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A product of strongly quasi-nonexpansive mappings in Hadamard spaces

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ABSTRACT. In this paper, we prove that the product of strongly quasi-nonexpansive Δ-demiclosed mappings is also a strongly quasi-nonexpansive orbitally Δ-demiclosed mapping in Hadamard spaces. Additionally, we establish the Δ-convergence theorem for approximating a common fixed point of infinite products of these mappings in Hadamard spaces. Our results have practical applications in convex function minimization, the minimization of the sum of finitely many convex functions, and solving the convex feasibility problem for finitely many sets in Hadamard spaces.

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