

Leray Schauder topological degree for nonlinear elliptic PDEs driven by the $\mathcal{A}_{p(\cdot)}$ -Laplacian operator

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ABSTRACT. This study explores a class of elliptic problems with variable exponents, governed by the $\mathcal{A}_{p(\cdot)}$ -Laplacian operator. A novel approach is proposed by reformulating the problem as an equivalent fixed-point problem within a suitable variable exponent space. By combining variational methods with Leray-Schauder topological degree theory, we establish the existence of weak solutions for the investigated problems.

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