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Applying convexificators to optimization problems with variational inequality constraints via a gap function approach

RACHID EL IDRISSI[1] AND LAHOUSSINE LAFHIM[2]

ABSTRACT. This study investigates an optimization problem with a variational inequality constraint, commonly referred to as a generalized bilevel optimization programming problem. We first employ a gap function to reformulate the original problem into a simpler equivalent formulation. Based on this reformulation, we derive a convexificator estimate of the gap function associated with the variational inequality. Using this estimate, we establish necessary optimality conditions via exact penalization techniques. Finally, we present an illustrative example to demonstrate the applicability of the obtained results.

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Corresponding author: Rachid El Idrissi; rachid.elidrissi1@usmba.ac.ma

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[1] Moroccan School of Engineering Sciences (EMSI), Casablanca, Morocco
Email address: elidrissirachid14@gmail.com

[2] L2MASI, Faculty of Sciences Dhar El Mahraz, Sidi Mohamed Ben Abdellah University, Fez, Morocco
Email address: lahoussine.lafhim@usmba.ac.ma