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Hardy-Littlewood-Pólya theorem of majorization in the framework of generalized convexity

CONSTANTIN P. NICULESCU [1,2] and IONEL ROVENŢA [1]

ABSTRACT. Based on a new concept of generalized relative convexity, a large extension of Hardy-Littlewood-Pólya theorem of majorization is obtained. Several applications escaping the classical framework of convexity are included.

1. INTRODUCTION

The important role played by the classical inequality of Jensen in mathematics, probability theory, economics, statistical physics, information theory etc. is well described in many books, including those by Niculescu and Persson [13], Pečarić, Proschan and Tong [19] and Simon [21]. The aim of this paper is to discuss the phenomenon of existence of points of convexity within the framework of convexity with respect to a pair of means. This makes inequality of Jensen available in a framework that includes a large variety of generalized convex functions. See Section 3. Based on this fact we prove in Section 4 the corresponding generalization of the Hardy-Littlewood-Pólya theorem of majorization. Its usefulness is illustrated by a number of examples.

The possibility to extend the inequality of Jensen outside the framework of convex functions was first noticed twenty years ago by Dragomirescu and Ivan [5]. Later, Pearce and Pečarić [18] and Czinder and Páles [4] have considered the special case of mixed convexity, assuming the symmetry of the graph with respect to the inflection point. Their work was extended by the results obtained by Niculescu and his collaborators [6], [10], [14], [16] and [17], leading to the present paper.

2. GENERALITIES ON MEANS

In this paper, the concepts of generalized convexity are associated to means. A *mean* on an interval I is an averaging process M which associates to each discrete random variable $X : \Omega \to I$ having the distribution

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \end{pmatrix},$$

a number $M(X) = M(x_1, ..., x_n; \lambda_1, ..., \lambda_n)$ such that

$$\inf \{x_1, ..., x_n\} \leq M(X) \leq \sup \{x_1, ..., x_n\}.$$

Notice that $\lambda_1, ..., \lambda_n \in [0, 1]$ and $\sum_{k=1}^n \lambda_k = 1$.

We make the convention to denote $M(x_1, ..., x_n; \lambda_1, ..., \lambda_n)$ by $M(x_1, ..., x_n)$ when $\lambda_1 = \cdots = \lambda_n = 1/n$.

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Corresponding author: Ionel Rovenţa; ionelroventa@yahoo.com

The most usual means are the *quasi-arithmetic means* $\mathfrak{M}_\varphi$; $\mathfrak{M}_\varphi$ is associated to a strictly monotonic function φ defined on an interval I via the formula

$$M_\varphi(x_1, ..., x_n; \lambda_1, ..., \lambda_n) = \varphi^{-1}\left(\sum_{k=1}^{n} \lambda_k \varphi(x_k)\right),$$

The quasi-arithmetic means include the *power means*, which are defined by the formulas

$$M_p(x_1, ..., x_n; \lambda_1, ..., \lambda_n) = \begin{cases} \left(\sum_{k=1}^{n} \lambda_k x_k^p\right)^{1/p} & \text{if } p \in \mathbb{R}\backslash\{0\} \\ \prod_{k=1}^{n} x_k^{\lambda_k} & \text{if } p = 0; \end{cases}$$

the power mean M_p corresponds to the choice $\varphi(x) = x^p$, if $p \neq 0$, and to $\varphi(x) = \log x$, if $p = 0$. The most used in applications are the power means of order 1, 0 and -1, usually known as the *arithmetic mean*, the *geometric mean* and the *harmonic mean.* They are denoted respectively A, G and H.

Two examples of non quasi-arithmetic means are the *logarithmic mean*,

$$L(x_1, x_2) = \begin{cases} \frac{x_1 - x_2}{\ln x_1 - \ln x_2} & \text{if } x_1 \neq x_2 \\ x_1 & \text{if } x_1 = x_2 \end{cases}$$

and the *identric mean*,

$$I(x_1, x_2) = \begin{cases} \frac{1}{e}\left(\frac{x_2^{x_2}}{x_1^{x_1}}\right)^{\frac{1}{x_2 - x_1}} & \text{if } x_1 \neq x_2 \\ x_1 & \text{if } x_1 = x_2. \end{cases}$$

Their extension to the case of arbitrary discrete random variables is by far nontrivial and the interested reader may found details in [11], [12] and [20].

Certain means admit analogues in higher dimensional spaces. The most notable case is that of arithmetic mean. Less known is the fact that the power means make sense in the framework of Banach lattices. See [7], Theorem 1.d.1, p. 42. For example, in the case of the Banach lattice $\mathbb{R}^N$ (endowed with the coordinatewise order),

$$\begin{aligned} &M_p(\mathbf{x}_1, ..., \mathbf{x}_n; \lambda_1, ..., \lambda_n) \\ &\qquad = \left(M_p(x_{11}, ..., x_{n1}; \lambda_1, ..., \lambda_n), ..., M_p(x_{1N}, ..., x_{nN}; \lambda_1, ..., \lambda_n)\right). \end{aligned}$$

3. Convexity according to a pair of means

n what follows I and J are closed order intervals (in suitable Banach lattices) on which there are defined the means M and N respectively. According to Aumann [2], a function $f : I \to J$ is said to be (M, N)-*convex* if it verifies the following analogue of Jensen's inequality:

$$(M, N) \qquad f\left(M\left(x_1, \ldots, x_n; \lambda_1, \ldots, \lambda_n\right)\right) \leq N\left(f(x_1), \ldots, f(x_n); \lambda_1, \ldots, \lambda_n\right)$$

for all $x_1, \ldots, x_n \in I$ and $\lambda_1, ..., \lambda_n \in [0, 1]$ with $\sum_{k=1}^{n} \lambda_k = 1$.

If the inequality (M, N) works in the opposite way, the function f is called (M, N)-*concave.*

The concept of *mid-*(M, N)*-convexity* verifies only the weaker condition

$$f\left(M\left(x_1, x_2\right)\right) \leq N\left(f(x_1), f(x_2)\right);$$

its usefulness is explained by the fact that in many cases *mid-*(M, N)*-convexity plus continuity imply* (M, N)*-convexity.* For example, this happens when M and N are power means. See [13], Ch. 2.

A moment's reflection shows that a function $f : I \to J$ is (M_r, M_s)-convex for $r, s \neq 0$ if and only if the function F defined by $F(y) = f^{1/s}(y^r)$ is convex in the usual sense on $I_r = \{y > 0 : y^r \in I\}$; for $r = s = 0$, this correspondence works between f and the function F defined by $F(y) = \log f(e^y)$ on $\log I = \{y \in \mathbb{R} : \ e^y \in I\}$. The other cases can be also described via a change of variable and a change of function.

The (M_r, M_s)-convex functions cover a very large variety of functions playing an important role in mathematics. The (A, A)-convex functions are precisely the usual convex functions, while the (A, G)-convex functions are the same with log-convex functions. The (G, G)-convex functions (also called *multiplicatively convex functions*) are those functions $f : I \to J$ acting on subintervals of $(0, \infty)$ such that

$$f(x^{1-\lambda} y^{\lambda}) \leq f(x)^{1-\lambda} f(y)^{\lambda} \quad \text{for all } x, y \in I,\ \lambda \in [0, 1];$$

equivalently, the functions f such that $\log f(\exp)$ is convex. Many special functions such as the Gamma function, the integral sinus, the logarithmic integral, the hyperbolic sinus and the hyperbolic cosine are multiplicatively convex (on appropriate intervals). See [13], Sections 2.3 and 2.4.

Notice that the Gamma function is (M_2, M_2)-convex on $(0, \infty)$, while the sine function is (M_2, M_2)-concave on $[0, \sqrt{\pi}]$. Other interesting examples of (M_r, M_s)-convex functions are available in the paper of Anderson, Vamanamurthy and Vuorinen [1].

The concept of convexity at a point (first considered in our paper [15] in the case of usual convex functions) makes available suitable Jensen type inequalities even in certain nonconvex environments. Its analogue in the case of (M, N)-convexity is formulated in Definition 3.1 below.

As above, I and J are two order intervals on which there are defined the means M and N respectively and $f : I \to J$ is a function.

Definition 3.1. A point $a \in I$ is a point of (M, N)-convexity for the function f if

$$(J) \qquad f(a) \leq N(f(x_1), \ldots, f(x_n); \lambda_1, \ldots, \lambda_n),$$

for every family of points $x_1, ..., x_n \in I$ and every family of positive weights $\lambda_1, \ldots, \lambda_n$ such that $\sum_{k=1}^{n} \lambda_k = 1$ and $M(x_1, \ldots, x_n; \lambda_1, \ldots, \lambda_n) = a$.

The point a is a point of concavity if it is a point of convexity for $-f$ (equivalently, if the inequality (J) works in the reversed way).

We make the convention to name a point of (A, A)-convexity simply a *point of convexity*.

When M and N are power means acting on real intervals one can indicate very simple geometric conditions under which a point a is a point of (M, N)-convexity. The basic observation is as follows:

Lemma 3.1. *If f admits a support line at the point point a, then a is a point of convexity of f.*

In other words, every point a at which the subdifferential of f is nonempty is a point of convexity.

Proof. Indeed, the existence of a support line at a is equivalent to the existence of an affine function $h(x) = \alpha x + \beta$ such that

$$f(a) = h(a) \text{ and } f(x) \geq h(x) \text{ for all } x \in I.$$

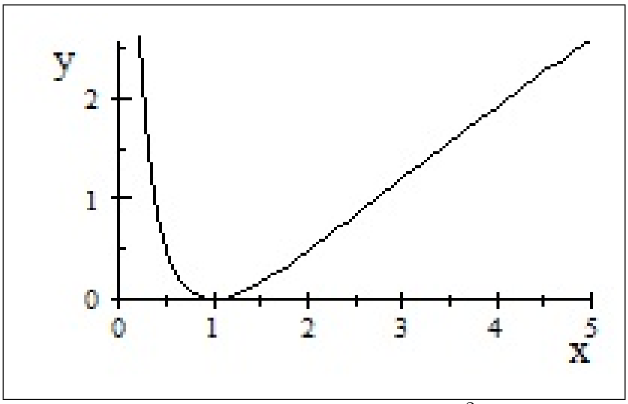
If $a = A(x_1, \ldots, x_n; \lambda_1, \ldots, \lambda_n) = \sum_{k=1}^{n} \lambda_k x_k$,

$$\begin{aligned} f(a) = h(a) = h\left(\sum_{k=1}^{n} \lambda_k x_k\right) &= \sum_{k=1}^{n} \lambda_k h(x_k) \leq \sum_{k=1}^{n} \lambda_k f(x_k) \\ &= A(f(x_1), \ldots, f(x_n); \lambda_1, \ldots, \lambda_n). \end{aligned}$$

□

Lemma 1 applies to a large variety of nonconvex functions such as xe^x, x^2e^{-x}, $\log^2 x$, $\frac{\log x}{x}$ etc.

For example, the function $\log^2 x$ is convex on $(0, e]$ and concave on $[e, \infty)$ (and attains a global minimum at $x = 1$). See Figure 1. All points $a \in (0, 1]$ are points of convexity for this function.



The graph of the function $\log^2 x$.

The other points of $[1, e]$ are point of convexity for $\log^2 x$ when restricted to appropriate intervals. For example, $a = 1.5$ is a point of convexity for $\log^2 x|_{(0,x^*]}$ (where $x^* = 14.\,256\,248...$ is the largest solution of the equation

$$\log^2 x = \log^2 1.5 + \frac{2\log 1.5}{1.5}(x - 1.5)),$$

while $a = e$ is a point of convexity for $\log^2 x|_{(0,e]}$.

The above discussion allows us to solve in an elementary way certain constrained non-convex optimization problems. For example,

$$\min_{\substack{x,y,z\in(0,14]\\ x+2y+3z=9}} \frac{\log^2 x + 2\log^2 y + 3\log^2 z}{6} = \log^2 1.5 = 0.164\,401\ldots\,.$$

This remark can be extended to the context of generalized convexity as follows:

Corollary 3.1. (i) *Suppose that $r, s \in \mathbb{R}\backslash\{0\}$. The point $a \in I$ is a point of (M_r, M_s)-convexity for the function $f : I \to J$ if the function $f^{1/s}(x^r)$ admits a support line at $a^{1/r}$.*

(ii) *Suppose that $r \in \mathbb{R}\backslash\{0\}$. The point $a \in I$ is a point of (M_r, G)-convexity for the function $f : I \to J$ if the function $\log f(x^r)$ admits a support line at $a^{1/r}$.*

(iii) *The point $a \in I$ is a point of (G, G)-convexity of a function $f : I \to J$ if the function $\log f(e^x)$ admits a support line at $\log a$.*

According to Corollary 3.1 (ii), applied for $r = 1$, the functions $x^{\log x} = \exp\left(\log^2 x\right)$ and $\log^2 x$ have the same points of convexity/concavity. Therefore

$$\min_{\substack{x,y,z\in(0,14]\\ x+2y+3z=9}} \frac{x^{\log x} + 2y^{\log y} + 3z^{\log z}}{6} = (1.5)^{\log^2 1.5} = 1.\,068\,931\ldots\,.$$

4. The extension of the Hardy-Littlewood-Pólya theorem of majorization

The notion of point of generalized convexity leads to a very large generalization of the Hardy-Littlewood-Pólya theorem of majorization. For convenience, we next recall this classical result. More details may be found in the monograph of Marshall, Olkin and Arnold [9].

Given a vector $\mathbf{x} = (x_1, ..., x_N)$ in $\mathbb{R}^N$, we denote by $\mathbf{x}^{\downarrow}$ the vector with the same entries as $\mathbf{x}$ but rearranged in decreasing order,

$$x_1^{\downarrow} \geq \cdots \geq x_N^{\downarrow}.$$

The vector $\mathbf{x}$ is said to be *majorized* by $\mathbf{y}$ (abbreviated, $\mathbf{x} \prec \mathbf{y}$) if

$$\sum_{i=1}^{k} x_i^{\downarrow} \leq \sum_{i=1}^{k} y_i^{\downarrow} \quad \text{for } k = 1, ..., N-1$$

and

$$\sum_{i=1}^{N} x_i^{\downarrow} = \sum_{i=1}^{N} y_i^{\downarrow}.$$

The concept of majorization admits an order-free characterization based on the notion of doubly stochastic matrix. Recall that a matrix $A \in \mathrm{M}_n(\mathbb{R})$ is *doubly stochastic* if it has nonnegative entries and each row and each column sums to unity.

Theorem 4.1. (Hardy, Littlewood and Pólya [9]). *Let $\mathbf{x}$ and $\mathbf{y}$ be two vectors in $\mathbb{R}^N$, whose entries belong to an interval I. Then the following statements are equivalent:*

a) $\mathbf{x} \prec \mathbf{y}$;

b) there is a doubly stochastic matrix $A = (a_{ij})_{1 \leq i,j \leq N}$ such that $\mathbf{x} = A\mathbf{y}$;

c) the inequality $\sum_{i=1}^{N} f(x_i) \leq \sum_{i=1}^{N} f(y_i)$ holds for every continuous convex function $f : I \to \mathbb{R}$.

The notion of majorization is generalized by weighted majorization, that refers to probability measures rather than vectors. This is done by identifying each vector $\mathbf{x} = (x_1, ..., x_N)$ in $\mathbb{R}^N$ with the discrete probability measure $\frac{1}{N}\sum_{i=1}^{N} \delta_{x_i}$, where δ_{x_i} denotes the Dirac measure concentrated at x_i. Precisely, the relation of majorization

$$\sum_{i=1}^{m} \lambda_i \delta_{\mathbf{x}_i} \prec \sum_{j=1}^{n} \mu_j \delta_{\mathbf{y}_j}, \tag{4.1}$$

between two discrete probability measures supported at points in $\mathbb{R}^N$, is defined by asking the existence of a $m \times n$-dimensional matrix $A = (a_{ij})_{i,j}$ such that

$$a_{ij} \geq 0, \text{ for all } i, j \tag{3}$$

$$\sum_{j=1}^{n} a_{ij} = 1, \quad i = 1, ..., m \tag{4}$$

$$\mu_j = \sum_{i=1}^{m} a_{ij} \lambda_i, \quad j = 1, ..., n \tag{5}$$

and

$$\mathbf{x}_i = \sum_{j=1}^{n} a_{ij} \mathbf{y}_j, \quad i = 1, ..., m \tag{6}$$

This leads to the following generalization of the Hardy-Littlewood-Pólya theorem of majorization.

Theorem 4.2. (Niculescu and Rovenţa [15]) *Suppose that f is a real-valued function defined on a compact convex subset K of $\mathbb{R}^N$ and $\sum_{i=1}^{m} \lambda_i \delta_{\mathbf{x}_i}$ and $\sum_{j=1}^{n} \mu_j \delta_{\mathbf{y}_j}$ are two positive discrete measures concentrated at points in K. such that*

$$\sum_{i=1}^{m} \lambda_i \delta_{\mathbf{x}_i} \prec \sum_{j=1}^{n} \mu_j \delta_{\mathbf{y}_j},$$

If $\mathbf{x}_1, ..., \mathbf{x}_m$ are points of convexity of f, then

$$\sum_{i=1}^{m} \lambda_i f(\mathbf{x}_i) \leq \sum_{j=1}^{n} \mu_j f(\mathbf{y}_j). \tag{7}$$

In order to extend this result to the context of means we have to notice that condition (5) is equivalent to

$$\mathbf{x}_i = A(\mathbf{y}_1, ..., \mathbf{y}_n; a_{i1}, ..., a_{in}) \quad \text{for } i = 1, ..., m,$$

while the conclusion (7) is equivalent to

$$A\left(f(\mathbf{x}_1), ..., f(\mathbf{x}_m); \lambda_1, ..., \lambda_m\right) \leq A\left(f(\mathbf{y}_1), ..., f(\mathbf{y}_n); \mu_1, ..., \mu_n\right).$$

These remarks suggest the following concept of majorization attached to an arbitrary mean M (defined on an order subinterval of a Banach lattice):

Definition 4.2. The relation

$$\sum_{i=1}^{m} \lambda_i \delta_{\mathbf{x}_i} \prec_M \sum_{j=1}^{n} \mu_j \delta_{\mathbf{y}_j},$$

between two discrete probability measures supported at points in I, means the existence of a $m \times n$-dimensional matrix $A = (a_{ij})_{i,j}$ that verifies the conditions *(*3*)*, *(*4*)* and *(*5*)* above and also the following condition whose significance is that the support of $\sum_{i=1}^{m} \lambda_i \delta_{\mathbf{x}_i}$ is "less spread out" than the support of $\sum_{j=1}^{n} \mu_j \delta_{\mathbf{y}_j}$:

$$\mathbf{x}_i = M(\mathbf{y}_1, ..., \mathbf{y}_n; a_{i1}, ..., a_{in}) \quad \text{for } i = 1, ..., m. \tag{8}$$

The simplest case of majorization covered by Definition 4.2 is

$$\delta_{M(\mathbf{x}_1,...,\mathbf{x}_n;1/n,...,1/n)} \prec_M \frac{1}{n} \sum_{i=1}^{n} \delta_{\mathbf{x}_i},$$

where $\mathbf{x}_1, ..., \mathbf{x}_n$ is an arbitrary finite family of points; for this, consider the matrix A whose all entries equal $1/n$.

We next discuss the case of power means on real intervals. For $p \in \mathbb{R} \backslash \{0\}$,

$$\frac{1}{n} \sum_{i=1}^{n} \delta_{x_i} \prec_{M_p} \frac{1}{n} \sum_{i=1}^{n} \delta_{y_i} \Leftrightarrow (x_1^p, ..., x_n^p) \prec (y_1^p, ..., y_n^p),$$

while for $p = 0$, M_p coincides with G and $(x_1, ..., x_n) \prec_G (y_1, ..., y_n)$ is equivalent to

$$(\log x_1, ..., \log x_n) \prec (\log y_1, ..., \log y_n),$$

and also to the concurrence of the relations

$$\prod_{i=1}^{k} x_i^{\downarrow} \leq \prod_{i=1}^{k} y_i^{\downarrow} \quad \text{for } k = 1, ..., n-1$$

and

$$\prod_{i=1}^{n} x_i^{\downarrow} = \prod_{i=1}^{n} y_i^{\downarrow}.$$

Simple examples show that the relations $\prec_{M_p}$ are distinct from $\prec$ when $p \neq 1$.

The extension of Hardy-Littlewood-Pólya theorem of majorization to the context of means is as follows:

Theorem 4.3. *Suppose that M is a mean defined on an order interval I of a Banach lattice and $\sum_{i=1}^m \lambda_i \delta_{\mathbf{x}_i}$ and $\sum_{j=1}^n \mu_j \delta_{\mathbf{y}_j}$ are two discrete probability measures concentrated at points in I such that*

$$\sum_{i=1}^m \lambda_i \delta_{\mathbf{x}_i} \prec_M \sum_{j=1}^n \mu_j \delta_{\mathbf{y}_j}.$$

If f is a real-valued function defined on I and φ is a strictly increasing function defined on an interval including the range of f, then

$$M_\varphi\left(f(\mathbf{x}_1), ..., f(\mathbf{x}_m); \lambda_1, ..., \lambda_m\right) \leq M_\varphi\left(f(\mathbf{y}_1), ..., f(\mathbf{y}_n); \mu_1, ..., \mu_n\right),$$

provided that $\mathbf{x}_1, ..., \mathbf{x}_m$ are points of (M, M_φ)-convexity of f.

The inequality works in the reversed way when $\mathbf{x}_1, ..., \mathbf{x}_m$ are points of (M, M_φ)-concavity of f.

Proof. By our hypotheses,

$$f(\mathbf{x}_i) \leq M_\varphi\left(f(\mathbf{y}_1), ..., f(\mathbf{y}_n); a_{i1}, ..., a_{in}\right) \quad \text{for } i = 1, ..., m.$$

which yields

$$\begin{aligned} M_\varphi&\left(f(\mathbf{x}_1), ..., f(\mathbf{x}_m); \lambda_1, ..., \lambda_m\right) \\ &\leq M_\varphi\left(M_\varphi\left(f(\mathbf{y}_1), ..., f(\mathbf{y}_n); a_{11}, ..., a_{1n}\right), ..., \right. \\ &\qquad \left. M_\varphi\left(f(\mathbf{y}_1), ..., f(\mathbf{y}_n); a_{m1}, ..., a_{mn}\right); \lambda_1, ..., \lambda_m\right). \end{aligned}$$

The proof ends by taking into account the formula

$$\begin{aligned} M_\varphi&\left(M_\varphi\left(f(\mathbf{y}_1), ..., f(\mathbf{y}_n); a_{11}, ..., a_{1n}\right), ..., \right. \\ &\qquad \left. M_\varphi\left(f(\mathbf{y}_1), ..., f(\mathbf{y}_n); a_{m1}, ..., a_{mn}\right); \lambda_1, ..., \lambda_m\right) \\ &= M_\varphi\left(f(\mathbf{y}_1), ..., f(\mathbf{y}_n); \sum_{i=1}^m \lambda_i a_{i1}, ..., \sum_{i=1}^m \lambda_i a_{in}\right) \\ &\qquad = M_\varphi\left(f(\mathbf{y}_1), ..., f(\mathbf{y}_n); \mu_1, ..., \mu_n\right). \end{aligned}$$

□

As was noticed above,

$$\delta_{M(\mathbf{x}_1, ..., \mathbf{x}_n; 1/n, ..., 1/n)} \prec_M \frac{1}{n} \sum_{i=1}^n \delta_{\mathbf{x}_i}$$

for every mean M. In this case Theorem 4.3 yields the following extension of *Jensen's Inequality*: If f is (M, M_φ)-convex, then

$$f(M(\mathbf{x}_1, ..., \mathbf{x}_n; 1/n, ..., 1/n)) \leq M_\varphi(f(\mathbf{x}_1), ..., f(\mathbf{x}_n); 1/n, ..., 1/n).$$

Other illustrations of Theorem 4.3 make the object of the next three examples.

Example 4.1. According to a remark made at the end of Section 2, the functions $x^{\log x} = \exp\left(\log^2 x\right)$ and $\log^2 x$ have the same points of convexity/concavity. Thus all points belonging to the interval $(0, 1.5]$ are points of convexity for the restriction of $x^{\log x}$ to $(0, 14]$. As a consequence, from Theorem 4.3 and Corollary 3.1 (ii) we infer that

$$\prod_{i=1}^n x_i^{\log x_i} \leq \prod_{i=1}^n y_i^{\log y_i}$$

for all $x_1, ..., x_n \in (0, 1.5]$, $y_1, ..., y_n \in (0, 14]$ and $(x_1, ..., x_n) \prec (y_1, ..., y_n)$.

Example 4.2. The exponential function is (L, L)-convex. See [13], Exercise 6, p. 91. Since $L \leq M_p$ for any $p \geq 1/3$ (see [8]), the exponential function is also (L, M_p)-convex. As a consequence, from Theorem 4.3 it follows that

$$\left(\sum_{i=1}^{m} \lambda_i e^{px_i}\right)^{1/p} \leq \left(\sum_{j=1}^{n} \mu_j e^{py_j}\right)^{1/p} \quad \text{for all } p \geq 1/3,$$

whenever $\sum_{i=1}^{m} \lambda_i \delta_{x_i} \prec_L \sum_{j=1}^{n} \mu_j \delta_{y_j}$.

Example 4.3. As was noticed by D. Borwein, J. Borwein, G. Fee and R. Girgensohn [3] the function

$$V_\alpha(p) = 2^\alpha \frac{\Gamma^\alpha(1 + 1/p)}{\Gamma(1 + \alpha/p)}, \quad p > 0,$$

is (H, G)-concave for each value $\alpha > 1$ of the parameter α. In other words,

$$V_\alpha^{1-\lambda}(p) V_\alpha^\lambda(q) \leq V_\alpha\left(\frac{1}{\frac{1-\lambda}{p} + \frac{\lambda}{q}}\right)$$

for all $p, q > 0$ and $\lambda \in [0, 1]$. According to Theorem 4.3, if $\frac{1}{n}\sum_{i=1}^{n} \delta_{x_i} \prec_H \frac{1}{n}\sum_{i=1}^{n} \delta_{y_i}$ (equivalently, $(1/x_1, ..., 1/x_n) \prec (1/y_1, ..., 1/y_n)$), then

$$\prod_{i=1}^{n} V_\alpha(x_i) \geq \prod_{i=1}^{n} V_\alpha(y_i) \quad \text{for all } \alpha > 1.$$

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[1]DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CRAIOVA
A. I. CUZA 13, 200585 CRAIOVA, ROMANIA
Email address: cpniculescu@gmail.com
Email address: ionelroventa@yahoo.com

[2]THE ACADEMY OF ROMANIAN SCIENTISTS
SPLAIUL INDEPENDENŢEI 54, BUCHAREST RO-050094, ROMANIA
Email address: cpniculescu@gmail.com