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Existence Results for Generalized Equilibrium Problems and Applications

BUI THE HUNG [1]

ABSTRACT. In this paper, we investigate a generalized equilibrium problem for multivalued mappings defined on the product of a convex subset of a topological vector space and an arbitrary nonempty set. Several existence results are established even when the underlying set is arbitrary and carries no specific algebraic or topological structure. Furthermore, we demonstrate that the sufficient conditions ensuring the existence of a solution can be imposed on a self-segment-dense subset rather than on the entire feasible set. The theoretical results are then applied to variational relation problems, vector equilibrium problems, and common fixed point problems, thereby unifying and extending a variety of known existence theorems in the literature.

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Department of Mathematics, Thai Nguyen University of Education, Viet Nam
Email address: hungbt@tnue.edu.vn