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Weak and strong convergence of multi-inertial forward-backward-forward methods for solving monotone inclusions

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ABSTRACT. In this work, we focus on the inclusion problem for the sum of two monotone operators in real Hilbert spaces. We establish weak and strong convergence theorems under standard and relatively mild assumptions. A notable contribution of this work lies in the relaxation of commonly imposed conditions. Specifically, the single-valued operator is required to be only monotone and Lipschitz continuous, without the stronger assumption of cocoercivity and without requiring prior information of the Lipschitz constant. Finally, we present several numerical experiments in both finite and infinite-dimensional settings to illustrate and support our main results.

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