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A Dual-Projective Double Inertial Forward-Backward Splitting Algorithm for Variational Inclusion Problems with Applications to External Validation in Breast Cancer Diagnosis

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ABSTRACT. This study proposes a novel algorithm that combines a double inertial technique with a two-step projection-based forward-backward splitting method to solve variational inclusion problems in real Hilbert spaces. A weak convergence theorem is established under suitable conditions, and its validity is demonstrated in an infinite-dimensional setting. The model is evaluated in a machine learning context using external validation—training on public data and testing on separate real-world clinical data for breast cancer prediction. Experimental results show promising classification performance: accuracy of 83.16%, precision of 86.56%, recall of 85.82%, and F1-score of 86.19%. The proposed method achieves superior computational efficiency, requiring only 47 iterations and 0.2998 seconds, compared to over 200 iterations and 2.3 seconds for baseline algorithms. Confusion matrix and ROC analysis confirm robust multi-class classification, with notable separability in Class 2 and Class 4. Training and loss curves further demonstrate model stability and generalizability, with no signs of overfitting. Overall, the algorithm presents a fast, accurate, and practical solution for real-world medical applications.

DATA AVAILABILITY

(i) The real data set from Phayao Hospital are available from the corresponding author, [W. Cholamjiak], upon reasonable request.
(ii) The mammographic mass dataset from UCI is available on the UCI website (http://archive.ics.uci.edu/ml/datasets/mammographic+mass).

INSTITUTIONAL REVIEW BOARD STATEMENT

This study was conducted in accordance with the Declaration of Helsinki, the Belmont report, CIOMS Guideline international conference on Harmonization in Good Cinical Pratice or ICH-GCP and with approval from the Ethics Committee and Institutional Review Board of Phayao Hospital (Institutional Review Board (IRB) approval, IRB Number: COA No.20966-02-26).

CONFLICTS OF INTEREST

The authors declare that they have no conflicts of interest regarding the publication.

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