

On estimates of the best M -term approximations of functions of the class $W_{q,\tau}^{\bar{r}}$ in the Lorentz space

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ABSTRACT. The article considers the Lorentz space $L_{p,\tau}(\mathbb{T}^m)$ of 2π -periodic functions with m variables and the Sobolev class $W_{p,\tau}^{\bar{r}}$ in $L_{p,\tau}(\mathbb{T}^m)$. The main goal of the article is to find the order of the best M -term trigonometric approximation $e_M(f)_{p,\tau_2}$ of the class $W_{q,\tau_1}^{\bar{r}}$ for a number of relations between the parameters $p, q, \tau_1, \tau_2 \in (1, \infty)$ and $r_j, j = 1, \dots, m, \bar{r} = (r_1, \dots, r_m)$. The article establishes order-sharp estimates for the value $e_M(W_{q,\tau_1}^{\bar{r}})_{p,\tau_2}$ in the cases $1 < q < 2 < p < \infty, r_1 > \frac{1}{q}$ and $r_1 = \frac{1}{q}, 1 < q < 2$ and $1 < \tau_1 < \infty$ or $q = 2$ and $2 < \tau_1 < \infty$. The exact order of this value are obtained for $p = q$ and $\tau_2 = \tau_1 = \tau$, as well as the upper estimate for $1 < p < q < \infty$.

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