

Two geometric constants of normed spaces related to new characterizations of inner product spaces

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ABSTRACT. In this paper, we first give new characterizations of inner product spaces and then introduce and calculate two related geometric constants of some normed spaces. We also present an open problem.

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