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Isosceles triple convexity

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ABSTRACT. A set S in $\mathbb{R}^d$ is called it-convex if, for any two distinct points in S, there exists a third point in S, such that one of the three points is equidistant from the others.

In this paper we first investigate nondiscrete it-convex sets, then discuss about the it-convexity of the eleven Archimedean tilings, and treat subsequently finite subsets of the square lattice. Finally, we obtain a lower bound on the number of isosceles triples contained in an n-point it-convex set.

1. INTRODUCTION

Three points $x,\ y,\ z \in \mathbb{R}^d$ (always $d \geq 2$) form an *isosceles triple* $\{x,\ y,\ z\}$ if one of them is equidistant from the others.

Let $S \subset \mathbb{R}^d$. A pair of points $x,\ y \in S$ is said to enjoy the *it-property in S* if there exists a third point $z \in S$, such that $\{x,\ y,\ z\}$ is an isosceles triple.

The set S is called *isosceles triple convex*, for short *it-convex*, if every pair of its points enjoys the it-property in S.

The study of it-convexity, to which is devoted this paper, is embedded in a more general theme. Let $\mathcal{F}$ be a family of sets in $\mathbb{R}^d$. A set $M \subset \mathbb{R}^d$ is called *$\mathcal{F}$-convex* if for any pair of distinct points $x,\ y \in M$ there is a set $F \in \mathcal{F}$ such that $x,\ y \in F$ and $F \subset M$.

The second author proposed at the 1974 meeting on Convexity in Oberwolfach the investigation of this very general kind of convexity. Usual convexity, affine linearity, arcwise connectedness, polygonal connectedness, are just some examples of $\mathcal{F}$-convexity (for suitably chosen families $\mathcal{F}$).

Blind, Valette and the second author [1], and also Böröczky, Jr. [2], investigated the rectangular convexity, Magazanik and Perles dealt with staircase connectedness [4], the second author studied the right convexity [8], the first two authors generalized the latter type of convexity and investigated the right triple convexity [7], [6].

For distinct $x,\ y \in \mathbb{R}^d$, let $\overline{xy}$ denote the line-segment from x to y. Furthermore, let H_{xy} be the hyperplane through the midpoint $(x+y)/2$ of $\overline{xy}$, orthogonal to $\overline{xy}$, and S_{xy} be the hypersphere of radius $\|x-y\|$ centred at x. Put $S'_{xy} = S_{xy} \setminus \{y\}$, and $W_{xy} = H_{xy} \cup S'_{xy} \cup S'_{yx}$.

Clearly, $S \subset \mathbb{R}^d$ is it-convex if and only if, for any two distinct points $x,\ y \in S$, $S \cap W_{xy} \neq \emptyset$.

For $S \subset \mathbb{R}^d$, let $\mathrm{diam} S = \sup_{x,y \in S} \|x - y\|$. The pair $(x, y) \in S \times S$ is called *diametral*, if $\|x - y\| = \mathrm{diam} S$.

Let $S \subset \mathbb{R}^d$. If there is a hyperplane H such that $H \cap S = \emptyset$, but S meets both open halfspaces determined by H, then S is said to be *strictly separated by H*.

Let $A, B \subset \mathbb{R}^d$ be compact and choose $x \in \mathbb{R}^d$. Put

$$r(x, A) = \max_{y \in A} \|x - y\|,\ \ n(x, A) = \min_{y \in A} \|x - y\|.$$

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Then $r(A) = \min_x r(x, A)$ is called the *radius* of A. Also,

$$h(A, B) = \max\{\max_{x \in A} n(x, B), \max_{y \in B} n(y, A)\}$$

is called the *Pompeiu-Hausdorff distance* between A and B.

2. NONDISCRETE it-CONVEXITY

For every cardinal number α satisfying $3 \leq \alpha \leq \mathfrak{c}$, there is an it-convex set of cardinality α. Indeed, consider a point x plus a set of cardinality $\alpha - 1$ included in a circle around x.

Also, remark that every open set in $\mathbb{R}^d$ is it-convex.

Theorem 2.1. *All sets in $\mathbb{R}^d$ which cannot be strictly separated by any hyperplane are it-convex.*

Proof. Assume $S \subset \mathbb{R}^d$ cannot be strictly separated by any hyperplane. For any two distinct points $x,\ y \in S$, consider the hyperplane H_{xy}. Clearly x and y are lying in the distinct half-spaces determined by H_{xy}, so $H_{xy} \cap S \neq \emptyset$. Now, for any $z \in H_{xy} \cap S$, $\|x - z\| = \|y - z\|$ and $\{x, y, z\}$ is an isosceles triple. □

Corollary 2.1. *All connected sets in $\mathbb{R}^d$ are it-convex.*

Now we study the it-convexity of sets with at least 2 connected components.

Theorem 2.2. *Let S be a set with at least 2 components. If the union of any two components is it-convex, then S is it-convex.*

Proof. Take two distinct points $x,\ y \in S$. If $x,\ y$ are in the same component, then by Corollary 2.1, they enjoy the *it*-property. Otherwise, $x,\ y$ are in distinct components, say S_1 and S_2. Since $S_1 \cup S_2$ is it-convex, $x,\ y$ enjoy for this reason the *it*-property. □

Theorem 2.3. *If the compact set K has two connected components A, B, and $h(A, B) < r(A)/2$, then K is it-convex.*

Proof. Take $x \in B$, $y \in A$. Let $w \in A$ be closest to x.

If $\|x - w\| \leq \|w - y\|$, see Figure 1 (a), then A meets the hyperplane H_{xy} in some point u, because A is connected. The triple $\{x, y, u\}$ is isosceles.

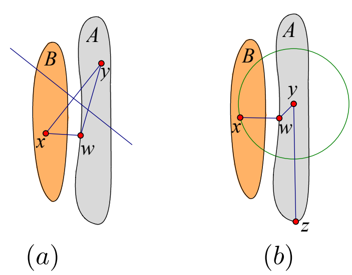


FIGURE 1. $h(A, B) < r(A)/2$.

If $\|x - w\| > \|w - y\|$, as shown in Figure 1 (b), take $z \in A$ farthest from y. We have

$$\|x - w\| \leq h(A, B) \leq \frac{r(A)}{2} \leq \frac{\|y - z\|}{2}.$$

Also,

$$\|x - y\| \leq \|x - w\| + \|w - y\| < 2\|x - w\|.$$

It follows that $\|x - y\| < \|y - z\|$, and we find some $v \in A \cap S_{yx}$, as S_{yx} separates in A the point y from z. Now, $\{x, y, v\}$ is an isosceles triple. □

The continua A, B are called *unseparable* if, for any hyperplane H disjoint from $A \cup B$, one of the two open half-spaces determined by H includes $A \cup B$.

Clearly, if A, B are unseparable continua, then $A \cup B$ is *it*-convex. This extends straightforwardly as follows. If the continua $A_1, A_2, ..., A_n$ are pairwise unseparable, then $\bigcup_{i=1}^n A_i$ is *it*-convex. But the hypothesis here can be relaxed.

Theorem 2.4. *Let $A_1, A_2, ..., A_n$ be continua, and G be a tree with vertex set $V(G) = \{v_1, ..., v_n\}$. Suppose for every edge $(v_i, v_j) \in E(G)$, the sets A_i, A_j are unseparable. Then $\bigcup_{i=1}^n A_i$ is it-convex.*

Proof. Take, for every i, $x_i \in A_i$. We have to show that, for any distinct i, j, there exists a $y \in \bigcup_{i=1}^n A_i$ such that $\{x_i, x_j, y\}$ is an isosceles triple.

We have a (unique) path P in G from v_i to v_j. Consider the hyperplane $H_{x_i x_j}$ and the broken line $Q \subset \mathbb{R}^d$ obtained as union of all line-segments $\overline{x_m x_n}$ corresponding to edges (v_m, v_n) of P. Since Q joins the points x_i, x_j lying on different sides of H, it must meet H. Let now $x_m x_n$ denote one of the line-segments of Q meeting H. Since $(v_m, v_n) \in E(P) \subset E(G)$, the sets A_m, A_n are unseparable. Hence,

$$H_{x_i x_j} \cap (A_m \cup A_n) \neq \emptyset,$$

and x_i, x_j enjoy the *it*-property. $\square$

A referee has asked the interesting question: "Is it easy to prove that most (in the Baire sense) compact sets of $\mathbb{R}^d$ are *it*-convex?" In fact, it is indeed easy to prove the contrary: *In most compact sets there is no pair of distinct pairs of points (a, b), (c, d), such that $\|a - b\| = \|c - d\|$. Consequently, most compact sets are not it-convex.*

3. it-CONVEXITY OF ARCHIMEDEAN TILINGS

For a general overview of tilings, see Grünbaum and Shephard's book [3]. A *plane tiling* $\mathcal{T}$ is a countable family of closed sets $\mathcal{T} = \{T_1, T_2, \cdots\}$ which cover the plane without gaps or overlaps. And every closed set $T_i \in \mathcal{T}$ is called a *tile of* $\mathcal{T}$. We consider a special case of tilings in which each tile is a polygon. If the corners and sides of a polygon coincide with the vertices and edges of the tiling, we call the tiling *edge-to-edge*. A so-called *type of vertex* describes its neighbourhood. If, for example, in some cyclic order around a vertex there are a triangle, then another triangle, then a square, next a third triangle, and last another square, then its type is $(3^2.4.3.4)$. We consider plane edge-to-edge tilings in which all tiles are regular polygons, and all vertices are of the same type. Thus, the vertex type will be defining our tiling.

There exist precisely eleven such tilings [3]. These are (3^6), $(3^4.6)$, $(3^3.4^2)$, $(3^2.4.3.4)$, $(3.4.6.4)$, $(3.6.3.6)$, (3.12^2), (4^4), $(4.6.12)$, (4.8^2), and (6^3). They are called *Archimedean tilings.*

We shall say that a tiling is *it-convex* if its vertex set is *it*-convex.

In this section, we investigate the it-convexity of all Archimedean tilings. We choose the length of edges in all tilings to be 1.

Theorem 3.5. *The Archimedean tilings (3^6), (4^4), (6^3), $(3.6.3.6)$ are it-convex.*

Proof. Let T denote the vertex set of the Archimedean tiling (3^6). For each $x \in T$, let f_x be a map from T to T formed by rotating T anticlockwise about x by an angle $\frac{\pi}{3}$, see Figure 2 (a). Obviously, f_x is a bijection from T to T, and $f_x(y) = y$ if and only if $y = x$. So for any two distinct points $x, y \in T$, $f_x(y) \in T \setminus \{x, y\}$ satisfies $\|x - y\| = \|x - f_x(y)\|$. Therefore T is it-convex.

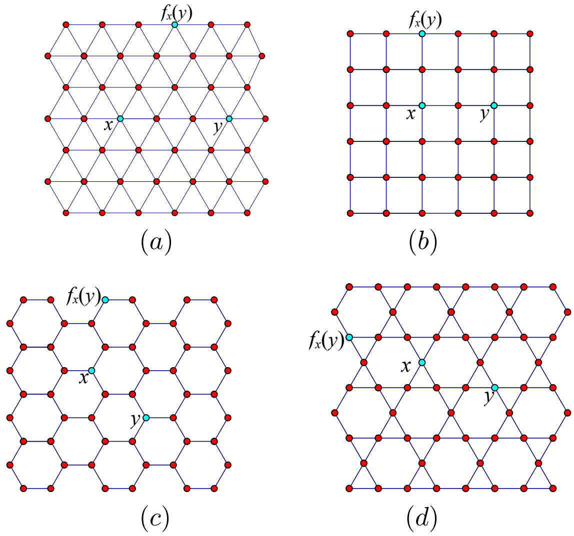


FIGURE 2. it-convexity of (3^6), (4^4), (6^3) and $(3.6.3.6)$.

For the Archimedean tilings (4^4), (6^3), $(3.6.3.6)$, we only need to define f_x by rotating the vertex sets about x by angles $\frac{\pi}{2}$, $\frac{2\pi}{3}$, π, respectively (see Figure 2 $(b) - (d)$). $\square$

Theorem 3.6. *The Archimedean tiling* $(3^2.4.3.4)$ *is* it*-convex.*

Proof. It is clear that a line passing through the common edge of any two adjacent triangular tiles is an axis of symmetry of the vertex set of the $(3^2.4.3.4)$ tiling. Furthermore, every vertex x is lying on precisely one axis of symmetry, ℓ_x (see Figure 3).

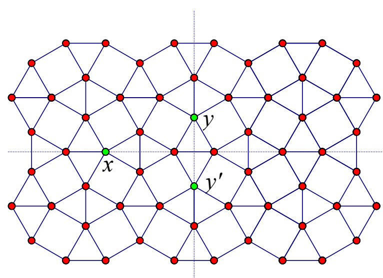


FIGURE 3. it-convexity of $(3^2.4.3.4)$.

For any two distinct vertices $x,\ y$, if $\ell_x \neq \ell_y$, let y' be the reflection image of y with respect to ℓ_x. Then $d(x,y) = d(x,y')$, and $\{x,\ y,\ y'\}$ is an isosceles triple.

If $\ell_x = \ell_y$, suppose they are horizontal. As the length of each edge is 1, the distance between two consecutive vertices lying on ℓ_x is 1 or $\sqrt{3}$, which appear alternately.

If $\|x - y\|$ is a multiple of $1 + \sqrt{3}$, the tiling has the vertex $2y - x$, and so (x, y) enjoys the *it*-property.

If not, the vertical axis of symmetry H_{xy} is also an axis of symmetry for the whole tiling, and meets its vertex set.

Consequently, the Archimedean tiling $(3^2.4.3.4)$ is it-convex. $\square$

Theorem 3.7. *The Archimedean tiling* $(3^4.6)$ *is* it*-convex.*

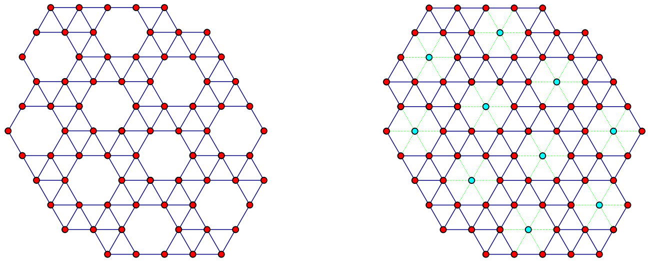


FIGURE 4. it-convexity of $(3^4.6)$.

Proof. Let S denote the vertex set of the tiling $(3^4.6)$ with side length 1, and C denote the set of the centers of all hexagon tiles. Then C is the vertex set of a triangular tiling with side length $\sqrt{7}$, and $S \cup C$ is the vertex set T of the triangular tiling (3^6) with side length 1. Clearly, $S \cap C = \emptyset$.

Given $x \in T$, let f_x be the map from the proof of Theorem 3.5. Then $\{f_x^0,\ f_x,\ f_x^2,\ f_x^3,\ f_x^4,\ f_x^5\}$ form a transformation group on T. For each $y \in T \setminus \{x\}$, the points $f_x(y)$, $f_x^2(y)$, $f_x^3(y)$, $f_x^4(y)$, $f_x^5(y)$ lie in T and are the vertices of a regular hexagon centred at x.

Assume now that $x,\ y \in S$. If all points $f_x(y)$, $f_x^2(y)$, $f_x^3(y)$, $f_x^4(y)$, $f_x^5(y)$ are in C, we have $f_{f_x(y)}(f_x^2(y)) \in C$, since C is the vertex set of a triangular tiling. But $f_{f_x(y)}(f_x^2(y)) = x$, so we get $x \in C$, contradicting our choice of x. Hence, there is a point $z \in \{f_x(y),\ f_x^2(y),\ f_x^3(y),\ f_x^4(y),\ f_x^5(y)\} \cap S$ and $\|x - y\| = \|x - z\|$. So $(3^4.6)$ is it-convex. $\square$

Despite the encouraging Theorems 3.5–3.7, not all Archimedean tilings are *it*-convex.

Theorem 3.8. *The Archimedean tilings* (4.8^2), $(3.4.6.4)$, $(3^3.4^2)$, $(4.6.12)$, (3.12^2) *are not it-convex.*

Proof. We only prove here the theorem for the tiling (4.8^2) (see Figure 5), the proof for the other tilings being very similar (see Figure 6 $(a) - (d)$). We denote the vertex set of (4.8^2) by V.

In order for V to be *it*-convex, the pair (x, y) from Figure 5 should enjoy the *it*-property, that is, $W_{xy} \cap V \neq \emptyset$. The line H_{xy} does not meet V. We show that $S'_{xy} \cap V = \emptyset$ too, which will end the proof.

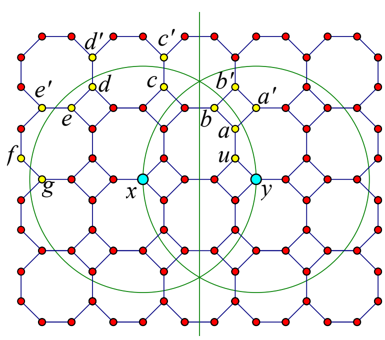


FIGURE 5. Non-it-convexity of (4.8^2).

The points of V which are "candidates" for belonging to S'_{xy} are a, a', b, b', c, c', d, d', e, e', f, g from Figure 5, and other points symmetrical to them with respect to $\overline{xy}$.

We have $\|x-y\| = 1+2\sqrt{2}$, so $\|x-y\|^2 = 9+4\sqrt{2}$.

But

$$\|x-a\|^2 = (1+\frac{3}{2}\sqrt{2})^2 + (1+\frac{1}{2}\sqrt{2})^2 = 7+4\sqrt{2} < \|x-y\|^2,$$

$$\|x-b\| < \|x-a\|, \text{as } \angle xba = \frac{\pi}{2},$$

$$\|x-b'\|^2 = 11+6\sqrt{2} > \|x-y\|^2, \qquad \|x-c\| = \|x-u\| < \|x-a\|,$$

$$\|x-c'\|^2 = 9+6\sqrt{2} > \|x-y\|^2, \qquad \|x-d'\| > \|x-c'\|,$$

$$\|x-d\| = \|x-a\|, \qquad \|x-e\| = \|x-b\|,$$

$$\|x-e'\|^2 = 9+6\sqrt{2} > \|x-y\|^2, \qquad \|x-a'\| > \|x-e'\|,$$

$$\|x-f\| = \|x-c'\|, \qquad \|x-g\| = 2+\sqrt{2} < \|x-y\|.$$

Thus, all candidates fail, and V is not *it*-convex. □

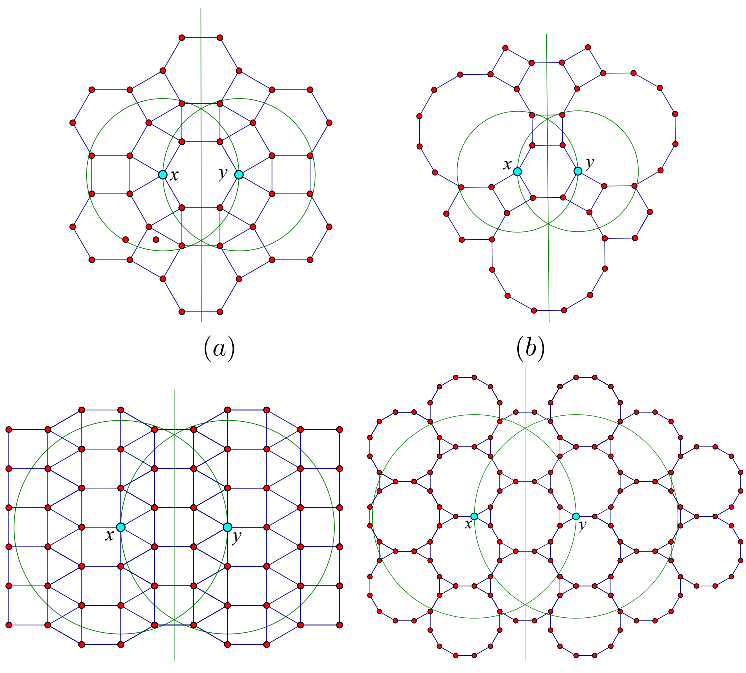


FIGURE 6. Non-it-convexity of $(3.4.6.4)$, $(4.6.12)$, $(3^3.4^2)$, (3.12^2).

4. it-CONVEXITY OF FINITE SUBSETS OF THE SQUARE LATTICE

We shall make use of the following well-known fact [5].

Lemma 4.1. *For the planar square lattice $\mathbb{Z}^2$, lines with rational slope either contain infinitely many lattice points, or contain no lattice points. A line*

$$y = \frac{m}{n}x + \frac{r}{s}$$

with $m,\ n,\ r,\ s \in \mathbb{Z}$ and $\gcd(m,n) = \gcd(r,s) = 1$, contains infinitely many lattice points if and only if $s \mid n$.

Let $x_{k,l}$ denote the point with Cartesian coordinates (k,l), where $k,l \in \mathbb{Z}$.

Lemma 4.2. *If $m+k+n+l \equiv 1 \pmod 2$, then $H_{x_{k,l}x_{m,n}}$ contains no lattice points.*

Proof. Assume w.l.o.g. $k=l=0$. Then $m+n \equiv 1 \pmod 2$. The equation of $H_{x_{0,0}x_{m,n}}$ is

$$y = \frac{-m}{n}x + \frac{n^2+m^2}{2n}.$$

If n is even, then $-m$ and n^2+m^2 are odd. Let $n = 2^p a$, where a is odd. The denominators of the irreducible form of $\frac{-m}{n}$ and $\frac{n^2+m^2}{2n}$ are $2^p b$ and $2^{p+1}c$ respectively, where b and c are odd. As $(2^{p+1}c) \nmid (2^p b)$, by Lemma 4.1, $H_{x_{0,0}x_{m,n}}$ contains no lattice points.

If n is odd, then m is even, and n^2+m^2 is odd. After reduction, the denominator of $\frac{-m}{n}$ is odd, but the denominator of $\frac{n^2+m^2}{2n}$ is even. By Lemma 4.1, $H_{x_{0,0}x_{m,n}}$ contains no lattice points. □

Lemma 4.3. *Let S be a finite subset of $\mathbb{Z}^2$. If S has a diametral pair (x,y), where $x = x_{k,l},\ y = x_{m,n}$, such that the following conditions are satisfied*
1) $m+k+n+l \equiv 1 \pmod 2$,
2) *no other diametral pair contains x or y,*
then (x,y) does not enjoy the it-property in S, and therefore S is not it-convex.

Proof. By Condition 1) and Lemma 4.2, H_{xy} contains no lattice points. Due to Condition 2), for each $z \in S \setminus \{x,\ y\}$, we have $\|x-z\| < \|x-y\|$, $\|y-z\| < \|x-y\|$. Therefore $S \cap W_{xy} = \emptyset$, which means that (x,y) does not enjoy the it-property in S, and hence S is not it-convex. □

Corollary 4.1. *Let $x = x_{k,l},\ y = x_{m,n}$. Consider the set $R \subset \mathbb{Z}^2$ of all points lying in the (possibly degenerate) rectangle with horizontal and vertical sides, admitting $\overline{xy}$ as a diagonal. If $m+k+n+l \equiv 1 \pmod 2$ and $\{x,\ y\} \subset S \subset R$, then (x,y) does not enjoy the it-property in S, whence S is not it-convex.*

Proof. The set S admits (x,y) as a diameter pair, and for this pair the conditions of Lemma 3 are satisfied. □

Let $x,\ y$ be points of the square lattice, and $P,\ Q$ be shortest paths from x to y in the graph defined by the lattice. These paths, considered as arcs in $\mathbb{R}^2$, form the boundary of an open set U, the unique unbounded component of $\mathbb{R}^2 \setminus (P \cup Q)$.

All lattice points in $\mathbb{R}^2 \setminus U$ form a set that we call *monotone*. We investigate now the *it*-convexity of monotone sets.

It is clear from the definition that both $P,\ Q$ and the whole monotone set determined by them lie in the rectangle (with horizontal and vertical sides) with diagonal $\overline{xy}$, and contain $x,\ y$, which are called *endpoints* of the set.

Thus, if $x = x_{k,l}$, $y = x_{m,n}$, we can suppose $k=l=0$, $m \ge n \ge 0$. This will be assumed from now on. Let $\mathcal{T}(m,n)$ be the family of all monotone sets with endpoints $x_{0,0}, x_{m,n}$, and $R(m,n)$ be the set of all lattice points lying in the rectangle (with horizontal and vertical sides) having $\overline{x_{0,0}x_{m,n}}$ as a diagonal.

Lemma 4.4. *For every monotone set $T \in \mathcal{T}(m,n)$, if $m+n \equiv 1 \pmod 2$, then $(x_{0,0}, x_{m,n})$ does not enjoy the it-property in T, and therefore T is not it-convex.*

Proof. Since $\{x_{0,0}, x_{m,n}\} \subset T \subset R(m,n)$, by Corollary 4.1, $(x_{0,0}, x_{m,n})$ does not enjoy the it-property in T, and hence T is not it-convex. □

Theorem 4.9. *There are precisely 8 pairwise non-congruent it-convex monotone sets in $\bigcup_{m,n=0}^{\infty} \mathcal{T}(m,n)$, namely one in $\mathcal{T}(2,0)$, two in $\mathcal{T}(1,1)$, three in $\mathcal{T}(2,2)$, one in $\mathcal{T}(4,4)$, and one in $\mathcal{T}(5,5)$ (see Figure 7).*

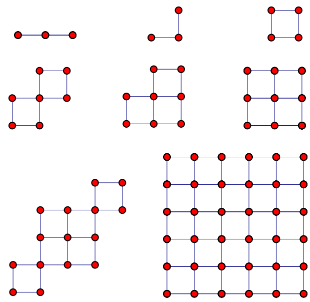
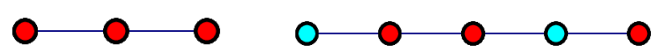


FIGURE 7. All the non-congruent it-convex monotone sets.

Proof. If $m+n \equiv 1 \pmod 2$, for each $T \in \mathcal{T}(m,n)$, by Lemma 4.4, T is not it-convex. If $m+n \equiv 0 \pmod 2$, there are 4 cases to be considered. We look for those $T \in \mathcal{T}(m,n)$ which are *it*-convex.

Case 1. $n=0$.



(a) (b)

FIGURE 8. Monotone sets in $\mathcal{T}(m,0)$.

Here m is even. $\mathcal{T}(m,0)$ contains only one monotone set, $R(m,0)$. Obviously, $R(2,0)$, shown in Figure 8 (a), is *it*-convex. For $m \geqslant 4$, we see that $(x_{0,0}, x_{m-1,0})$ does not enjoy the *it*-property in $R(m,0)$.

Case 2. $n=1$.

First suppose $m > n = 1$. In this case m is odd. For $T \in \mathcal{T}(m,1)$, $x_{m,0}$ or $x_{m-1,1}$ must be in T. As $(x_{0,0}, x_{m,0})$ does not enjoy the *it*-property in $R(m,1)$, we have $x_{m,0} \notin T$ and $x_{m-1,1} \in T$. By Lemma 4.4, $W_{x_{0,0}x_{m-1,1}} \cap (T \setminus \{x_{m,1}\}) = \emptyset$. Since $x_{m,1} \notin W_{x_{0,0}x_{m-1,1}}$, too, we get $W_{x_{0,0}x_{m-1,1}} \cap T = \emptyset$, and T is not *it*-convex.

If $m=n=1$, there are precisely two non-congruent monotone sets in $\mathcal{T}(1,1)$, as shown in Figure 9, and they are both it-convex.

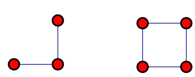


FIGURE 9. it-convex monotone sets in $\mathcal{T}(1,1)$.

Case 3. $n=2$.

Subcase 3.1. $m > n = 2$.

Now m is even. For $T \in \mathcal{T}(m,2)$, $x_{m,1}$ or $x_{m-1,2}$ must be in T. Suppose $x_{m-1,2} \in T$. Then $(T \setminus \{x_{m,0},\ x_{m,1},\ x_{m,2}\}) \in \mathcal{T}(m-1,2)$. By Lemma 4.4,

$$W_{x_{0,0}x_{m-1,2}} \cap T \setminus \{x_{m,0},\ x_{m,1},\ x_{m,2}\} = \emptyset.$$

We also can verify that

$$(S'_{x_{0,0}x_{m-1,2}} \cup S'_{x_{m-1,2}x_{0,0}}) \cap \{x_{m,0},\ x_{m,1},\ x_{m,2}\} = \emptyset.$$

By Lemma 4.2, $H_{x_{0,0}x_{m-1,2}}$ contains no lattice points. Consequently,

$$W_{x_{0,0}x_{m-1,2}} \cap \{x_{m,0},\ x_{m,1},\ x_{m,2}\} = \emptyset.$$

Therefore $W_{x_{0,0}x_{m-1,2}} \cap T = \emptyset$ and T is not it-convex. Hence, $x_{m,1} \in T$ and $(T \setminus \{x_{m,2}\}) \in \mathcal{T}(m, 1)$. By Lemma 4.4, $W_{x_{0,0}x_{m,1}} \cap (T \setminus \{x_{m,2}\}) = \emptyset$. Since $x_{m,2} \notin W_{x_{0,0}x_{m-1,2}}$, we have $T \cap W_{x_{0,0}x_{m-1,2}} = \emptyset$, and T is not it-convex.

Subcase 3.2. $m = n = 2$.

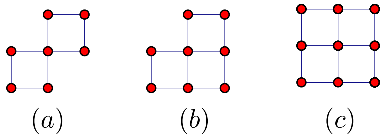


FIGURE 10. it-convex monotone sets in $\mathcal{T}(2, 2)$.

For $T \in \mathcal{T}(2, 2)$, $x_{1,2}$ or $x_{2,1}$ must be in T. If one of them, say $x_{1,2}$, belongs to T, then the it-property at $(x_{0,0}, x_{1,2})$ implies that the other, $x_{2,1}$, is also in T. Symmetrically, $x_{1,0}$ and $x_{0,1}$ are both in T. Therefore $T \supset S$, where S is the set shown in Figure 10 (a). We can easily verify that each $T \in \mathcal{T}(2, 2)$ including S is it-convex. See Figure 10.

Case 4. $n \geqslant 3$.

Subcase 4.1. $m > n \geqslant 3$.

For $T \in \mathcal{T}(m, n)$, $x_{m,n-1}$ or $x_{m-1,n}$ must be in T. By a method similar to that used in the proof of Subcase 3.1, we can verify that $(x_{0,0}, x_{m,n-1})$ does not enjoy the it-property in T, hence $x_{m,n-1} \notin T$, and $x_{m-1,n} \in T$, which imply that $(x_{0,0}, x_{m-1,n})$ does not enjoy the it-property in T.

Subcase 4.2. $m = n = 3$.

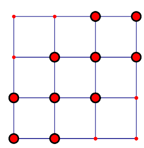


FIGURE 11. Monotone sets in $\mathcal{T}(3, 3)$.

We prove that there is no it-convex set in $\mathcal{T}(3, 3)$. Suppose $T \in \mathcal{T}(3, 3)$; as before, $x_{1,0}, x_{0,1}, x_{2,3}, x_{3,2} \in T$. We observe that the it-property is not satisfied at $(x_{1,0}, x_{1,3})$ in $R(3, 3)$, and at the other pairs symmetrical to it. This implies that T must be included in the set S shown in Figure 11. But $(x_{0,1}, x_{3,2})$ does not enjoy the it-property in S, contradicting the fact that T is it-convex.

Subcase 4.3. $m = n = 4$.

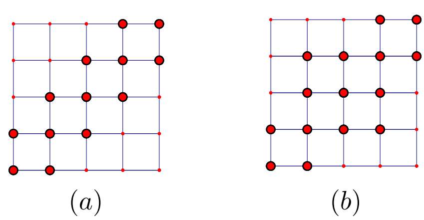
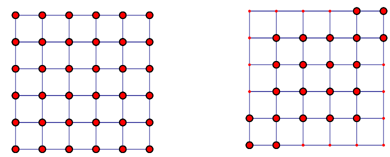


FIGURE 12. Monotone sets in $\mathcal{T}(4, 4)$.

We can show that there is exactly one it-convex set $S_2 \in \mathcal{T}(4, 4)$, shown in Figure 12 (b). Of course, as before, $x_{1,0}, x_{0,1}, x_{4,3}, x_{3,4} \in T$. Therefore $x_{3,3}$ and $x_{1,1}$ are also in T, which implies that $x_{2,3}$ or $x_{3,2}$ must be in T, and $x_{1,2}$ or $x_{2,1}$ must be in T.

If, say, $x_{2,3} \in T$, then $x_{2,0} \notin T$ because the *it*-property is not verified at $(x_{2,0}, x_{2,3})$ in $R(4,4)$. Hence $x_{4,0} \notin T$. The *it*-property at $(x_{0,0}, x_{2,3})$ implies that $x_{3,2}$ must be in T. Therefore, by symmetry, $S_1 \subset T \subset S_2$, where S_1 and S_2 are shown in Figure 12 (a) and (b), respectively. The *it*-property of the pairs $(x_{1,0}, x_{2,3})$ and $(x_{0,1}, x_{3,2})$ implies that $T = S_2$. We can see that S_2 is *it*-convex.

Subcase 4.4. $m = n = 5$.



(a) (b)

FIGURE 13. Monotone sets in $\mathcal{T}(5,5)$.

We will prove that $R(5,5)$, shown in Figure 13 (a), is the only it-convex set in $\mathcal{T}(5,5)$. Again $x_{1,0}, x_{0,1}, x_{1,1}, x_{5,4}, x_{4,5}, x_{4,4} \in T$.

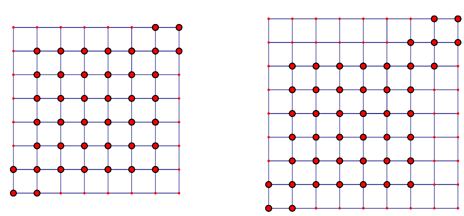
If $\{x_{0,2},\ x_{2,0},\ x_{3,5},\ x_{5,3}\} \cap T \neq \emptyset$, then $T = R(5,5)$. Indeed, if, say $x_{3,5} \in T$, the *it*-property at $(x_{1,0}, x_{3,5})$ implies that $x_{5,0} \in T$, whence $x_{5,3} \in T$. Now the *it*-property at $(x_{0,1}, x_{5,3})$, forces $x_{0,5} \in T$. Therefore $T = R(5,5)$. It is easy to verify that $R(5,5)$ is indeed *it*-convex.

If $\{x_{0,2},\ x_{2,0},\ x_{3,5},\ x_{5,3}\} \cap T = \emptyset$, then $T \subset S$, the set S being shown in Figure 13 (b). But $(x_{1,0},\ x_{4,5})$ does not enjoy the *it*-property in S, contradicting the it-convexity of T.

Subcase 4.5. $m = n \geqslant 6$.

We prove that there is no it-convex set in $\mathcal{T}(n,n)$, if $n \geqslant 6$.

As before, $x_{1,0}, x_{0,1}, x_{1,1}, x_{n,n-1}, x_{n-1,n}, x_{n-1,n-1} \in T$. We observe that the it-property is not satisfied at $(x_{1,0}, x_{n-2,n})$ in $R(n,n)$, and at the other pairs symmetrical to it. This implies that $T \subset S_1$, where S_1 is shown in Figure 14 (a).



(a) (b)

FIGURE 14. Monotone sets in $\mathcal{T}(n,n)$ $(n \geqslant 6)$.

If $n \equiv 1 (\text{mod } 2)$, we can verify that $(x_{1,0}, x_{n-1,n})$ does not enjoy the *it*-property in S_1, contradicting the fact that T is *it*-convex. Hence $n \equiv 0 (\text{mod } 2)$. Now, $x_{n-1,n-2}$ or $x_{n-2,n-1}$ must be in T, and if one of them, say $x_{n-1,n-2}$, is in T, then the *it*-property at $(x_{0,0}, x_{n-1,n-2})$ implies that the other, $x_{n-2,n-1}$, is also in T. Since $(x_{0,1} x_{n-1,n-3})$ does not enjoy the *it*-property in S_1, $x_{n-1,n-3} \notin T$. Symmetrically, $x_{n-3,n-1} \notin T$. Thus, $T \subset S_2$, where S_2 is shown in Figure 14 (b). Unfortunately, $(x_{1,0}, x_{n-2,n-1})$ does not enjoy the *it*-property in S_2, contradicting the it-convexity of T. □

5. it-CONVEXITY OF OTHER DISCRETE POINT SETS

Are the vertex sets of all regular polygons *it*-convex?

Theorem 5.10. *The vertex set of a regular n-polygon is it-convex if and only if $n \not\equiv 2 \pmod 4$.*

Proof. Let R_n denote the vertex set of a regular n-polygon. We label the vertices in R_n as $x_0,\ x_1,\ \cdots,\ x_{n-1}$, in counterclockwise direction.

Suppose $n \equiv 0,\ 1,\ 3 \pmod 4$. For any $x_i,\ x_j \in R_n\ (i < j)$, if $j - i \neq \frac{n}{2}$, let $k = 2j - i \pmod n$. Then $\|x_i - x_j\| = \|x_k - x_j\|$. If $j - i = \frac{n}{2}$, let $k = i + \frac{n}{4}$. Then $\|x_i - x_k\| = \|x_k - x_j\|$. So, x_i and x_j enjoy the it-property, and R_n is it-convex.

If $n \equiv 2$ (mod4), it is clear that $W_{x_0 x_{n/2}} \cap R_n = \emptyset$, so R_n is not *it*-convex. □

Also the vertex sets of the five Platonic polyhedra behave differently. While the vertex sets of the regular tetrahedron and regular octahedron are *it*-convex, those of the cube, regular dodecahedron and regular icosahedron are not.

Let $S = \{x_1,\ x_2,\ \cdots,\ x_n\} \subset \mathbb{R}^d$. A matrix $A(S) = [a_{ijk}]_{n\times n\times n}$ is called the *it-trimatrix of S* in case $a_{ijk} = 1$ if and only if $\{x_i,\ x_j,\ x_k\}$ form an isosceles triple, otherwise $a_{ijk} = 0$. Particularly, $a_{ijk} = 0$, when at least two of i, j, k are equal.

Thus, $S = \{x_1,\ x_2,\ \cdots,\ x_n\}$ is *it*-convex if and only if for any distinct $i,\ j \in \{1,\ 2,\ \cdots,\ n\}$, there is a k such that, in $A(S)$, $a_{ijk} \neq 0$.

Let $A(S) = [a_{ijk}]_{n\times n\times n}$ be the *it*-trimatrix of $S = \{x_1,\ x_2,\ \cdots,\ x_n\}$. The matrix $B(S) = [b_{ij}]_{n\times n}$ is called the *it-matrix of S*, if $b_{ij} = \sum\limits_{k=1}^{n} a_{ijk}$.

Obviously, $S = \{x_1,\ x_2,\ \cdots,\ x_n\}$ is it-convex if and only if it is non-zero outside the main diagonal.

An n-point it-convex set is called *poor* if the number of isosceles triples in it is minimal among all n-point it-convex sets. The number of isosceles triples in a poor n-point it-convex set is denoted by $N(n)$.

Theorem 5.11. $N(n) \geqslant \left\lceil \frac{n(n-1)}{6} \right\rceil$, *when n is odd;* $N(n) \geqslant \left\lceil \frac{n(n-1)}{6} \right\rceil + 1$, *when n is even.*

Proof. Let $S = \{x_1,\ x_2,\ \cdots,\ x_n\}$ be it-convex, with *it*-matrix $B(S) = [b_{ij}]_{n\times n}$. Thus, for $i \neq j$, $b_{ij} \geqslant 1$. Hence, $\sum\limits_{i,j} b_{ij} \geqslant n(n-1)$. For every isosceles triple $\{x_i, x_j, x_k\}$, we have $a_{ijk} = a_{ikj} = a_{jik} = a_{jki} = a_{kij} = a_{kji} = 1$. Thus, the contribution of each isosceles triple to b_{ij}, b_{ji}, b_{ik}, b_{ki}, b_{jk}, b_{kj} is 1, and therefore its contribution to $\sum\limits_{i,j} b_{ij}$ is 6. Hence, the number of isosceles triples in S is not less than $\left\lceil \frac{n(n-1)}{6} \right\rceil$, which means that $N(n) \geqslant \left\lceil \frac{n(n-1)}{6} \right\rceil$.

Now, let n be even, and put $m = \left\lceil \frac{n(n-1)}{6} \right\rceil$, $r = 6m - n(n-1)$. As $6m$ and $n(n-1)$ are even, $r = 0,\ 2,$ or 4. To show that $N(n) > m$, assume, on the contrary, $N(n) = m$.

Since for each isosceles triple $\{x_i, x_j, x_k\}$, its contribution to b_{ij}, b_{ji}, b_{ik}, b_{ki}, b_{jk}, b_{kj} is 1, the contribution to $\sum\limits_{l=1}^{n} b_{il}$, $\sum\limits_{l=1}^{n} b_{jl}$, $\sum\limits_{l=1}^{n} b_{kl}$ is 2. Hence, for every $i \in \{1,\ 2,\ \cdots,\ n\}$, $\sum\limits_{j=1}^{n} b_{ij}$ is even.

If $r = 0$, then for any distinct $i, j \in \{1,\ 2,\ \cdots,\ n\}$, $b_{ij} = 1$. So, for every i, $\sum_{j=1}^{n} b_{ij} = n - 1$ is even, and a contradiction is obtained.

If $r = 2$, then the number of pairs (i, j) satisfying $b_{ij} > 1$ is at most 2. By the symmetry of $B(S)$, this number of pairs is exactly 2. Suppose they are $(i_0, j_0), (j_0, i_0)$, so $b_{i_0 j_0} = b_{j_0 i_0} = 2$. As $n \geqslant 4$, there exists an $i \in \{1,\ 2,\ \cdots,\ n\} \setminus \{i_0,\ j_0\}$, such that $\sum\limits_{j=1}^{n} b_{ij} = n - 1$, leading again to a contradiction.

If $r = 4$, we have $n \geqslant 8$. Now, for any distinct $i, j \in \{1,\ 2,\ \cdots,\ n\}$, the number of pairs (i, j) that satisfy $b_{ij} > 1$ is 4. Let the 4 pairs be $(i_1, j_1),\ (j_1, i_1),\ (i_2, j_2),\ (j_2, i_2)$. As $n \geqslant 8$, for some i, $\sum_{j=1}^{n} b_{ij} = n - 1$, a contradiction.

Hence $N(n) \geqslant \left\lceil \frac{n(n-1)}{6} \right\rceil + 1$. □

We exhibit several poor *it*-convex sets to prove our last theorem.

Theorem 5.12. $N(3) = 1$, $N(4) = 3$, $N(5) = 4$, $N(6) = 6$.

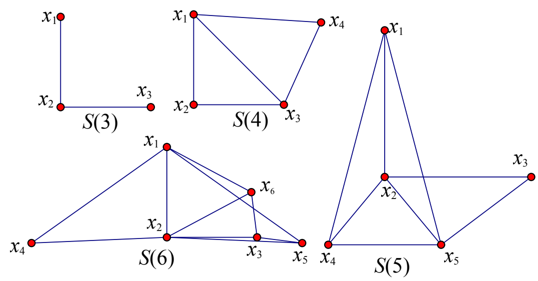


FIGURE 15. Examples of poor n-point *it*-convex sets for $3 \leq n \leq 6$.

Proof. $N(3) = 1$, because there is just one triple.

Let $S(4) = \{x_1,\ x_2,\ x_3,\ x_4\}$, where $x_1 = (0, 4)$, $x_2 = (0, 0)$, $x_3 = (4, 0)$, $x_4 = (\sqrt{7} + 3, \sqrt{7} + 1)$. In this case of an obviously *it*-convex 4-point set, all 4 triples are isosceles but one, $\{x_1, x_2, x_4\}$. By Theorem 5.11, $N(4) \geqslant 3$. Consequently, $N(4) = 3$, see Figure 15.

Let $S(5) = \{x_1,\ x_2,\ x_3,\ x_4,\ x_5\}$, where $x_1 = (0, 6.5)$, $x_2 = (0, 0)$, $x_3 = (6.5, 0)$, $x_4 = (-2.5, -3)$, $x_5 = (2.5, -3)$. The it-matrix of $S(5)$ is

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 3 \\ 1 & 1 & 1 & 3 & 0 \end{pmatrix},$$

which shows that $S(5)$ is an it-convex 5-point set with $24/6 = 4$ isosceles triples. Combining this with Theorem 5.11, we get $N(5) = 4$.

Let $S(6) = \{x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6\}$, where $x_1 = (0, 2)$, $x_2 = (0, 0)$, $x_3 = (2, 0)$, $x_4 = (-3, \sqrt{15} - 4)$, $x_5 = (3, \sqrt{15} - 4)$, $x_6 = (\sqrt{15} - 2, 1)$. Then $S(6)$ has the it-matrix

$$\begin{pmatrix} 0 & 2 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 2 \\ 1 & 1 & 1 & 0 & 2 & 1 \\ 1 & 1 & 1 & 2 & 0 & 1 \\ 1 & 1 & 2 & 1 & 1 & 0 \end{pmatrix}.$$

Clearly $S(6)$ is an it-convex set containing $36/6 = 6$ isosceles triples. Again by Theorem 5.11, we have $N(6) = 6$. □

Theorem 5.12 shows that the bounds in Theorem 5.11 are best possible for $n \leqslant 6$. This motivates us to end the paper with the following.

Open problem. Are the inequalities of Theorem 5.11 in fact equalities?

As pointed out by a referee, a planar version of this open problem would also be of interest.

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