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Approximating fixed points of nonexpansive mappings by Nakajo–Takahashi type iteration in Hadamard spaces

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ABSTRACT. In this paper, we show a fixed point theorem in the setting of Hadamard spaces, which corresponds to a projection method proposed by Nakajo and Takahashi in 2003 in the setting of Hilbert spaces. In an iteration with a projection method, we need to assume that a given mapping has a fixed point to guarantee the generation of its approximate sequence. However, in a work by Matsushita and Takahashi in 2008, they proved the well-definedness of the iterative sequence without the assumption that a given mapping has a fixed point. In this work, we obtain Matsushita–Takahashi's theorem in a Hadamard space.

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