

Inertial viscosity fixed point algorithms for monotone inclusion, convex minimization, and image restoration problems

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ABSTRACT. A novel inertial viscosity iterative algorithm approximates a common fixed point of an infinite family of nonexpansive mappings in a Hilbert space is introduced and studied. The proposed method's strong convergence theorem was proved under suitable control conditions. Furthermore, we applied our results to solve monotone inclusion, convex minimization, and image restoration problems. We also achieve the viscosity approximation method with inertial effect in a novel accelerated image restoration form. To illustrate the efficacy of our method, we provide a number of numerical examples. We emphasize that the results accounted for in the manuscript extend and complement various results in this field of study.

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