

# Approximate Solutions for Multi-Objective Optimization Problems via Scalarizing and Nonscalarizing Methods <sup>1</sup>

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**ABSTRACT.** In this paper, max-ordering and weighted compromising methods are employed to investigate approximate Pareto solutions for a class of multi-objective optimization problems with an infinite number of constraints. Approximate optimality conditions for  $\varepsilon$ -quasi Pareto solutions and almost  $\varepsilon$ -quasi Pareto solutions of the considered problems are established. The results are derived using a new notion of  $\varepsilon$ -quasi subdifferentials for locally Lipschitz functions and  $\varepsilon$ -quasi normal sets. Approximate duality theorems are also introduced. In particular, the relationships between the original multi-objective optimization problem and its dual are analyzed via a corresponding pair of primal-dual scalar problems. Several examples are provided to illustrate the proposed notions and to demonstrate the applicability of the obtained results.

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