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Bézier variant of genuine-Durrmeyer type operators based on Pólya distribution

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ABSTRACT. In this paper we introduce the Bézier variant of genuine-Durrmeyer type operators having Pólya basis functions. We give a global approximation theorem in terms of second order modulus of continuity, a direct approximation theorem by means of the Ditzian-Totik modulus of smoothness and a Voronovskaja type theorem by using the Ditzian-Totik modulus of smoothness. The rate of convergence for functions whose derivatives are of bounded variation is obtained. Further, we show the rate of convergence of these operators to certain functions by illustrative graphics using the Maple algorithms.

1. INTRODUCTION

In 1968, Stancu [23] introduced a sequence of positive linear operators $P_n^{(\alpha)} : C[0,1] \longrightarrow C[0,1]$, depending on a non negative parameter α as

$$P_n^{(\alpha)}(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}^{(\alpha)}(x), \tag{1.1}$$

where $p_{n,k}^{(\alpha)}(x)$ is the Pólya distribution with density function given by

$$p_{n,k}^{(\alpha)}(x) = \binom{n}{k} \frac{\prod_{v=0}^{k-1}(x+v\alpha)\prod_{\mu=0}^{n-k-1}(1-x+\mu\alpha)}{\prod_{\lambda=0}^{n-1}(1+\lambda\alpha)}, \ \ x \in [0,1].$$

Lupaş and Lupaş [18] considered a special case of the operators given by (1.1) for $\alpha = \dfrac{1}{n}$ which can be expressed as

$$P_n^{(\frac{1}{n})}(f;x) = \frac{2(n!)}{(2n)!} \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) (nx)_k (n-nx)_{n-k}, \tag{1.2}$$

where the rising factorial $(x)_n$ is given by $(x)_n = x(x+1)(x+2).....(x+n-1)$ with $(x)_0 = 1$. Gupta and Rassias [13] introduced the Durrmeyer type integral modification of the operators (1.2) and established local and global direct error estimates and a Voronovskaja type asymptotic formula. Recently in [4], to approximate Lebsegue integrable functions, a Kantorovich variant of the operators given by (1.1) was introduced and the properties of local and global approximation were investigated in univariate and bivariate cases. Recently, Gupta [12] defined a genuine-Durrmeyer type modification of the operators given by (1.2) and obtained a Voronovskaja type asymptotic theorem and a local

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Table 1.1 *Error of approximation for* U_n^ρ *and* $\overline{U}_n^\rho$

x	$\lvert U_n^\rho(f;x)-f(x)\rvert$	$\lvert\overline{U}_n^\rho(f;x)-f(x)\rvert$
0.60	0.001239490900	0.001509158700
0.61	0.000351701000	0.001971924300
0.62	0.0004378093000	0.002365328800
0.63	0.001135096600	0.002695352000
0.64	0.001746098000	0.002967653800
0.65	0.002276601100	0.003187565800
0.66	0.002732224700	0.003360091000
0.67	0.003118397000	0.003489901500
0.68	0.003440343200	0.003581349400

approximation theorem. Very recently in [20], for a function $f \in C[0,1]$, a genuine-Durrmeyer type integral modification of the operators given by (1.2) was introduced as follows:

$$U_n^\rho(f;x) = \sum_{k=0}^{n} F_{n,k}^\rho p_{n,k}^{(\frac{1}{n})}(x), \ \rho > 0, \tag{1.3}$$

where

$$F_{n,k}^\rho = \begin{cases} \int_0^1 f(t)\mu_{n,k}^\rho(t)dt, & 1 \le k \le n-1 \\ f(0), & k=0 \\ f(1), & k=n, \end{cases}$$

and

$$\mu_{n,k}^\rho(t) = \frac{t^{k\rho-1}(1-t)^{(n-k)\rho-1}}{B(k\rho,(n-k)\rho)},$$

$B(m,n)$ being the Euler's Beta-function. A Voronovskaja type asymptotic theorem, global and local approximation theorems have been proved in this paper.

Remark 1.1. Let us consider the class of operators $\overline{U}_n^\rho : C[0,1] \to \prod_n$ introduced in [21] by Păltănea as follows

$$\overline{U}_n^\rho(f;x) := \sum_{k=1}^{n-1}\left(\int_0^1 \frac{t^{k\rho-1}(1-t)^{(n-k)\rho-1}}{\beta(k\rho,(n-k)\rho)} f(t)dt\right)\overline{p}_{n,k}(x) + f(0)(1-x)^n + f(1)x^n,$$

where $\rho > 0$, $x \in [0,1]$ and $\overline{p}_{n,k}(x) = \binom{n}{k} x^k(1-x)^{n-k}$.

In the Table 1.1 for $\rho=1$, $n=20$ and the function $f:[0,1]\to\mathbb{R}$, $f(x)=\begin{cases} x^2\sin\frac{1}{x},\ x\neq 0, \\ 0,\ x=0, \end{cases}$ we computed the error of approximation for U_n^ρ and $\overline{U}_n^\rho$ at certain points from $[0.6, 0.7]$. From the above results it follows that the error of approximation for U_n^ρ is better than $\overline{U}_n^\rho$ to the function f at the points $x_i = 0.6 + 0.01\cdot i$, $i=\overline{0,8}$.

It is a well known fact that Bézier curves play an important role in computer aided designs and computer graphics systems. These curves were invented by Pierre Etienne Bézier, a French engineer at Renault. Zeng and Piriou [26] pioneered the study of Bézier variant of Bernstein operators. Subsequently, the research work on different positive linear Bézier operators motivated us to study further in this direction (cf. [4], [7], [11],

[24] and [25] etc.). We propose a Bézier variant of the operators given by (1.3) as

$$U_{n,\alpha}^{\rho}(f;x)=\sum_{k=0}^{n}F_{n,k}^{\rho}Q_{n,k}^{(\alpha)}(x), \tag{1.4}$$

where $Q_{n,k}^{(\alpha)}(x)=[J_{n,k}(x)]^{\alpha}-[J_{n,k+1}(x)]^{\alpha}$, $\alpha\geq 1$ with $J_{n,k}(x)=\sum_{j=k}^{n}p_{n,j}^{(1/n)}(x)$, when $k\leq n$ and 0 otherwise. Clearly, $U_{n,\alpha}^{\rho}$ is a sequence of linear positive operators. If $\alpha=1$, then the operators $U_{n,\alpha}^{\rho}$ reduce to the operators U_n^{ρ}.

The study of the rate of convergence for the functions with a derivative of bounded variation is an active area of research in approximation theory. Recently Ispir et al. [16] considered the Kantorovich modification of Lupaş operators based on Pólya distribution and estimated the rate of convergence for absolutely continuous functions having a derivative equivalent with a function of bounded variation. Very recently, the same problem has been investigated for the Bézier variant of summation integral type operators having Pólya and Bernstein basis functions and for the modified Srivastava-Gupta operators by Agrawal et al. [4] and Maheshwari [19] respectively. In this direction, significant contribution are due to (cf. [1], [2], [5], [6], [14], [15], and [17] etc.)

The aim of this paper is to study some approximation properties of the operators (1.4), to investigate a direct approximation result, a global approximation theorem, a quantitative Voronovskaja type theorem and the rate of convergence for functions f whose derivative f' are of bounded variation on $[0,1]$. Lastly, we show the rate of convergence of these operators by some graphics to certain functions.

Throughout this paper, C denotes a positive constant not necessarily the same at each occurrence.

2. Auxiliary results

In what follows let $\|.\|$ denotes uniform norm on $C[0,1]$.

Lemma 2.1. [20] *For $U_n^{\rho}(t^m;x)$, $m=0,1,2$, one has*

(i) $U_n^{\rho}(1;x)=1$,

(ii) $U_n^{\rho}(t;x)=x$,

(iii) $U_n^{\rho}(t^2;x)=\dfrac{n\rho}{n\rho+1}\left(x^2+\dfrac{2x(1-x)}{n+1}\right)+\dfrac{x}{n\rho+1}$.

Consequently, for every $x\in[0,1]$ it follows

$$U_n^{\rho}((t-x)^2;x)\leq\frac{2\rho+1}{n\rho+1}\phi^2(x)=\delta_{n,\rho}^2(x),$$

where $\phi^2(x)=x(1-x)$.

Lemma 2.2. *For every $f\in C[0,1]$, we have*

$$\|U_n^{\rho}f\|\leq\|f\|.$$

Applying Lemma 2.1 , the proof of this lemma easily follows. Hence the details are omitted.

Lemma 2.3. *a) Let $f\in C[0,1]$. Then we have $\|U_{n,\alpha}^{\rho}f\|\leq\|f\|$.*
b) Let $x\in[0,1]$ and $f\in C[0,1]$ such that $f\geq 0$ on $[0,1]$. Then we have $U_{n,\alpha}^{\rho}(f;x)\leq\alpha U_n^{\rho}(f;x)$.

Proof. a) By (1.4) and (1.3), we obtain

$$|U^{\rho}_{n,\alpha}(f;x)| \leq \sum_{k=0}^{n} |F^{\rho}_{n,k}| Q^{(\alpha)}_{n,k}(x) \leq \|f\| \sum_{k=0}^{n} Q^{(\alpha)}_{n,k}(x) = \|f\|,$$

because

$$U^{\rho}_{n,\alpha}(1;x) = \sum_{k=0}^{n} Q^{(\alpha)}_{n,k}(x) = \sum_{k=0}^{n} \{[J_{n,k}(x)]^{\alpha} - [J_{n,k+1}(x)]^{\alpha}\} = [J_{n,0}(x)]^{\alpha} = \left[\sum_{j=0}^{n} p^{(1/n)}_{n,j}(x)\right]^{\alpha} = 1.$$

b) Using the inequality $|a^{\alpha} - b^{\alpha}| \leq \alpha|a-b|$, where $0 \leq a, b \leq 1$ and $\alpha \geq 1$, from the definition of $Q^{(\alpha)}_{n,k}(x)$ we obtain

$$0 < [J_{n,k}(x)]^{\alpha} - [J_{n,k+1}(x)]^{\alpha} \leq \alpha(J_{n,k}(x) - J_{n,k+1}(x)) = \alpha p^{(1/n)}_{n,k}(x).$$

Hence, in view of the definition of $U^{\rho}_{n,\alpha}$ and the positivity of f, we get $U^{\rho}_{n,\alpha}(f;x) \leq \alpha U^{\rho}_{n}(f;x)$, which was to be proved. □

The operators $U^{\rho}_{n,\alpha}$ can be expressed in an integral form as follows:

$$U^{\rho}_{n,\alpha}(f;x) = \int_0^1 K^{\rho}_{n,\alpha}(x,t) f(t) dt, \tag{2.5}$$

where the kernel $K^{\rho}_{n,\alpha}$ is given by

$$K^{\rho}_{n,\alpha}(x,t) = \sum_{k=1}^{n-1} Q^{(\alpha)}_{n,k}(x) \mu^{\rho}_{n,k}(t) + Q^{(\alpha)}_{n,0}(x)\delta(t) + Q^{(\alpha)}_{n,n}(x)\delta(1-t),$$

$\delta(u)$ being the Dirac-delta function.

Lemma 2.4. *For a fixed $x \in (0,1)$ and sufficiently large n, we have*

(i) $\xi^{\rho}_{n,\alpha}(x,y) = \int_0^y K^{\rho}_{n,\alpha}(x,t)dt \leq \alpha \dfrac{\delta^2_{n,\rho}(x)}{(x-y)^2}$, $0 \leq y < x$,

(ii) $1 - \xi^{\rho}_{n,\alpha}(x,z) = \int_z^1 K^{\rho}_{n,\alpha}(x,t)dt \leq \alpha \dfrac{\delta^2_{n,\rho}(x)}{(z-x)^2}$, $x < z < 1$,

where $\delta_{n,\rho}(x)$ is defined in Lemma 2.1.

Proof. (i) Using Lemmas 2.1 and 2.3, we get

$$\begin{aligned} \xi^{\rho}_{n,\alpha}(x,y) &= \int_0^y K^{\rho}_{n,\alpha}(x,t)dt \leq \int_0^y \left(\frac{x-t}{x-y}\right)^2 K^{\rho}_{n,\alpha}(x,t)dt \\ &\leq \frac{U^{\rho}_{n,\alpha}((t-x)^2;x)}{(x-y)^2} \leq \alpha \frac{U^{\rho}_{n}((t-x)^2;x)}{(x-y)^2} \leq \alpha \frac{\delta^2_{n,\rho}(x)}{(x-y)^2}. \end{aligned}$$

The proof of (ii) is similar, therefore the details are omitted. □

3. Main results

First we establish a direct approximation theorem for the operators $U^{\rho}_{n,\alpha}$, using the second order modulus of smoothness and the classical modulus of continuity.
Let

$$W^2[0,1] = \left\{ g \in C[0,1] : g'' \in C[0,1] \right\}.$$

For $f \in C[0,1]$ and $\delta > 0$, the corresponding Peetre's K-functional [22] is defined by

$$K_2(f;\delta) = \inf_{g \in W^2[0,1]} \Big\{ \|f-g\| + \delta \|g''\| \Big\}.$$

From [8], there exists an absolute constant $C > 0$ such that

$$K_2(f;\delta) \le C\omega_2(f;\sqrt{\delta}), \tag{3.6}$$

where ω_2 is the second order modulus of smoothness of $f \in C[0,1]$, defined as

$$\omega_2(f;\delta) = \sup_{0<h\le\delta} \sup_{x,x+2h\in[0,1]} |f(x+2h) - 2f(x+h) + f(x)|.$$

The usual modulus of continuity of $f \in C[0,1]$ is given by

$$\omega(f;\delta) = \sup_{0<h\le\delta} \sup_{x,x+h\in[0,1]} |f(x+h) - f(x)|.$$

Theorem 3.1. *Let $f \in C[0,1]$ and $x \in [0,1]$. Then there exists an absolute constant $C > 0$ such that*

$$|U^{\rho}_{n,\alpha}(f;x) - f(x)| \le C\alpha\omega_2(f;\delta_{n,\rho}(x)) + \omega(f;\sqrt{\alpha}\delta_{n,\rho}(x)),$$

where $\delta_{n,\rho}(x)$ is defined in Lemma 2.1.

Proof. In view of (1.4) and (1.3), we have

$$\begin{aligned} 0 \le U^{\rho}_{n,\alpha}(t;x) &= \sum_{k=1}^{n-1} Q^{(\alpha)}_{n,k}(x) \int_0^1 t\mu^{\rho}_{n,k}(t)dt + Q^{(\alpha)}_{n,n}(x) \\ &= \sum_{k=1}^{n-1} Q^{(\alpha)}_{n,k}(x) \frac{B(k\rho+1,(n-k)\rho)}{B(k\rho,(n-k)\rho)} + Q^{(\alpha)}_{n,n}(x) \\ &= \sum_{k=1}^{n-1} \frac{k}{n} Q^{(\alpha)}_{n,k}(x) + Q^{(\alpha)}_{n,n}(x) \le \sum_{k=0}^{n} Q^{(\alpha)}_{n,k}(x) = 1 \end{aligned}$$

for all $x \in [0,1]$. This means that we can introduce the auxiliary operators $\overline{U}^{\rho}_{n,\alpha}$ defined by

$$\overline{U}^{\rho}_{n,\alpha}(f;x) = U^{\rho}_{n,\alpha}(f;x) - f(U^{\rho}_{n,\alpha}(t;x)) + f(x),\ x \in [0,1]. \tag{3.7}$$

Then the operators $\overline{U}^{\rho}_{n,\alpha}$ are linear and preserve the linear functions:

$$\overline{U}^{\rho}_{n,\alpha}(at+b;x) = ax+b.$$

Let $g \in W^2[0,1]$ and $t \in [0,1]$. Then, by Taylor's expansion, we have

$$g(t) - g(x) = (t-x)g'(x) + \int_x^t (t-u)g''(u)du.$$

Now applying $\overline{U}^{\rho}_{n,\alpha}$ to both sides of the previous equation, we get, by (3.7), that

$$\begin{aligned} \overline{U}^{\rho}_{n,\alpha}(g;x) - g(x) &= \overline{U}^{\rho}_{n,\alpha}\left(\int_x^t (t-u)g''(u)du;x\right) \\ &= U^{\rho}_{n,\alpha}\left(\int_x^t (t-u)g''(u)du;x\right) - \int_x^{U^{\rho}_{n,\alpha}(t;x)} \left(U^{\rho}_{n,\alpha}(t;x) - u\right) g''(u)du. \end{aligned}$$

Hence

$$|\overline{U}^{\rho}_{n,\alpha}(g;x)-g(x)|\leq \|g''\|U^{\rho}_{n,\alpha}\left((t-x)^2;x\right)+\|g''\|\left(U^{\rho}_{n,\alpha}(t;x)-x\right)^2$$
$$=\|g''\|\left\{U^{\rho}_{n,\alpha}\left((t-x)^2;x\right)+\left(U^{\rho}_{n,\alpha}(t-x;x)\right)^2\right\}.$$

Using Cauchy-Schwarz inequality and Lemmas 2.1 and 2.3, we obtain

$$|\overline{U}^{\rho}_{n,\alpha}(g;x)-g(x)|\leq \|g''\|\left\{\alpha\delta^2_{n,\rho}(x)+U^{\rho}_{n,\alpha}\left((t-x)^2;x\right)\right\}\leq 2\alpha\delta^2_{n,\rho}(x)\|g''\|. \tag{3.8}$$

On the other hand, by (3.7) and Lemma 2.3, we have for each $f\in C[0,1]$ and $x\in[0,1]$ that

$$|\overline{U}^{\rho}_{n,\alpha}(f;x)|\leq |U^{\rho}_{n,\alpha}(f;x)|+|f(U^{\rho}_{n,\alpha}(t;x))|+|f(x)|\leq 3\|f\|.$$

Hence

$$\|\overline{U}^{\rho}_{n,\alpha}f\|\leq 3\|f\|. \tag{3.9}$$

Now, combining (3.7)-(3.9), we obtain

$$\begin{aligned}|U^{\rho}_{n,\alpha}(f;x)-f(x)| &\leq |\overline{U}^{\rho}_{n,\alpha}(f;x)-f(x)|+|f(U^{\rho}_{n,\alpha}(t;x))-f(x)|\\ &\leq |\overline{U}^{\rho}_{n,\alpha}(f-g;x)-(f-g)(x)|+|\overline{U}^{\rho}_{n,\alpha}(g;x)-g(x)|+\omega(f;|U^{\rho}_{n,\alpha}(t;x)-x|)\\ &\leq 4\|f-g\|+2\alpha\delta^2_{n,\rho}(x)\|g''\|+\omega(f;U^{\rho}_{n,\alpha}(|t-x|;x))\\ &\leq 4\|f-g\|+2\alpha\delta^2_{n,\rho}(x)\|g''\|+\omega(f;(U^{\rho}_{n,\alpha}((t-x)^2;x))^{1/2})\\ &\leq 4\alpha\left\{\|f-g\|+\delta^2_{n,\rho}(x)\|g''\|\right\}+\omega(f;\sqrt{\alpha}\delta_{n,\rho}(x)).\end{aligned}$$

Taking the infimum on the right-hand side over all $g\in W^2[0,1]$ and using (3.6), we arrive at the assertion of the theorem. □

To describe our next result, we recall the definitions of the Ditzian-Totik first order modulus of smoothness and the K-functional [9]. Let $\phi(x)=\sqrt{x(1-x)}$ and $f\in C[0,1]$. The first order modulus of smoothness is given by

$$\omega_\phi(f;t)=\sup_{0<h\leq t}\left\{\left|f\left(x+\frac{h\phi(x)}{2}\right)-f\left(x-\frac{h\phi(x)}{2}\right)\right|, x\pm\frac{h\phi(x)}{2}\in[0,1]\right\}. \tag{3.10}$$

Further, the corresponding K-functional to (3.10) is defined by

$$K_\phi(f;t)=\inf_{g\in W_\phi[0,1]}\{||f-g||+t||\phi g'||\}\ (t>0),$$

where $W_\phi[0,1]=\{g: g\in AC_{loc}[0,1],\|\phi g'\|<\infty\}$ and $g\in AC_{loc}[0,1]$ means that g is absolutely continuous on every interval $[a,b]\subset[0,1]$. It is well known ([9], p.11) that there exists a constant $C>0$ such that

$$K_\phi(f;t)\leq C\omega_\phi(f;t). \tag{3.11}$$

Now, we establish a global approximation theorem by means of Ditzian-Totik modulus of smoothness.

Theorem 3.2. *Let f be in $C[0,1]$ and $\phi(x)=\sqrt{x(1-x)}$, then for every $x\in[0,1]$, we have*

$$|U^{\rho}_{n,\alpha}(f;x)-f(x)|\leq C\omega_\phi\left(f;\sqrt{\frac{2\rho+1}{n\rho+1}}\right),$$

where C is a constant independent of n and x.

Proof. Using the representation

$$g(t) = g(x) + \int_x^t g'(u)du,$$

we get

$$\left|U^{\rho}_{n,\alpha}(g;x) - g(x)\right| \quad = \quad \left|U^{\rho}_{n,\alpha}\bigg(\int_x^t g'(u)du;x\bigg)\right|. \tag{3.12}$$

For any $x \in (0,1)$ and $t \in [0,1]$ we find that

$$\bigg|\int_x^t g'(u)du\bigg| \leq ||\phi g'||\bigg|\int_x^t \frac{1}{\phi(u)}du\bigg|. \tag{3.13}$$

But,

$$\begin{aligned}\bigg|\int_x^t \frac{1}{\phi(u)}du\bigg| &= \bigg|\int_x^t \frac{1}{\sqrt{u(1-u)}}du\bigg| \leq \bigg|\int_x^t \Big(\frac{1}{\sqrt{u}} + \frac{1}{\sqrt{1-u}}\Big)du\bigg| \\ &\leq 2\bigg(\ |\ \sqrt{t} - \sqrt{x}\ | + |\ \sqrt{1-t} - \sqrt{1-x}\ |\ \bigg) \\ &= 2|t-x|\bigg(\frac{1}{\sqrt{t}+\sqrt{x}} + \frac{1}{\sqrt{1-t}+\sqrt{1-x}}\bigg) \\ &< 2|t-x|\bigg(\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{1-x}}\bigg) \leq \frac{2\sqrt{2}\ |t-x|}{\phi(x)}.\end{aligned} \tag{3.14}$$

Combining (3.12)-(3.14) and using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}|U^{\rho}_{n,\alpha}(g;x) - g(x)| &< 2\sqrt{2}||\phi g'||\phi^{-1}(x)U^{\rho}_{n,\alpha}(|t-x|;x) \\ &\leq 2\sqrt{2}||\phi g'||\phi^{-1}(x)\bigg(U^{\rho}_{n,\alpha}((t-x)^2;x)\bigg)^{1/2} \\ &\leq 2\sqrt{2}||\phi g'||\phi^{-1}(x)\bigg(\alpha\ U^{\rho}_{n}((t-x)^2;x)\bigg)^{1/2}.\end{aligned}$$

Now using Lemma 2.1, we get

$$|U^{\rho}_{n,\alpha}(g;x) - g(x)| \leq 2\sqrt{2\alpha}\sqrt{\frac{2\rho+1}{n\rho+1}}\|\phi g'\|. \tag{3.15}$$

Using Lemma 2.3 and (3.15) we can write

$$\begin{aligned}|\ U^{\rho}_{n,\alpha}(f;x) - f(x)\ | &\leq |\ U^{\rho}_{n,\alpha}(f-g;x)\ | + |f(x)-g(x)| + |\ U^{\rho}_{n,\alpha}(g;x) - g(x)\ | \\ &\leq 2\sqrt{2\alpha}\bigg(||f-g|| + \sqrt{\frac{2\rho+1}{n\rho+1}}||\phi g'||\bigg).\end{aligned}$$

Taking infimum on the right hand side of the above inequality over all $g \in W_\phi[0,1]$, we get

$$|U^{\rho}_{n,\alpha}(f;x) - f(x)| \leq 2\sqrt{2\alpha}K_\phi\bigg(f;\sqrt{\frac{2\rho+1}{n\rho+1}}\bigg).$$

Using the relation (3.11) this theorem is proven. □

In the following we prove a quantitative Voronovskaja type theorem for the operator $U^{\rho}_{n,\alpha}$. This result is given using the first order Ditzian-Totik modulus of smoothness.

Theorem 3.3. *For any $f \in C^2[0,1]$ the following inequalities hold*

i) $\left|n\mathcal{E}\left(U^{\rho}_{n,\alpha};f;x\right)\right| \le C\omega_{\phi}\left(f'',\phi(x)n^{-1/2}\right),$

ii) $\left|n\mathcal{E}\left(U^{\rho}_{n,\alpha};f;x\right)\right| \le C\phi(x)\omega_{\phi}\left(f'',n^{-1/2}\right),$

where

$$\mathcal{E}\left(U^{\rho}_{n,\alpha};f;x\right) := U^{\rho}_{n,\alpha}(f;x)-f(x)-f'(x)U^{\rho}_{n,\alpha}(t-x;x)-\frac{1}{2}f''(x)U^{\rho}_{n,\alpha}\left((t-x)^2;x\right).$$

Proof. Let $f \in C^2[0,1]$ be given and $t, x \in [0,1]$. Then by Taylor's expansion, we have

$$f(t)-f(x) = (t-x)f'(x) + \int_x^t (t-u)f''(u)du.$$

Hence

$$\begin{aligned} f(t)-f(x)-(t-x)f'(x)-\frac{1}{2}(t-x)^2f''(x) &= \int_x^t (t-u)f''(u)du - \int_x^t (t-u)f''(x)du \\ &= \int_x^t (t-u)[f''(u)-f''(x)]du. \end{aligned}$$

Applying $U^{\rho}_{n,\alpha}(\cdot;x)$ to both sides of the above relation, we get

$$\left|\mathcal{E}\left(U^{\rho}_{n,\alpha};f;x\right)\right| \le U^{\rho}_{n,\alpha}\left(\left|\int_x^t |t-u||f''(u)-f''(x)|du\right|;x\right). \tag{3.16}$$

The quantity $\left|\int_x^t |f''(u)-f''(x)|\,|t-u|du\right|$ was estimated in [10, p. 337] as follows:

$$\left|\int_x^t |f''(u)-f''(x)||t-u|du\right| \le 2\|f''-g\|(t-x)^2 + 2\|\phi g'\|\phi^{-1}(x)|t-x|^3, \tag{3.17}$$

where $g \in W_{\phi}[0,1]$.

We have

$$U^{\rho}_{n}\left((t-x)^4;x\right) = \phi^2(x)\frac{A(\rho,n)\phi^2(x)+B(\rho,n)}{(n+1)(n+2)(n+3)(1+\rho n)(2+\rho n)(3+\rho n)},$$

where

$$\begin{aligned} A(\rho,n) &= -3\left[-\rho(2\rho+1)^2n^4 + 2(14\rho^3+14\rho^2+9\rho+3)n^3 + (120\rho^2+109\rho+36)n^2 \right. \\ &\quad \left. +6(23\rho+11)n+36\right], \\ B(\rho,n) &= 2\left[(13\rho^3+18\rho^2+11\rho+3)n^3 + (-\rho^3+54\rho^2+55\rho+18)n^2 + 33(2\rho+1)n + 18\right]. \end{aligned}$$

Therefore, there exists a constant $C > 0$ such that

$$U^{\rho}_{n}\left((t-x)^4;x\right) \le \frac{C}{n^2}\phi^2(x). \tag{3.18}$$

Now combining (3.16)-(3.18) and applying Lemmas 2.3 and 2.1, the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
&\left|\mathcal{E}\left(U_{n,\alpha}^{\rho};f;x\right)\right| \leq 2\|f''-g\|U_{n,\alpha}^{\rho}\left((t-x)^2;x\right)+2\|\phi g'\|\phi^{-1}(x)U_{n,\alpha}^{\rho}\left(|t-x|^3;x\right)\\
&\leq 2\|f''-g\|\alpha\delta_{n,\rho}^2(x)+2\alpha\|\phi g'\|\phi^{-1}(x)\left\{U_n^{\rho}(t-x)^2;x\right\}^{1/2}\left\{U_n^{\rho}\left((t-x)^4;x\right)\right\}^{1/2}\\
&\leq 2\|f''-g\|\alpha\delta_{n,\rho}^2(x)+2\alpha\frac{C}{n}\|\phi g'\|\delta_{n,\rho}(x)\leq C\left\{\delta_{n,\rho}^2(x)\|f''-g\|+\frac{1}{n}\delta_{n,\rho}(x)\|\phi g'\|\right\}\\
&= C\left\{\frac{2\rho+1}{n\rho+1}\phi^2(x)\|f''-g\|+\frac{1}{n}\sqrt{\frac{2\rho+1}{n\rho+1}}\phi(x)\|\phi g'\|\right\}\\
&\leq \frac{C}{n}\left\{\phi^2(x)\|f''-g\|+n^{-1/2}\phi(x)\|\phi g'\|\right\}.
\end{aligned}$$

Since $\phi^2(x)\leq\phi(x)\leq 1, x\in[0,1]$, we obtain

$$\left|\mathcal{E}\left(U_{n,\alpha}^{\rho};f;x\right)\right|\leq\frac{C}{n}\left\{\|f''-g\|+n^{-1/2}\phi(x)\|\phi g'\|\right\}.$$

Also, the following inequality can be obtained

$$\left|\mathcal{E}\left(U_{n,\alpha}^{\rho};f;x\right)\right|\leq\frac{C}{n}\phi(x)\left\{\|f''-g\|+n^{-1/2}\|\phi g'\|\right\}.$$

Taking the infimum on the right hand side of the above relations over $g\in W_\phi[0,1]$, we get

$$\left|n\mathcal{E}\left(U_{n,\alpha}^{\rho};f;x\right)\right|\leq\begin{cases} CK_\phi\left(f'';\phi(x)n^{-1/2}\right),\\ C\phi(x)K_\phi\left(f'';n^{-1/2}\right).\end{cases}$$

Using (3.11) the theorem is proved. □

Lastly we discuss the approximation of functions with a derivative of bounded variation on $[0,1]$. Let $DBV[0,1]$ denote the class of differentiable functions f defined on $[0,1]$, whose derivatives are of bounded variation on $[0,1]$. The functions $f\in DBV[0,1]$ has the following representation

$$f(x)=\int_0^x g(t)dt+f(0),$$

where $g\in BV[0,1]$, i.e., g is a function of bounded variation on $[0,1]$.

Theorem 3.4. *Let $f\in DBV[0,1]$. Then, for every $x\in(0,1)$ and sufficiently large n, we have*

$$\begin{aligned}
|U_{n,\alpha}^{\rho}(f;x)-f(x)| &\leq \{|f'(x+)+\alpha f'(x-)|+\alpha|f'(x+)-f'(x-)|\}\frac{\sqrt{\alpha}}{\alpha+1}\delta_{n,\rho}(x)\\
&+\frac{\alpha\delta_{n,\rho}^2(x)}{x}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{x-x/k}^{x}f'_x\right)+\frac{x}{\sqrt{n}}\left(\bigvee_{x-x/\sqrt{n}}^{x}f'_x\right)\\
&+\frac{\alpha\delta_{n,\rho}^2(x)}{1-x}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{x}^{x+(1-x)/k}f'_x\right)+\frac{1-x}{\sqrt{n}}\left(\bigvee_{x}^{x+(1-x)/\sqrt{n}}f'_x\right),
\end{aligned}$$

where $\bigvee_a^b f$ denotes the total variation of f on $[a,b]$ and f'_x is defined by

$$f'_x(t)=\begin{cases} f'(t)-f'(x-), & 0\leq t<x\\ 0, & t=x\\ f'(t)-f'(x+) & x<t\leq 1.\end{cases} \tag{3.19}$$

Proof. Since $U^{\rho}_{n,\alpha}(1;x)=1$, using (2.5), for every $x\in(0,1)$ we get

$$U^{\rho}_{n,\alpha}(f;x)-f(x)=\int_0^1 K^{\rho}_{n,\alpha}(x,t)(f(t)-f(x))dt=\int_0^1 K^{\rho}_{n,\alpha}(x,t)\int_x^t f'(u)du\,dt. \tag{3.20}$$

For any $f\in DBV[0,1]$, from (3.19) we may write

$$\begin{aligned} f'(u)&=f'_x(u)+\frac{1}{\alpha+1}(f'(x+)+\alpha f'(x-))\\ &+\frac{1}{2}(f'(x+)-f'(x-))\left(sgn(u-x)+\frac{\alpha-1}{\alpha+1}\right)+\delta_x(u)[f'(u)-\frac{1}{2}(f'(x+)+f'(x-))], \end{aligned} \tag{3.21}$$

where

$$\delta_x(u)=\begin{cases}1\ ,\ u=x\\ 0\ ,\ u\neq x\end{cases}.$$

Obviously,

$$\int_0^1\left(\int_x^t\left(f'(u)-\frac{1}{2}(f'(x+)+f'(x-))\right)\delta_x(u)du\right)K^{\rho}_{n,\alpha}(x,t)dt=0. \tag{3.22}$$

Using (2.5), we get

$$\begin{aligned} A_1&=\int_0^1\left(\int_x^t\frac{1}{\alpha+1}(f'(x+)+\alpha f'(x-))du\right)K^{\rho}_{n,\alpha}(x,t)dt\\ &=\frac{1}{\alpha+1}(f'(x+)+\alpha f'(x-))\int_0^1(t-x)K^{\rho}_{n,\alpha}(x,t)dt\\ &=\frac{1}{\alpha+1}(f'(x+)+\alpha f'(x-))U^{\rho}_{n,\alpha}((t-x);x) \end{aligned} \tag{3.23}$$

and

$$\begin{aligned} A_2&=\int_0^1 K^{\rho}_{n,\alpha}(x,t)\left(\int_x^t\frac{1}{2}(f'(x+)-f'(x-))\left(sgn(u-x)+\frac{\alpha-1}{\alpha+1}\right)du\right)dt\\ &=\frac{1}{2}\left(f'(x+)-f'(x-)\right)\Bigg[-\int_0^x\left(\int_t^x\left(sgn(u-x)+\frac{\alpha-1}{\alpha+1}\right)du\right)K^{\rho}_{n,\alpha}(x,t)dt\\ &+\int_x^1\left(\int_x^t\left(sgn(u-x)+\frac{\alpha-1}{\alpha+1}\right)du\right)K^{\rho}_{n,\alpha}(x,t)dt\Bigg]\\ &=\frac{\alpha}{\alpha+1}\left(f'(x+)-f'(x-)\right)\int_0^1|t-x|\,K^{\rho}_{n,\alpha}(x,t)dt\\ &=\frac{\alpha}{\alpha+1}\left(f'(x+)-f'(x-)\right)U^{\rho}_{n,\alpha}\left(|t-x|\,;x\right). \end{aligned} \tag{3.24}$$

Using Lemma 2.3, the equalities (3.20-3.24) and applying Cauchy-Schwarz inequality, we obtain

(3.25)

$$\begin{aligned} |U^{\rho}_{n,\alpha}(f;x)-f(x)|&\le\frac{1}{\alpha+1}|f'(x+)+\alpha f'(x-)|\sqrt{\alpha}\delta_{n,\rho}(x)+\frac{\alpha}{\alpha+1}|f'(x+)-f'(x-)|\sqrt{\alpha}\delta_{n,\rho}(x)\\ &+\left|\int_0^x\left(\int_x^t f'_x(u)du\right)K^{\rho}_{n,\alpha}(x,t)dt\right|+\left|\int_x^1\left(\int_x^t f'_x(u)du\right)K^{\rho}_{n,\alpha}(x,t)dt\right|. \end{aligned}$$

Now, let

$$A^{\rho}_{n,\alpha}(f'_x,x)=\int_0^x\left(\int_x^t f'_x(u)du\right)K^{\rho}_{n,\alpha}(x,t)dt,$$

and

$$B^{\rho}_{n,\alpha}(f'_x,x)=\int_x^1\left(\int_x^t f'_x(u)du\right)K^{\rho}_{n,\alpha}(x,t)dt.$$

Thus our problem is reduced to calculate the estimates of the terms $A^{\rho}_{n,\alpha}(f'_x,x)$ and $B^{\rho}_{n,\alpha}(f'_x,x)$. From the definition of $\xi^{\rho}_{n,\alpha}$ given in Lemma 2.4, we can write

$$A^{\rho}_{n,\alpha}(f'_x,x)=\int_0^x\left(\int_x^t f'_x(u)\right)\frac{\partial}{\partial t}\xi^{\rho}_{n,\alpha}(x,t)dt.$$

Applying the integration by parts, we get

$$\begin{aligned}\left|A^{\rho}_{n,\alpha}(f'_x,x)\right|&\le\int_0^x|f'_x(t)|\xi^{\rho}_{n,\alpha}(x,t)dt\\&\le\int_0^{x-\frac{x}{\sqrt{n}}}|f'_x(t)|\xi^{\rho}_{n,\alpha}(x,t)dt+\int_{x-\frac{x}{\sqrt{n}}}^x|f'_x(t)|\xi^{\rho}_{n,\alpha}(x,t)dt:=I_1+I_2.\end{aligned}$$

Since $f'_x(x)=0$ and $\xi^{\rho}_{n,\alpha}(x,t)\le 1$, we have

$$\begin{aligned}I_2&:=\int_{x-\frac{x}{\sqrt{n}}}^x|f'_x(t)-f'_x(x)|\,\xi^{\rho}_{n,\alpha}(x,t)dt\le\int_{x-\frac{x}{\sqrt{n}}}^x\left(\bigvee_t^x f'_x\right)dt\\&\le\left(\bigvee_{x-\frac{x}{\sqrt{n}}}^x f'_x\right)\int_{x-\frac{x}{\sqrt{n}}}^x dt=\frac{x}{\sqrt{n}}\left(\bigvee_{x-\frac{x}{\sqrt{n}}}^x f'_x\right).\end{aligned}$$

By applying Lemma 2.4 and considering $t=x-\dfrac{x}{u}$, we get

$$I_1\le\alpha\delta^2_{n,\rho}(x)\int_0^{x-\frac{x}{\sqrt{n}}}|f'_x(t)-f'_x(x)|\frac{dt}{(x-t)^2}\le\alpha\delta^2_{n,\rho}(x)\int_0^{x-\frac{x}{\sqrt{n}}}\left(\bigvee_t^x f'_x\right)\frac{dt}{(x-t)^2}$$

$$=\frac{\alpha\delta^2_{n,\rho}(x)}{x}\int_1^{\sqrt{n}}\left(\bigvee_{x-\frac{x}{u}}^x f'_x\right)du\le\frac{\alpha\delta^2_{n,\rho}(x)}{x}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{x-\frac{x}{k}}^x f'_x\right).$$

Therefore,

$$|A^{\rho}_{n,\alpha}(f'_x,x)|\le\frac{\alpha\delta^2_{n,\rho}(x)}{x}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{x-\frac{x}{k}}^x f'_x\right)+\frac{x}{\sqrt{n}}\left(\bigvee_{x-\frac{x}{\sqrt{n}}}^x f'_x\right).\tag{3.26}$$

Also, using integration by parts in $B_n^\rho(f'_x, x)$ and applying Lemma 2.4 with $z = x + (1-x)/\sqrt{n}$, we have

$$
\begin{aligned}
|B^\rho_{n,\alpha}(f'_x,x)| &= \left| \int_x^1 \left(\int_x^t f'_x(u)du \right) K^\rho_{n,\alpha}(x,t)dt \right| \\
&= \left| \int_x^z \left(\int_x^t f'_x(u)du \right) \frac{\partial}{\partial t}(1-\xi^\rho_{n,\alpha}(x,t))dt + \int_z^1 \left(\int_x^t f'_x(u)du \right) \frac{\partial}{\partial t}(1-\xi^\rho_{n,\alpha}(x,t))dt \right| \\
&= \left| \left[\int_x^t (f'_x(u)du)(1-\xi^\rho_{n,\alpha}(x,t)) \right]_x^z - \int_x^z f'_x(t)(1-\xi^\rho_{n,\alpha}(x,t))dt \right. \\
&\quad \left. + \int_z^1 \int_x^t (f'_x(u)du) \frac{\partial}{\partial t}(1-\xi^\rho_{n,\alpha}(x,t))dt \right| \\
&= \left| \int_x^z (f'_x(u)du)(1-\xi^\rho_{n,\alpha}(x,z)) - \int_x^z f'_x(t)(1-\xi^\rho_{n,\alpha}(x,t))dt \right. \\
&\quad \left. + \left[\int_x^t (f'_x(u)du)(1-\xi^\rho_{n,\alpha}(x,t)) \right]_z^1 - \int_z^1 f'_x(t)(1-\xi^\rho_{n,\alpha}(x,t))dt \right| \\
&= \left| \int_x^z f'_x(t)(1-\xi^\rho_{n,\alpha}(x,t))dt + \int_z^1 f'_x(t)(1-\xi^\rho_{n,\alpha}(x,t))dt \right| \\
&\leq \alpha\delta^2_{n,\rho}(x) \int_z^1 \left(\bigvee_x^t f'_x \right)(t-x)^{-2}dt + \int_x^z \bigvee_x^t f'_x dt \\
&\leq \alpha\delta^2_{n,\rho}(x) \int_{x+(1-x)/\sqrt{n}}^1 \left(\bigvee_x^t f'_x \right)(t-x)^{-2}dt + \frac{1-x}{\sqrt{n}} \left(\bigvee_x^{x+(1-x)/\sqrt{n}} f'_x \right).
\end{aligned}
$$

By substituting $u = (1-x)/(t-x)$, we get

$$
\begin{aligned}
|B^\rho_{n,\alpha}(f'_x,x)| &\leq \alpha\delta^2_{n,\rho}(x) \int_1^{\sqrt{n}} \left(\bigvee_x^{x+(1-x)/u} f'_x \right)(1-x)^{-1}du + \frac{1-x}{\sqrt{n}} \left(\bigvee_x^{x+(1-x)/\sqrt{n}} f'_x \right) \\
&\leq \frac{\alpha\delta^2_{n,\rho}(x)}{1-x} \sum_{k=1}^{[\sqrt{n}]} \left(\bigvee_x^{x+(1-x)/k} f'_x \right) + \frac{1-x}{\sqrt{n}} \left(\bigvee_x^{x+(1-x)/\sqrt{n}} f'_x \right). \qquad (3.27)
\end{aligned}
$$

Collecting the estimates (3.25 - 3.27), we get the required result. This completes the proof. □

Example 3.1. Let us consider the following two functions

$$
f:[0,1]\to\mathbb{R},\, f(x)=\begin{cases} x^2\sin\frac{1}{x}, x\neq 0 \\ 0,\ x=0 \end{cases} \quad \text{and } g:[0,1]\to\mathbb{R},\ g(x)=\begin{cases} (1-x)\cos\frac{\pi}{1-x},\ x\neq 1 \\ 0,\ x=1 \end{cases}
$$

The function f is differentiable and of bounded variation on $[0,1]$, while g is continuous but is not of bounded variation on $[0,1]$.

For $n = 20$, $\rho = 1$ and $\alpha \in \left\{1, \frac{3}{2}, 2\right\}$, the convergence of $U^\rho_{n,\alpha}f$ to f and $U^\rho_{n,\alpha}g$ to g is illustrated in Figure 1 and Figure 2, respectively.

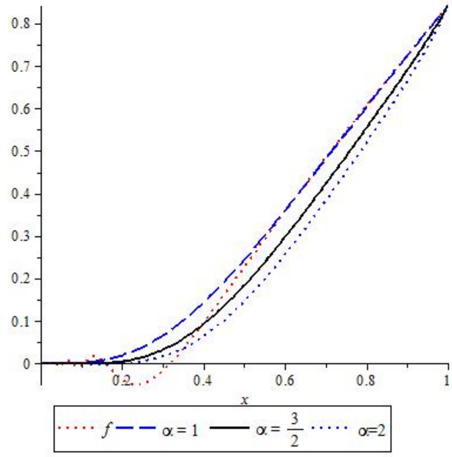

Figure 1: The convergence of $U^{\rho}_{n,\alpha}(f;x)$ to $f(x)$

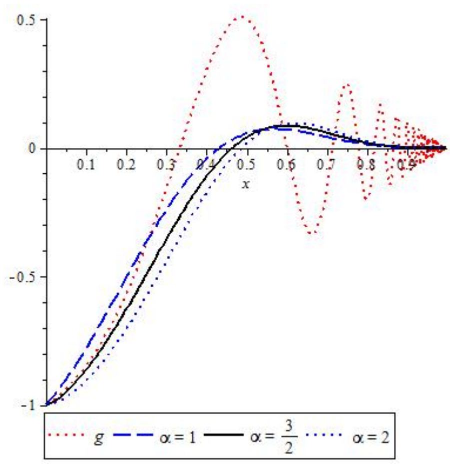

Figure 2: The convergence of $U^{\rho}_{n,\alpha}(g;x)$ to $g(x)$

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