_n) \leq \tau$ for all $n \in \mathbb{N}$, then $p(x,y) \leq \tau$.
($w3$) For any sequences (x_n) and (y_n) in M and $z \in M$. If there exist sequences (α_n) and (β_n) of positive real converging to 0 satisfying the estimates $p(x_n, y_n) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for all $n \in \mathbb{N}$, then (y_n) converges to the point z.

Note that the above concept is more general than the original w-distance of Kada et al. [9] or the extended version used by Lin and Du [11]. Now, we are ready to state our EVPs.

Theorem 3.1 (Generalized EVP: Strong form)**.** *Let $G := (V(G), E(G))$ be a reflexive and acyclic graph endowed with a G-complete metric d satisfying the (OSC) property. Suppose that p is a generalized w-distance on $V(G)$. Let $\varphi : V(G) \to \mathbb{R}$ and assume that there is $u \in V(G)$ such that $K(u)$ is G-sequentially closed and φ is G-lsca and bounded below on $K(u)$. Let $r, \varepsilon > 0$ and $\widetilde{x} \in K(u)$ be given such that*

$$\varphi(\widetilde{x}) < \inf_{K(u)} \varphi + r\varepsilon.$$

Then, there exists $z \in K(\widetilde{x})$ such that

(i) $p(\widetilde{x}, z) \leq r$,
(ii) $\varphi(z) + \varepsilon p(\widetilde{x}, z) \leq \varphi(\widetilde{x})$,
(iii) $\varphi(x) + \varepsilon p(z,x) > \varphi(z)$ *for all $x \in K(z) \setminus \{z\}$.*

Proof. For any $x, y \in V(G)$, we adopt the notations

$$x \preccurlyeq_{\varepsilon,\varphi} y \iff p(y,x) \leq \frac{1}{\varepsilon}(\varphi(y) - \varphi(x))$$

and

$$\Gamma(x) := \{z \in V(G) \,|\, z \preccurlyeq_{\varepsilon,\varphi} x\}.$$

From $(w1)$, the relation $\preccurlyeq_{\varepsilon,\varphi}$ defines a partial ordering on $V(G)$. Put $x_0 := \widetilde{x}$ and let $S_0 := \Gamma(x_0) \cap K(x_0)$. Note that S_0 is nonempty as $x_0 \in S_0$. Take $x_1 \in S_0$ such that

$$\varphi(x_1) \leq \inf_{S_0} \varphi + \frac{1}{2}\left[\varphi(x_0) - \inf_{S_0} \varphi\right] = \frac{1}{2}\left[\varphi(x_0) + \inf_{S_0} \varphi\right] \leq \varphi(x_0).$$

We continue constructing the sequence (x_n) inductively as follows. Suppose that x_n is already defined for some $n \in \mathbb{N}$. Then we take a nonempty set $S_n := \Gamma(x_n) \cap K(x_n)$ and pick $x_{n+1} \in S_n$ satisfying

$$\varphi(x_{n+1}) \leq \inf_{S_n} \varphi + \frac{1}{2}\left[\varphi(x_n) - \inf_{S_n} \varphi\right] = \frac{1}{2}\left[\varphi(x_n) + \inf_{S_n} \varphi\right] \leq \varphi(x_n). \tag{3.1}$$

We may notice now that $(\varphi(x_n))$ is a nonincreasing real sequence and (x_n) satisfies $x_n \in K(x_{n-1})$ for all $n \in \mathbb{N}$. From the transitivity of G, the sequence (x_n) is in $K(u)$ and since φ is bounded below on $K(u)$, it follows that $(\varphi(x_n))$ converges to a finite limit $\varphi^* \in \mathbb{R}$. For any $n \in \mathbb{N}$, we have $x_{n+1} \in \Gamma(x_n)$ which means

$$p(x_n, x_{n+1}) \leq \frac{1}{\varepsilon}(\varphi(x_n) - \varphi(x_{n+1})).$$

Adding up this inequality for $n = i$ up to $n = j-1 > i$ and using $(w1)$, where $i, j \in \mathbb{N} \cup \{0\}$, we get

$$\begin{aligned} p(x_i, x_j) &\leq p(x_i, x_{i+1}) + \cdots + p(x_{j-1}, x_j) \\ &\leq \frac{1}{\varepsilon}(\varphi(x_i) - \varphi(x_j)) \leq \frac{1}{\varepsilon}(\varphi(x_i) - \varphi^*). \end{aligned} \tag{3.2}$$

If we let $\alpha_n := \frac{1}{\varepsilon}(\varphi(x_n) - \varphi^*)$ for each $n \in \mathbb{N}$, then one may see that $\alpha_n \to 0$ and the above inequality shows that (x_n) is Cauchy. The G-completeness of $V(G)$ and the G-sequential closedness of $K(u)$ implies that (x_n) is convergent to a limit point $z \in K(u)$. Moreover, we have $z \in K(\widetilde{x})$ by (OSC1). Letting $i = 0$ and $j \to \infty$ in (3.2), using $(w2)$ and the fact that φ is G-lsca on $K(u)$, we obtain

$$\begin{aligned} p(x_0, z) &\leq \frac{1}{\varepsilon}(\varphi(x_0) - \varphi^*) \leq \frac{1}{\varepsilon}(\varphi(x_0) - \varphi(z)) \\ &\leq \frac{1}{\varepsilon}\left(\inf_{K(u)} \varphi + r\varepsilon - \inf_{K(u)} \varphi\right) = r. \end{aligned}$$

The above inequalities imply the conclusions (i) and (ii).

To show (iii), let us assume to the contrary that (iii) is false. Then there exists $x \in K(z)$ with $x \neq z$ and $x \preccurlyeq_{\varepsilon,\varphi} z$. Let $n \in \mathbb{N} \cup \{0\}$. Observe from (3.2) with $i = n \in \mathbb{N} \cup \{0\}$ as $j \to \infty$, we get $z \preccurlyeq_{\varepsilon,\varphi} x_n$. Hence, the transitivity of $\preccurlyeq_{\varepsilon,\varphi}$ implies that $x \preccurlyeq_{\varepsilon,\varphi} x_n$ for any $n \in \mathbb{N} \cup \{0\}$. On the other hand, using (OSC1) and the transitivity of G, we have $x \in K(x_n)$ for all $n \in \mathbb{N} \cup \{0\}$, or that $x \in \bigcap_{n=0}^{\infty} K(x_n)$. Take any $n \in \mathbb{N} \cup \{0\}$. We have $x \in S_n$ and therefore

$$p(x_{n+1}, x) \leq \frac{1}{\varepsilon}(\varphi(x_{n+1}) - \varphi(x)) \leq \frac{1}{\varepsilon}\left(\varphi(x_{n+1}) - \inf_{S_n} \varphi\right). \tag{3.3}$$

From (3.1), we have

$$\varphi(x_{n+1}) + \varphi(x_{n+1}) = 2\varphi(x_{n+1}) \leq \varphi(x_n) + \inf_{S_n} \varphi$$

and so $\varphi(x_{n+1}) - \inf_{S_n} \varphi \leq \varphi(x_n) - \varphi(x_{n+1})$. Combining this with (3.3), we get

$$p(x_{n+1}, x) \leq \frac{1}{\varepsilon}(\varphi(x_n) - \varphi(x_{n+1})) \to 0.$$

By $(w3)$, we obtain $x_n \to x$ and since $x \neq z$, this is a contradiction. Therefore we conclude that (iii) holds true. □

The following direct byproduct is known as the weak formulation of the EVP.

Theorem 3.2 (Generalized EVP: Weak form). *Let $G := (V(G), E(G))$ be a reflexive and acyclic graph endowed with a G-complete metric d satisfying the (OSC) property. Suppose that p is a generalized w-distance on $V(G)$. Let $\varphi : V(G) \to \mathbb{R}$ and assume that there is $u \in V(G)$ such that φ is G-lsca and bounded below on $K(u)$. Then, for any given $\varepsilon > 0$ and $\widetilde{x} \in K(u)$, there exists $z \in K(\widetilde{x})$ such that*

(i) $\varphi(z) + \varepsilon p(\widetilde{x}, z) \leq \varphi(\widetilde{x})$,
(ii) $\varphi(x) + \varepsilon p(z, x) > \varphi(z)$ *for all* $x \in K(z) \setminus \{z\}$.

Proof. Pick $r > 0$ such that $\varphi(\widetilde{x}) < \inf_{K(u)} \varphi + r\varepsilon$ and apply Theorem 3.1 to obtain the desired result. □

4. Quasi-equilibrium problems

The class of QEP generalizes the class of equilibrium problems (EP) by allowing for the variable constraints. Particularly on a nonempty set X, a QEP consists of an objective bifunction $\Phi : X \times X \to \mathbb{R}$ and a constraint map $T : X \rightrightarrows X$ and concerns with finding a point $z \in X$ such that $z \in T(z)$ and

$$\Phi(z, y) \geq 0, \qquad (\forall y \in T(z)).$$

This specific problem will be further denoted briefly as $QEP(\Phi, T)$.

In this section, we discuss some existence results for QEPs by first drawing the variational principle adapted for the problem and then apply it to confirm the existence of its solution.

4.1. A variational principle for QEPs.

The following result is the variational principle that is stated for a QEP. It follows from the EVP we proved in the preceding section.

Theorem 4.3. *Let $G := (V(G), E(G))$ be a reflexive and acyclic graph endowed with a G-complete metric d satisfying the (OSC) property. Suppose that p is a generalized w-distance on $V(G)$. Let $\Phi : V(G) \times V(G) \to \mathbb{R}$ and assume that there is $u \in V(G)$ satisfying the following properties on $X := K(u)$:*

(a) $\Phi(x, z) \leq \Phi(x, y) + \Phi(y, z)$ *for every* $x, y, z \in X$, *and*
(b) *there exists* $v \in V(G)$ *such that* $\Phi(v, \cdot)$ *is G-lsca and bounded below on* X.

Then for any $\varepsilon > 0$ and $\widetilde{x} \in X$, there exists $z \in K(\widetilde{x})$ such that

(i) $\varepsilon p(\widetilde{x}, z) \leq \Phi(v, \widetilde{x}) - \Phi(v, z) \leq \Phi(z, \widetilde{x})$,
(ii) $\Phi(z, x) + \varepsilon p(z, x) \geq \Phi(v, x) - \Phi(v, z) + \varepsilon p(z, x) > 0$ *for all* $x \in K(z) \setminus \{z\}$.

Proof. Immediately from (b), the function $\varphi := \Phi(v, \cdot)$ is G-lsca and bounded below on X. By (a), it follows that

$$\Phi(x, y) \geq \Phi(v, y) - \Phi(v, x) = \varphi(y) - \varphi(x)$$

for any $x, y \in X$. For any $\varepsilon > 0$, the above inequality and Theorem 3.2 with $\varphi := \Phi(v, \cdot)$ on X yield the desired results. □

4.2. **An existence theorem for QEPs.** To strengthen the above result to the existence of an exact solution, a common criterion often involves some compactness assumptions. Since the metric completeness can be rather restrictive in infinite dimensional settings, we deem to relax such condition for a more convenient applicability. Our approach involves introducing a simple unifying concept of a convergence class.

Given any set M. A relation $\mathcal{C}$ defined on $M \times M^{\mathbb{N}}$ is said to be a *sequential convergence class* (or plainly a *convergence class* for this paper) on M if $(x,(x_n)) \in \mathcal{C}$ implies $(x,(x_{n_k})) \in \mathcal{C}$ for all subsequences (x_{n_k}) of (x_n). If $(x,(x_n)) \in \mathcal{C}$, we say that (x_n) is *$\mathcal{C}$-convergent* (or that it *$\mathcal{C}$-converges*) to x. In this case, we also adopt the tradition of calling x a *$\mathcal{C}$-limit* of (x_n). If every sequence has at most one $\mathcal{C}$-limit, then the convergence class $\mathcal{C}$ is said to be *Hausdorff*. Given two convergence classes $\mathcal{C}_1$ and $\mathcal{C}_2$ on M, we say that $\mathcal{C}_1$ is *weaker* than $\mathcal{C}_2$ if $\mathcal{C}_2 \subseteq \mathcal{C}_1$, i.e., if (x_n) is $\mathcal{C}_2$-convergent to x implies that it is also $\mathcal{C}_1$-convergent to x. If M is a topological space, then its natural convergence class $\mathcal{C}_M$ is defined by $(x,(x_n)) \in \mathcal{C}_M$ if and only if (x_n) converges to x in the given topology of M. We then say that $\mathcal{C}$ is a *weaker convergence class* of M if it is a convergence class which is weaker than $\mathcal{C}_M$.

Example 4.1. Let M be either a Banach space or a complete $\mathrm{CAT}(0)$ space. Then the weak convergence in such spaces constitutes a weaker convergence class on M, denoted with $\mathcal{C}_w$. Since whether the weak convergence in a $\mathrm{CAT}(0)$ space emerges from a topology remains an open question, the notion of a weaker convergence class is a relevant concept for the present situation.

We can adapt the sequential characterizations of several topological concepts to a convergence class. Here, we shall only do so with the combined graphical structure only. Let G be the graph with a metric d as in the previous sections, then the following definitions are used.

Definition 4.2. Let $\mathcal{C}$ be a convergence class on $V(G)$. A subset $C \subseteq V(G)$ is said to be

(1) *G-$\mathcal{C}$-compact* if every sequence (x_n) in C such that $x_n \in K(x_{n-1})$ for each $n \in \mathbb{N}$ contains a subsequence that is $\mathcal{C}$-convergent to an element in C;
(2) *G-$\mathcal{C}$-closed* if every $\mathcal{C}$-convergent sequences in C such that $x_n \in K(x_{n-1})$ for each $n \in \mathbb{N}$ have their $\mathcal{C}$-limits contained in C.

Definition 4.3. A function $\varphi : V(G) \to \mathbb{R}$ is said to be *G-$\mathcal{C}$-lower semicontinuous from above* (or *G-$\mathcal{C}$-lsca*) on $C \subseteq V(G)$ if for any $x \in C$, the inequality $\varphi(x) \leq \lim_n \varphi(x_n)$ holds for every sequence (x_n) in C which satisfies $(x,(x_n)) \in \mathcal{C}$, $x_n \in K(x_{n-1})$ for all $n \in \mathbb{N}$, and $(\varphi(x_n))$ is decreasing.

Apparently, a G-$\mathcal{C}$-compact set is G-$\mathcal{C}$-closed and a G-$\mathcal{C}$-closed subset of a G-$\mathcal{C}$-compact set is again G-$\mathcal{C}$-compact. If $\mathcal{C}$ is a weaker convergence class on $V(G)$, then every G-$\mathcal{C}$-closed subsets are G-sequentially closed and every G-sequentially compact subsets are G-$\mathcal{C}$-compact. Moreover, every $\mathcal{C}$-lsca functions are lsca.

We also need to strengthen the (OSC) property. A graph G is said to satisfy the $(\mathcal{C}$-$OSC)$ property if it satisfies already the (OSC) property and additionally the condition (OSC1) holds for any $\mathcal{C}$-convergent sequence (x_n) in $V(G)$ with $\mathcal{C}$-limit $x \in V(G)$ and $x_n \in K(x_{n-1})$ for all $n \in \mathbb{N}$. In particular, this implies that $K(u)$ is G-$\mathcal{C}$-closed for every $u \in V(G)$

Now we are ready to present the main existence theorem for QEPs. Note that the result is independent of p, which clearly emphasize that the generalized w-distance p only plays a part in regularizing the perturbed problem and renders a more relaxed geometry for the supporting structures.

Theorem 4.4. *Let $G := (V(G), E(G))$ be a reflexive and acyclic graph endowed with a G-complete metric d. Suppose that $\mathcal{C}$ is a weaker convergence class on $V(G)$ such that the ($\mathcal{C}$-OSC) property holds. Let $\Phi : V(G) \times V(G) \to \mathbb{R}$ be a bifunction. Suppose that there is $u \in V(G)$ for which the following properties are satisfied on $X := K(u)$:*

(a) $\Phi(x,z) \leq \Phi(x,y) + \Phi(y,z)$ *for every* $x, y, z \in X$,
(b) *there exists* $v \in V(G)$ *such that* $\Phi(v,\cdot)$ *is G-$\mathcal{C}$-lsca and bounded below on* X,
(c) $K(q)$ *is bounded and G-$\mathcal{C}$-compact for some point* $q \in X$.

Then $QEP(\Phi, K)$ has a solution.

Proof. Take any generalized w-distance p on $V(G)$ such that p is bounded on bounded sets in its first argument (such p exists as we can always choose $p = d$). Recall that a G-$\mathcal{C}$-closed set is G-sequentially closed and a G-$\mathcal{C}$-lsca function is G-lsca. Hence, the Theorem 4.3 allows us to pick $z_1 \in K(q)$ such that $\Phi(z_1,x) + p(z_1,x) \geq 0$ for all $x \in K(z_1)$. Now, using again the Theorem 4.3, we pick $z_2 \in K(z_1)$ such that $\frac{1}{2}p(z_1,z_2) \leq \Phi(v,z_1) - \Phi(v,z_2)$ and $\Phi(v,x) - \Phi(v,z_2) + \frac{1}{2}p(z_2,x) \geq 0$ for all $x \in K(z_2)$. Continue applying the Theorem 4.3 inductively to construct a sequence (z_n) as follows: Suppose that $z_n \in K(z_{n-1})$ has been defined for some $n \in \mathbb{N}$. Then we pick $z_{n+1} \in K(z_n)$ such that

$$\frac{1}{n+1}p(z_n, z_{n+1}) \leq \Phi(v,z_n) - \Phi(v,z_{n+1}) \tag{4.4}$$

and

$$\Phi(v,x) - \Phi(v,z_{n+1}) \geq -\frac{1}{n+1}p(z_{n+1},x), \qquad (\forall x \in K(z_{n+1})). \tag{4.5}$$

The ($\mathcal{C}$-OSC) property of G implies that all the sets $K(z_n)$ are G-$\mathcal{C}$ closed. Invoking the G-$\mathcal{C}$-compactness for $K(u)$, we can extract a convergent subsequence (z_{n_i}) from (z_n) and denote its limit by $z \in X$. Since (z_{n_i}) is eventually in $K(z_n)$ for every $n \in \mathbb{N}$, the G-$\mathcal{C}$-closedness of each $K(z_n)$ implies that $z \in \bigcap_{n=1}^{\infty} K(z_n)$ so that $\bigcap_{n=1}^{\infty} K(z_n)$ is nonempty. We also have $K(z) \subseteq \bigcap_{n=1}^{\infty} K(z_n)$. Combining this with (4.5), we deduce for any $i \in \mathbb{N}$ that

$$\Phi(v,x) - \Phi(v,z_{n_i}) \geq -\frac{1}{n_i}p(z_{n_i},x), \qquad (\forall x \in K(z)). \tag{4.6}$$

From (4.4), we may see that $(\Phi(v,z_n))$ is decreasing and has a finite limit by (b). Since p is bounded on bounded sets in the first argument and (z_{n_i}) is a sequence in a bounded set $K(q)$, the sequence $(\frac{1}{n_i}p(z_{n_i},x))$ vanishes as $i \to \infty$. Let $x \in K(z)$. Then (4.6) implies

$$\Phi(z,x) \geq \Phi(v,x) - \Phi(v,z) \geq \Phi(v,x) - \lim_i \Phi(v,z_{n_i}) \geq -\lim_i \frac{1}{n_i}p(z_{n_i},x) = 0,$$

and therefore the theorem is proved. □

We end this subsection by stating the compact case which is obtained by simply choosing the convergence class to be the one induced from the metric convergence.

Corollary 4.1. *Let $G := (V(G), E(G))$ be a reflexive and acyclic graph endowed with a G-complete metric d such that the (OSC) property holds and $\Phi : V(G) \times V(G) \to \mathbb{R}$ is a bifunction. Suppose that there is $u \in V(G)$ for which the following properties are satisfied on $X := K(u)$:*

(a) $\Phi(x,z) \leq \Phi(x,y) + \Phi(y,z)$ *for every* $x, y, z \in X$,
(b) *there exists* $v \in V(G)$ *such that* $\Phi(v,\cdot)$ *is G-lsca and bounded below on* X,
(c) $K(u)$ *is bounded and G-compact for some point* $u \in X$.

Then $QEP(\Phi, K)$ has a solution.

4.3. **Particular weakly compact cases.** In this part, we deduce some direct consequences of Theorem 4.4 by focusing on the convergence class $\mathcal{C}_w$ induced by the weak convergence on either a Banach space or a complete $\mathrm{CAT}(0)$ space. The properties of weak convergence in Banach spaces used here are fundamental, and one may consult any related texts, e.g. [17], if necessary. One should refer to Proposition 2.1 in the case of a complete $\mathrm{CAT}(0)$ space.

Assume that $V(G)$ is either a Banach space or a complete $\mathrm{CAT}(0)$ space with the weaker convergence class $\mathcal{C}_w$ induced from the weak convergence. Then we simply call a G-$\mathcal{C}_w$-compact set a G-weakly compact set. Similarly, a G-$\mathcal{C}_w$-closed set will be simply called a G-weakly sequentially closed set and a G-$\mathcal{C}_w$-lsca function is simplfied to a G-weakly lsca function. Lastly, the $(\mathcal{C}_w\text{-}OSC)$ property is reduced to the $(w\text{-}OSC)$ property.

Corollary 4.2. *Let $G := (V(G), E(G))$ be a reflexive and acyclic graph whose vertex set $V(G)$ is either a Banach space or a complete $\mathrm{CAT}(0)$ space such that the $(w\text{-}OSC)$ property is satisfied. Assume that $\Phi : V(G) \times V(G) \to \mathbb{R}$ is a bifunction and there is $u \in V(G)$ for which $X := K(u)$ is bounded and the following properties are satisfied:*

(a) $\Phi(x, z) \leq \Phi(x, y) + \Phi(y, z)$ *for every $x, y, z \in X$,*
(b) *there exists $v \in V(G)$ such that $\Phi(v, \cdot)$ is G-weakly lsca and bounded below on X.*

Then $QEP(\Phi, K)$ has a solution.

Proof. Since the $(w\text{-}OSC)$ property holds, X is G-weakly sequentially closed. By the boundedness of X, any sequence (x_n) in X has a convergent subsequence to some weak sequential limit point, say $x \in V(G)$. If additionally $x_n \in K(x_{n-1})$ is satisfied for all $n \in \mathbb{N}$, then x belongs to X which yeilds the G-weak compactness of X. Now, apply Theorem 4.4 with the convergence class $\mathcal{C}_w$ induced from the weak convergence to obtain the desired conclusion. □

Next, we show that the above corollary can be utmost simplified when the convexity is in the play. Let us first consider the following primitive property before stating the next corollary.

Proposition 4.2. *If $C \subseteq V(G)$ is G-weakly closed and convex, and $\varphi : C \to \mathbb{R}$ is convex and G-lsca on C, then it is G-weakly lsca on C.*

Proof. Let (x_n) be a sequence in C which is weakly convergent to $x^* \in C$, $x_n \in K(x_{n-1})$ for all $n \in \mathbb{N}$, and $(\varphi(x_n))$ is decreasing. Let $\varphi^* := \lim_n \varphi(x_n)$ and assume that $\varphi(x^*) > \varphi^*$. Then there exists $N > 0$ such that $\varphi(x_n) \leq \frac{1}{2}[\varphi(x^*) + \varphi^*]$ for all $n > N$. Since $x^* \in \overline{\mathrm{conv}}\{x_n \,|\, n > N\}$, the convexity of φ yields $\varphi(x^*) \leq \frac{1}{2}[\varphi(x^*) + \varphi^*] < \varphi(x^*)$, which is a contradiction. □

Corollary 4.3. *Let $G := (V(G), E(G))$ be a reflexive and acyclic graph whose vertex set $V(G)$ is either a Banach space or a complete $\mathrm{CAT}(0)$ space such that the $(w\text{-}OSC)$ property is satisfied. Assume that $\Phi : V(G) \times V(G) \to \mathbb{R}$ is a bifunction and there is $u \in V(G)$ for which $X := K(u)$ is bounded and the following properties are satisfied:*

(a) $\Phi(x, z) \leq \Phi(x, y) + \Phi(y, z)$ *for every $x, y, z \in X$,*
(b) *there exists $v \in V(G)$ such that $\Phi(v, \cdot)$ is convex, G-lsca and bounded below on X.*

Then $QEP(\Phi, K)$ has a solution.

Proof. The results follow from Corollary 4.2 and Proposition 4.2. □

We say that a subset $C \subseteq V(G)$ is *G-convex* if for any $u \in C$, the set $K(u) \cap C$ is convex. This concept allows us to relax the $(w\text{-}OSC)$ property to the ordinary (OSC) property, as follows:

Proposition 4.3. *Suppose that $V(G)$ is either a reflexive Banach space or a complete $\mathrm{CAT}(0)$ space and that the (OSC) property is satisfied. If $C \subseteq V(G)$ is closed, bounded, and G-convex, then it is G-weakly compact. Moreover, the $(w\text{-}OSC)$ property holds on the subgraph G_C induced by C.*

Proof. Take any sequence (x_n) in C with $x_n \in K(x_{n-1})$ for all $n \in \mathbb{N}$. Then (x_n) also belongs to $K(x_0) \cap C$, which is weakly compact. Therefore (x_n) contains a weakly convergent subsequence whose weak sequential limit point is in $K(x_0) \cap C$. The second conclusion follows from the weak compactness of C. □

Finally, we arrive at the following corollary which is greatly simplified with the help of convexity.

Corollary 4.4. *Let $G := (V(G), E(G))$ be a reflexive and acyclic graph whose vertex set $V(G)$ is either a reflexive Banach space or a complete $\mathrm{CAT}(0)$ space such that the (OSC) property is satisfied. Assume that $\Phi : V(G) \times V(G) \to \mathbb{R}$ is a bifunction and there is $u \in V(G)$ such that $X := K(u)$ is closed, bounded, and G-convex subset, and the following properties hold:*

(a) $\Phi(x, z) \leq \Phi(x, y) + \Phi(y, z)$ *for every* $x, y, z \in X$,
(b) *there exists $v \in V(G)$ such that $\Phi(v, \cdot)$ is G-lsca, convex, and bounded below on X.*

Then $QEP(\Phi, K)$ has a solution.

Proof. Apply Corollary 4.2 to the induced subgraph G_X by taking into account Proposition 4.3. □

5. Generalized Nash equilibrium problems

As an application of Section 4, we deduce a few existence theorems for a Generalized Nash Equilibrium Problem (GNEP, for short). Recall that a GNEP comprises of k players (for some $k \in \mathbb{N}$), where each player $i = 1, \cdots, k$ admits a personal cost $c^i(\cdot)$ that is affected as well by the decisions of all other players. Each player $i = 1, \cdots, k$ assumes a control exclusively over her own decision x^i from some decision space X^i under the constraint $K^i(\cdot)$, which is also affected by all the players' decisions. If we denote by $\mathbf{X} := \prod_{i=1}^k X^i$ the space of all decision vectors, then the cost functions are given by $c^i : \mathbf{X} \to \mathbb{R}$ and the constraints by $K^i : \mathbf{X} \to 2^{X^i}$ for $i = 1, \cdots, k$. We also adopt the component representation $\mathbf{x} = (x^1, \cdots, x^k)$ with $x^i \in X^i$ for all $i \in 1, \cdots, k$. If $\mathbf{x} \in \mathbf{X}$ and $y^i \in X^i$ is a decision of some player i, we use the convention $(y^i|\mathbf{x}^{-i}) = (x^1, \cdots, x^{i-1}, y^i, x^{i+1}, \cdots, x^k)$. The GNEP with the components given aforementioned will be denoted by $GNEP(k, \{X^i\}, \{c^i\}, \{K^i\})$ and aims to solve for a decision vector $\hat{\mathbf{x}} \in \mathbf{X}$, called a *generalized Nash equilibrium*, such that $\hat{\mathbf{x}} \in \mathbf{K}(\hat{\mathbf{x}})$ and

$$c^i(\hat{\mathbf{x}}) \leq c^i(x^i|\hat{\mathbf{x}}^{-i})$$

holds for all $\mathbf{x} = (x^1, \cdots, x^k) \in \mathbf{K}(\hat{\mathbf{x}})$ and all players $i = 1, \cdots, k$. Here, we use $\mathbf{K}(\cdot) := \prod_{i=1}^k K^i(\cdot)$. Intuitively, a GNEP asks every players to make the decisions that is feasible and cannot be improved once everybody played rationally.

In this paper, we focus on a specific type of GNEP where the constraint maps are given globally by some authorities called a *Dictated GNEP* (for short, a *DGNEP*). In such a game, the global feasibility map $\mathbf{K} : \mathbf{X} \to 2^{\mathbf{X}}$ is pre-given by a dictator. Each player i then follows their dictator by using the constraint map

$$K^i(\mathbf{x}) := \{u^i \in X^i \mid (u^i|\mathbf{x}^{-i}) \in \mathbf{K}(\mathbf{x})\} \tag{5.7}$$

for $\mathbf{x} \in \mathbf{X}$. We shall henceforth refer to this game by $DGNEP(k, \{X^i\}, \{c^i\}, \mathbf{K})$. Moreover, we assume that the cost c^i for each player $i = 1, \cdots, k$ takes the form

$$c^i(y^i|\mathbf{x}^{-i}) := \theta^i(y^i) + \eta^i(\mathbf{x}^{-i}) \tag{5.8}$$

for $y^i \in X^i$ and $\mathbf{x}^{-i} \in \mathbf{X}^{-i}$, where $\theta^i : X^i \to \mathbb{R}$ is the individual cost, $\eta^i : \mathbf{X}^{-i} \to \mathbb{R}$ is the aggregate cost, and $\mathbf{X}^{-i} := \prod_{j \neq i} X^j$. This way, the difference $c^i(y^i|\mathbf{x}^{-i}) - c^i(\mathbf{x})$ equals to $\theta^i(y^i) - \theta^i(x^i)$ which represents the surplus cost that occurs to the player i when she replaces her decision x^i with y^i.

A DGNEP (in fact, any GNEP) can be equivalently stated as a QEP. Suppose that we were given a $DGNEP(k, \{X^i\}, \{c^i\}, \mathbf{K})$, we define $\mathbf{\Phi} : \mathbf{X} \times \mathbf{X} \to \mathbb{R}$ by

$$\mathbf{\Phi}(\mathbf{x}, \mathbf{y}) := \sum_{i=1}^{k} \left[c^i(y^i|\mathbf{x}^{-i}) - c^i(\mathbf{x})\right] \tag{5.9}$$

for $\mathbf{x}, \mathbf{y} \in \mathbf{X}$. Then $\hat{\mathbf{x}} \in \mathbf{X}$ is a solution of $GNEP(k, \{X^i\}, \{c^i\}, \mathbf{K}\})$ if and only if it solves $QEP(\mathbf{\Phi}, \mathbf{K})$.

Next we deduce some existence results for a DGNEP via its QEP reformulation, where the decision space of each player is prescibed with a graph whose directed edge represents the feasibility. To this end, note that a $DGNEP(k, \{X^i\}, \{c^i\}, \mathbf{K})$ induces a *global feasibility graph* $\mathbf{G} := (\mathbf{X}, \mathbf{E})$, where the edge $\mathbf{E}$ is given by the relation

$$(\mathbf{y}, \mathbf{x}) \in \mathbf{E} \iff \mathbf{y} \in \mathbf{K}(\mathbf{x}). \tag{5.10}$$

For this graph $\mathbf{G}$, $\mathbf{K}$ coincides with the inward adjacency map. If $(\mathbf{x}_n)$ is a sequence in $\mathbf{X}$, then we write $(x^i_n)_n$ the sequence obtained by the i^{th} coordinate projection, which is $\pi_i(\mathbf{x}) := x^i$.

Theorem 5.5. *Suppose that* $DGNEP(k, \{X^i\}, \{c^i\}, \mathbf{K})$ *takes the form* (5.8) *and induces a global feasibility graph* $\mathbf{G} := (\mathbf{X}, \mathbf{E})$ *which is reflexive and transitive and* $\mathbf{X}$ *is equipped with a* $\mathbf{G}$-*complete metric* $\mathbf{d}$ *and a weaker convergence class* $\mathbf{C}$ *so that* $\mathbf{G}$ *has the* ($\mathbf{C}$-*OSC*) *property. Suppose that there exists* $\mathbf{q} \in \mathbf{X}$ *such that* $\mathbf{K}(\mathbf{q})$ *is bounded* $\mathbf{G}$-$\mathbf{C}$-*compact and each* $\theta^i \circ \pi_i$ *is bounded below and* $\mathbf{G}$-$\mathbf{C}$-*lsca on* $K^i(\mathbf{q})$. *Then* $DGNEP(k, \{X^i\}, \{c^i\}, \mathbf{K})$ *has a solution.*

Proof. Recall that $DGNEP(k, \{X^i\}, \{c^i\}, \mathbf{K})$ is equivalent to $QEP(\mathbf{\Phi}, \mathbf{K})$, where $\mathbf{\Phi}$ is defined by (5.9). The conclusion follows by verifying all the requirements of Theorem 4.4. □

The following results are consequences of the above theorem, which are in line with the Corollaries 4.3 and 4.4, respectively. In both results, $\mathbf{X}$ is either a Banach or a complete $\mathrm{CAT}(0)$ space and the convergence class is induced with the corresponding weak convergence. The product $\mathbf{X}$ is then equipped with the product metric and the weak convergence on $\mathbf{X}$ is therefore coordinatewise. Their proofs are similar to the above theorem and the proofs are thus omitted.

Corollary 5.5. *Suppose that* $DGNEP(k, \{X^i\}, \{c^i\}, \mathbf{K})$ *takes the form* (5.8) *and induces a global feasibility graph* $\mathbf{G} := (\mathbf{X}, \mathbf{E})$ *which is reflexive and transitive and* $\mathbf{X}$ *is either a reflexive Banach space or a complete* $\mathrm{CAT}(0)$ *space satisfying the* (w-*OSC*) *property. Suppose that there exists* $\mathbf{q} \in \mathbf{X}$ *such that* $\mathbf{K}(\mathbf{q})$ *is bounded* $\mathbf{G}$-*weakly compact and each* $\theta^i \circ \pi_i$ *is bounded below and* $\mathbf{G}$-*weakly lsca on* $K^i(\mathbf{q})$. *Then* $DGNEP(k, \{X^i\}, \{c^i\}, \mathbf{K})$ *has a solution.*

Corollary 5.6. *Suppose that* $DGNEP(k, \{X^i\}, \{c^i\}, \mathbf{K})$ *takes the form* (5.8) *and induces a global feasibility graph* $\mathbf{G} := (\mathbf{X}, \mathbf{E})$ *which is reflexive and transitive and* $\mathbf{X}$ *is either a reflexive Banach space or a complete* $\mathrm{CAT}(0)$ *space satisfying the* (*OSC*) *property. Suppose that there exists* $\mathbf{q} \in \mathbf{X}$ *such that* $\mathbf{K}(\mathbf{q})$ *is bounded, closed, and* $\mathbf{K}(\mathbf{u})$ *is convex for all* $\mathbf{u} \in \mathbf{K}(\mathbf{q})$, *and each* $\theta^i \circ \pi_i$ *is bounded below and* $\mathbf{G}$-*lsca on* $K^i(\mathbf{q})$. *Then* $DGNEP(k, \{X^i\}, \{c^i\}, \mathbf{K})$ *has a solution.*

We conclude the section with a simple example of a GNEP to illustrate our results. Note that we shall verify directly the conditions of Theorem 4.1. The process will be the same with the derivation of GNEP results throughout this section.

Example 5.2. Consider a generalized game of $k = 2$ players with $X^1 = X^2 = [0,1]$. Let I_0 be the line segment joining $(0,1) \in \mathbb{R}^2$ to $(0,0) \in \mathbb{R}^2$ and for each $n \in \mathbb{N}$, let I_n be the line segment joining the point $\left(1, \frac{1}{n}\right) \in \mathbb{R}^2$ to the origin $(0,0) \in \mathbb{R}^2$. Put $\mathbf{Z} := \bigcup_{n=0}^{\infty} I_n$ and equip it with the length metric $\mathbf{d_Z}$ induced by the Euclidean distance, i.e. $\mathbf{d_Z}(\mathbf{y}, \widetilde{\mathbf{y}}) = \|\mathbf{y} - \widetilde{\mathbf{y}}\|$ if $\mathbf{y}$ and $\widetilde{\mathbf{y}}$ belongs to the same segment I_n (for some $n \in \mathbb{N}$) and $\mathbf{d_Z}(\mathbf{y}, \widetilde{\mathbf{y}}) = \|\mathbf{y}\| + \|\widetilde{\mathbf{y}}\|$ otherwise, where $\|\cdot\|$ denotes the Euclidean norm. Define a metric $\mathbf{d}$ on $\mathbf{X} := X^1 \times X^2$ by $\mathbf{d}(\mathbf{y}, \widetilde{\mathbf{y}}) := \mathbf{d_Z}(\mathbf{y}, \widetilde{\mathbf{y}})$ if both $\mathbf{y}$ and $\widetilde{\mathbf{y}}$ belong to $\mathbf{Z}$, and $\mathbf{d}(\mathbf{y}, \widetilde{\mathbf{y}}) := 3$ if at least one of $\mathbf{y}$ and $\widetilde{\mathbf{y}}$ is not in $\mathbf{Z}$ and $\mathbf{y} \neq \widetilde{\mathbf{y}}$. The metric space $(\mathbf{X}, \mathbf{d})$ is complete.

Next, we define the dictator's global feasibility map $\mathbf{K}(\cdot)$ for $(x,y) \in \mathbf{X}$ by

$$\mathbf{K}(x,y) := \{(u,v) \in \mathbf{Z} \mid u = x,\ v \leq y\}$$

if $(x,y) \in \mathbf{Z}$, and $\mathbf{K}(x,y) = \{(x,y)\}$ otherwise. Observe that $\mathbf{K}$ is compact-valued. Obviously, the graph $\mathbf{G} := (\mathbf{X}, \mathbf{E})$, where $\mathbf{E}$ is defined by (5.10), is reflexive and transitive. It also satisfies the (OSC) property.

Let us simply define the linear costs by

$$c^1(x,y) := x - y \tag{5.11}$$

$$c^2(x,y) := y - x \tag{5.12}$$

for all $x = (x_1, x_2) \in X^1$ and $y = (y_1, y_2) \in X^2$. We will now show that all the requirements of Corollary 4.1 are met. Since the costs satisfy (5.8), the bifunction $\mathbf{\Phi}$ defined by (5.9) verifies (c). Moreover, the condition (b) is true by the very definition of (5.11) and (5.12). By Corollary 4.1, the problem $QEP(\mathbf{\Phi}, \mathbf{K})$, which is equivalent to $DGNEP(k, \{X^i\}, \{c^i\}, \mathbf{K})$, has a solution.

Finally let us pick one of such a solution explicitly. Recall that K^1 and K^2 are given in (5.7). If $(x,y) \in \mathbf{Z}$ is fixed with $x > 0$, then $c^2(x,\cdot)$ has no minimizer over $K^2(x,y)$. Thus, the only possible optimal decision for the player $i = 1$ is $x^* = 0$. Since the player $i = 2$ seeks to minimize her cost as well, her optimal decision is $y^* = 0$. Therefore, a desired solution is $(x^*, y^*) = (0,0)$. One should observe that the optimality concept from a non-cooperative game perspective is quite different from a minimization problem, since in that case the player $i = 1$ would want to increase y and the player $i = 2$ would want to increase x which is completely opposite to what actually happened here.

CONCLUDING REMARKS

In this paper, we have successfully introduced a version of Ekeland's variational principle which incorperates the graphical structure and a generalized w-distance function. The graphical structure allows a quasi-optimization view and the generalized w-distance allows a broader supporting structure. We then took advantage of the former aspect and develop existence theorems for quasi-equilibrium problems where the constraint map is induced directly from the graphical structure itself. We also deduce a number of consequences in game theory together with a supporting example. Our example also illustrates that the Nash equilibrium is much different from the forceful minimization over all the variables.

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[1]Department of Mathematics, Faculty of Science
[2]Center of Excellence in Theoretical and Computational Science
King Mongkut's University of Technology Thonburi
126 Pracha Uthit Rd., Bang Mod, Thung Khru, Bangkok 10140, Thailand.
Email address: parin.cha@kmutt.ac.th

[3] DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE AND TECHNOLOGY
BANSOMDEJCHAOPRAYA RAJABHAT UNIVERSITY
1061 ISARAPHAB 15 RD., DHONBURI, BANGKOK 10600, THAILAND.
Email address: nantaporn.ch@bsru.ac.th

[4] DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE
FACULTY OF SCIENCE AND TECHNOLOGY
RAJAMANGALA UNIVERSITY OF TECHNOLOGY THANYABURI (RMUTT)
39 MOO 1, KHLONG HOK, KHLONG LUANG, PATHUMTHANI 12110, THAILAND.
Email address: printaporn_s@rmutt.ac.th