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# New results related to cutters and to an extrapolated block-iterative method for finding a common fixed point of a collection of them

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ABSTRACT. Given a Hilbert space and a finite family of operators defined on the space, the common fixed point problem (CFPP) is to find a point in the intersection of the fixed point sets of these operators. Instances of the problem have numerous applications in science and engineering. We consider an extrapolated block-iterative method with dynamic weights for solving the CFPP assuming the operators belong to a wide class of operators called cutters. Global convergence is proved in two different scenarios, one of them is under a seemingly new condition on the weights which is less restrictive than a condition suggested in previous works. In order to establish convergence, we derive various new results of independent interest related to cutters, some of them extend, generalize and clarify previously published results.

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