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Iterative methods for generalized split feasibility problems in Banach spaces

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ABSTRACT. Inspired by the recent work of Takahashi et al. [W. Takahashi, H.-K. Xu and J.-C. Yao, *Iterative methods for generalized split feasibility problems in Hilbert spaces*, Set-Valued Var. Anal., **23** (2015), 205–221], in this paper, we study generalized split feasibility problems (GSFPs) in the setting of Banach spaces. We propose iterative algorithms to compute the approximate solutions of such problems. The weak convergence of the sequence generated by the proposed algorithms is studied. As applications, we derive some algorithms and convergence results for some problems from nonlinear analysis, namely, split feasibility problems, equilibrium problems, etc. Our results generalize several known results in the literature including the results of Takahashi et al. [W. Takahashi, H.-K. Xu and J.-C. Yao, *Iterative methods for generalized split feasibility problems in Hilbert spaces*, Set-Valued Var. Anal., **23** (2015), 205–221].

1. INTRODUCTION AND FORMULATIONS

The split feasibility problem (in short, SFP) is formulated as:

$$\text{Find } x^* \in C \text{ such that } Ax^* \in Q, \tag{1.1}$$

where C and Q are nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively, and $A : H_1 \to H_2$ is a bounded linear operator. There has been growing interest in recent years in the theory of split feasibility problems. It has been considered by many authors in several different directions because of its applications to medical image reconstruction [7], intensity-modulated radiation therapy [12], signal processing and image reconstruction [8], etc. For further details on SFP, we refer [3, 10, 11, 17, 25, 27] and the references therein. In the recent past, several split type problems have been introduced and studied. Byrne et al. [9] considered and studied the split common null point problem (in short, SCNPP) in the setting of Hilbert spaces. They developed some algorithms for finding the approximate solutions of SCNPP. Very recently, Takahashi and Yao [26, 28] introduced SCNPP in the setting of Banach spaces. By using hybrid method and Halpern-type method, they proposed some iterative algorithms for computing the approximate solutions of SCNPP. They also established some strong and weak convergence theorems for such algorithms under some suitable conditions.

In this paper, we study the following generalized split feasibility problems (in short, GSFP) in the setting of Banach spaces. Let B_1 and B_2 be uniformly convex and uniformly smooth real Banach spaces. Let $K : B_1 \rightrightarrows B_1^*$ be a maximal monotone set-valued mapping such that $K^{-1}0 \neq \emptyset$, and $S : B_2 \to B_2$ be a nonexpansive mapping such that $\mathrm{Fix(S)} \neq \emptyset$ and $A : B_1 \to B_2$ be a bounded linear operator, where $\mathrm{Fix(S)}$ denotes the set of fixed points of S. The generalized split feasibility problem in the setting of Banach spaces is

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defined as follows:

$$\text{Find } x^* \in \text{Fix}(V) \cap K^{-1}0 \text{ such that } Ax^* \in \text{Fix}(S), \tag{1.2}$$

where $V : C \to C$ is a mapping such that $\text{Fix}(V) \neq \emptyset$ and C is a nonempty closed convex subset of B_1. If we consider $V \equiv I$ the identity mapping, then problem (1.2) reduces to the following generalized split feasibility problem:

$$\text{Find } x^* \in K^{-1}0 \text{ such that } Ax^* \in \text{Fix}(S). \tag{1.3}$$

We denote by Λ and Γ the solution set of problem (1.2) and (1.3), respectively, and assume that $\Lambda \neq \emptyset$ and $\Gamma \neq \emptyset$.

When $B_1 = H_1$ is a real Hilbert space and $B_2 = H_2$ is another real Hilbert space, then problems (1.2) and (1.3) are considered and studied by Takahashi et al. [27].

In this paper, we propose iterative algorithms for finding the approximate solutions of problems (1.2) and (1.3) in the setting of Banach spaces. We study the weak convergence of proposed algorithms under some suitable conditions. At the end, we derive some algorithms and convergence results for some problems from nonlinear analysis, namely, split feasibility problems, equilibrium problems, etc.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let B be a real Banach space with its topological dual space B^* and $\langle ., . \rangle$ denote the duality pairing between B and B^*. When $\{x_n\}$ is a sequence in B, we denote the strong convergence of $\{x_n\}$ to x by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$.

Let $S(B)$ be the unit sphere centered at the origin of B. The norm of B is said to be Gâteaux differentiable if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.4}$$

exists for each $x, y \in S(B)$. The space B is said to be smooth if its norm is Gâteaux differentiable. The norm of B is said to be uniformly Gâteaux differentiable if for each $y \in S(B)$, the limit in (2.4) is attained uniformly for all $x \in S(B)$. The space B is said to be uniformly smooth if the limit in (2.4) is attained uniformly for all $x, y \in S(B)$. It is well known that if B is uniformly smooth, then norm of B is uniformly Gâteaux differentiable. A Banach space B is said to be strictly convex if $\|(x + y)/2\| < 1$ whenever $x, y \in S(B)$ and $x \neq y$. The space B is said to be uniformly convex if for all $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that $x, y \in S(B)$ and $\|x - y\| \geq \varepsilon$ imply $\|(x + y)/2\| \leq 1 - \delta$.

Lemma 2.1. [18, 30] *Let B be a uniformly convex Banach space. Then for any given number $r > 0$, there exists a continuous strictly increasing function $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$ and $\|tx + (1 - t)y\|^2 \leq t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)g(\|x - y\|)$ for all $x, y \in B$ with $\|x\| \leq r$ and $\|y\| \leq r$, $t \in [0, 1]$.*

A function $\rho : [0, \infty) \to [0, \infty)$, defined by

$$\rho(\tau) = \sup \left\{ \frac{1}{2}(\|x + y\| + \|x - y\|) - 1 \,:\, x,\ y \in B, \|x\| = 1,\ \|y\| = \tau \right\},$$

is said to be the modulus of smoothness of B [13]. It is known that B is uniformly smooth [13] if and only if $\lim_{\tau \to 0} \rho(\tau)/\tau = 0$. For $q > 1$, a Banach space B is said to be q-uniformly smooth [13] if there exists a constant $c > 0$ such that $\rho(\tau) \leq c\tau^q$ for all $\tau > 0$. It can be easily seen that every q-uniformly smooth space is uniformly smooth.

The normalized duality mapping $J : B \rightrightarrows B^*$ is defined as

$$J(x) := \left\{ f^* \in B^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2 \text{ for all } x \in B \right\}.$$

It is well-known that the normalized duality mapping $J : B \rightrightarrows B^*$ is single-valued if B is smooth (see, Theorem 4.3.1 in [23]).

The normalized duality mapping J from a smooth Banach space B into B^* is said to be weakly sequentially continuous if $Jx_n \overset{*}{\rightharpoonup} Jx$ whenever $x_n \rightharpoonup x$, where $\overset{*}{\rightharpoonup}$ means the weak* convergence in the dual space. For further details on geometry of Banach spaces, we refer to [2, 13, 23].

Lemma 2.2. [23] *Let B be a smooth Banach space and J be the normalized duality mapping on B. Then, $\langle x-y, Jx-Jy\rangle \geq 0$ for all $x, y \in B$. Furthermore, if B is strictly convex and $\langle x-y, Jx-Jy\rangle = 0$, then $x = y$.*

Lemma 2.3. [30] *Let B be a 2-uniformly smooth Banach space with best smooth constant $\kappa > 0$ and J be the normalized duality mapping on B. Then $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, Jx\rangle + 2\|\kappa y\|^2$ for all $x, y \in B$.*

Let B be a smooth Banach space. Following Alber [1] and Kamimura and Takahashi [16], let $\phi : B \times B \to \mathbb{R}$ be the mapping defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \text{for all } x, y \in B. \tag{2.5}$$

If B is a Hilbert space, then we have $\phi(x,y) = \|x-y\|^2$ for all $x, y \in B$. We know that

$$(\|x\| - \|y\|)^2 \leq \phi(x,y) \leq (\|x\| + \|y\|)^2, \quad \text{for all } x, y \in B. \tag{2.6}$$

If B is strictly convex, then

$$\phi(x,y) = 0 \quad \Leftrightarrow \quad x = y. \tag{2.7}$$

If C is a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space B, then, for all $x \in B$, there exists a unique point $x_0 \in C$ such that

$$\|x - x_0\| = \min_{y \in C} \|x - y\|. \tag{2.8}$$

We denote such a point x_0 by $P_C x$. The mapping P_C is called the metric projection from B onto C.

We also have

$$\phi(x, x_0) = \min_{y \in C} \phi(y, x). \tag{2.9}$$

Following Alber [1], we denote such a point x_0 by $\Pi_C x$. The mapping Π_C is called the generalized projection from B onto C.

The following lemmas on metric projection and generalized projection are well known.

Lemma 2.4. [23] *Let B be a smooth, strictly convex, and reflexive Banach space. Let C be a nonempty closed convex subset of B, $x \in B$ and $z \in C$. Then the following conditions are equivalent.*

(i) $z = P_C x$;
(ii) $\langle z - y, J(x - z)\rangle \geq 0$ *for all* $y \in C$.

Lemma 2.5. [1] (see also [16]) *Let B be a reflexive, strictly convex, and smooth Banach space, C be a nonempty closed convex subset of B, and $x \in B$. Then*

(i) $x_0 = \Pi_C x \Leftrightarrow \langle x_0 - y, Jx - Jx_0\rangle \geq 0$ *for each* $y \in C$;
(ii) $\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x)$ *for each* $y \in C$.

It is well known that the normalized duality mapping J of a Hilbert space is the identity mapping. In the setting of Hilbert spaces, $P_C = \Pi_C$.

Lemma 2.6. [16] *Let B be a smooth and uniformly convex Banach space and $\{x_n\}$ and $\{y_n\}$ be sequences in B such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\phi(x_n, y_n) = 0$, then $\|x_n - y_n\| = 0$.*

Lemma 2.7. [16] *Let* $r > 0$ *and* B *be a uniformly convex and smooth Banach space. Then* $g(\|y - z\|) \leq \phi(y, z)$ *for all* $y, z \in_r = \{w \in B : \|w\| \leq r\}$*, where* $g : [0, \infty) \to [0, \infty)$ *is a continuous, strictly increasing, and convex function with* $g(0) = 0$.

Definition 2.1. Let C be a nonempty closed convex subset of a smooth Banach space B and $T : C \to C$ be an operator. A point $a \in C$ is called an asymptotic fixed point [22] of T if there exists a sequence $\{x_n\}$ such that $x_n \rightharpoonup a$ and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T is denoted by $\widehat{\text{Fix}}(T)$.

The operator $T : C \to C$ is said to be:

(a) nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$;
(b) firmly nonexpansive type [18] if $\phi(Tx, Ty) + \phi(Ty, Tx) + \phi(Tx, x) + \phi(Ty, y) \leq \phi(Tx, y) + \phi(Ty, x)$ for all $x, y \in C$;
(c) relatively nonexpansive (see [18]) if the following properties are satisfied:
 (i) $\text{Fix}(T) \neq \emptyset$;
 (ii) $\phi(p, Tx) \leq \phi(p, x)$ for all $p \in \text{Fix(T)}$, $x \in C$;
 (iii) $\widehat{\text{Fix}}(T) = \text{Fix}(T)$.
(d) strongly relatively nonexpansive (see [18, 22]) if the following properties are satisfied:
 (i) T is relative nonexpansive;
 (ii) $\lim\limits_{n\to\infty} \phi(Tx_n, x_n) = 0$ whenever $\{x_n\}$ is bounded sequence in C and $\lim\limits_{n\to\infty} (\phi(p, x_n) - \phi(p, Tx_n)) = 0$ for some $p \in \text{Fix}(T)$.
(e) nonspreading [19] if $\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(Tx, y) + \phi(Ty, x)$ for all $x, y \in C$;
(f) generalized nonspreading [15, 20] if there are $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\alpha\phi(Tx, Ty) + (1 - \alpha)\phi(x, Ty) + \gamma\{\phi(Ty, Tx) - \phi(Ty, x)\} \leq \beta\phi(Tx, y) + (1 - \beta)\phi(x, y) + \delta\{\phi(y, Tx) - \phi(y, x)\}, \text{ for all } x, y \in C. \tag{2.10}$$

Remark 2.1. A generalized nonspreading mapping is nonspreading if $\alpha = 1, \gamma = 1, \beta = 1$ and $\delta = 0$.

Remark 2.2. If B is a real Hilbert space, then $\phi(x, y) = \|x - y\|$, and therefore we obtain the following inequality from (2.10)

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 + \gamma\{\|Tx - Ty\|^2 - \|x - Ty\|^2\} \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2 + \delta\{\|Tx - y\|^2 - \|x - y\|^2\}, \quad \text{for all } x, y \in C. \tag{2.11}$$

This implies that

$$(\alpha + \gamma)\|Tx - Ty\|^2 + \{1 - (\alpha + \gamma)\}\|x - Ty\|^2 \leq (\beta + \delta)\|Tx - y\|^2 + \{1 - (\beta + \delta)\}\|x - y\|^2, \quad \text{for all } x, y \in C. \tag{2.12}$$

So, from (2.12), T is a generalized hybrid mapping on a Hilbert space (see, [15, 20]). Observe that if $\text{Fix(T)} \neq \emptyset$, then $\phi(p, Ty) \leq \phi(p, y)$ for all $p \in \text{Fix(T)}$ and $y \in C$. Indeed, putting $x = p \in \text{Fix(T)}$ in (2.10), we obtain

$$\alpha\phi(p, Ty) + (1 - \alpha)\phi(p, Ty) + \gamma\{\phi(Ty, p) - \phi(Ty, p)\} \leq \beta\phi(p, y) + (1 - \beta)\phi(p, y) + \delta\{\phi(y, p) - \phi(y, p)\}. \tag{2.13}$$

So, we have

$$\phi(p, Ty) \leq \phi(p, y). \tag{2.14}$$

Lemma 2.8. [15, 20] *Let B be a strictly convex Banach space with a uniformly Gâteaux differentiable norm and C be a nonempty closed convex subset of B. Let $T : C \to C$ be a generalized nonspreading mapping such that* $\mathrm{Fix(T)} \neq \emptyset$*. Then* $\widehat{\mathrm{Fix}}(\mathrm{T}) = \mathrm{Fix(T)}$ *and* $\mathrm{Fix(T)}$ *is closed and convex.*

Remark 2.3. In view of Lemma 2.8 and inequality (2.14), we can say that every generalized nonspreading mapping is relative nonexpansive provided $\mathrm{Fix(T)} \neq \emptyset$ (see [20]).

Let $K : B \rightrightarrows B^*$ be a set-valued mapping. The domain, range, graph and inverse of K are denoted by

$$D(K) = \{x \in B : K(x) \neq \emptyset\}, \quad R(K) = \{x^* \in B^* : x^* \in Kx\},$$
$$G(K) = \{(x, x^*) : x^* \in Kx\} \quad \text{and} \quad K^{-1}0 = \{x \in D(K) : 0 \in Kx\},$$

respectively.

Definition 2.2. [18] A set-valued mapping $K : B \rightrightarrows B^*$ is said to be

(a) monotone if $\langle x - y, x^* - y^* \rangle \geq 0$ whenever $(x, x^*), (y, y^*) \in G(K)$.
(b) maximal monotone if its graph is not properly contained in the graph of any other monotone operator.

Remark 2.4. If K is maximal monotone, then the solution set $K^{-1}0$ is closed and convex.

Lemma 2.9. [21] Let B be a smooth, strictly convex and reflexive Banach space and $K : B \rightrightarrows B^*$ be a monotone operator. Then K is maximal monotone if and only if $R(J+\lambda K) = B^*$ for all $\lambda > 0$.

Let B be a smooth, strictly convex and reflexive Banach space and $K : B \rightrightarrows B^*$ be a maximal monotone operator. Then for $\lambda > 0$ and $x \in B$, consider

$$J_\lambda^K x := \{z \in B : Jx \in Jz + \lambda K(z)\}.$$

In other words, $J_\lambda^K = (J + \lambda K)^{-1} J$. Also, J_λ^K is known as relative resolvent of K for $\lambda > 0$. Following [18], we know the following properties:

(i) $J_\lambda^K : B \to D(K)$ is a single-valued mapping;
(ii) $K^{-1}0 = \mathrm{Fix}(\mathrm{J}_\lambda^\mathrm{K})$ for each $\lambda > 0$;
(iii) J_λ^K is strongly relative nonexpansive.

We close this section by mentioning the closedness principle in the setting of uniformly convex Banach spaces.

Lemma 2.10. [6] *Let C be a nonempty closed convex subset of a uniformly convex Banach space B and $T : C \to B$ be a nonexpansive mapping. If $\{x_n\}$ is a sequence of C such that $x_n \rightharpoonup x$ and $\|(I - T)x_n\| \to 0$, then $(I - T)x = 0$, that is, x is a fixed point of T, where I is the identity mapping on B.*

3. Algorithms and convergence results

Throughout this section, unless otherwise specified, we assume that B_1 and B_2 are uniformly convex and 2-uniformly smooth real Banach spaces having smoothness constant κ satisfying $0 < \kappa \leq \frac{1}{\sqrt{2}}$. Let $K : B_1 \rightrightarrows B_1^*$ be a maximal monotone set-valued mapping such that $K^{-1}0 \neq \emptyset$, $S : B_2 \to B_2$ be a nonexpansive mapping such that $\mathrm{Fix(S)} \neq \emptyset$ and $A : B_1 \to B_2$ be a bounded linear operator whose adjoint is denoted by A^*. Let J_λ^K be a relative resolvent operator of K for $\lambda > 0$ and $V : C \to C$ be a mapping such that $\mathrm{Fix}(V) \neq \emptyset$. We denote by J_{B_1} and J_{B_2} the normalized duality mappings on B_1 and B_2, respectively.

We propose the following algorithm to solve the problem (1.2).

Algorithm 3.1. *Choose arbitrary $x_1 \in C$ and $\beta_n \in (0,1)$, compute*

$$x_{n+1} = J_{B_1}^{-1}\left(\beta_n J_{B_1}(x_n) + (1-\beta_n) J_{B_1} V J_\lambda^K \left(J_{B_1}^{-1}\left(J_{B_1}(x_n) - \gamma A^* J_{B_2}(I-S)Ax_n\right)\right)\right), \tag{3.15}$$

for all $n \in \mathbb{N}$, $0 < c \leq \beta_n \leq d < 1$, $\gamma \in \left(0, \frac{1}{\|A\|^2}\right)$ and $\lambda > 0$.

We also propose the following algorithm to solve the problem (1.3).

Algorithm 3.2. *Choose arbitrary $x_1 \in B_1$ and compute*

$$x_{n+1} = J_\lambda^K\left(J_{B_1}^{-1}\left(J_{B_1}(x_n) - \gamma A^* J_{B_2}(I-S)Ax_n\right)\right), \quad \text{for all } n \in \mathbb{N}, \tag{3.16}$$

where $\gamma \in \left(0, \frac{1}{\|A\|^2}\right)$ and $\lambda > 0$.

We first establish weak convergence of the sequence generated by Algorithm 3.2 to a solution of problem (1.3).

Theorem 3.1. *If J_{B_1} is weakly sequentially continuous, then the sequence $\{x_n\}$ generated by Algorithm 3.2 converges weakly to an element $z \in \Gamma$.*

Proof. Let $p \in \Gamma$. Then $J_\lambda^K p = p$ and $S(Ap) = Ap$. Let

$$y_n = J_{B_1}^{-1}(J_{B_1}(x_n) - \gamma A^* J_{B_2}(I-S)Ax_n).$$

In view of equation (2.5) and Lemma 2.3, we have

$$\begin{aligned}
\phi(p, y_n) &= \phi(p, J_{B_1}^{-1}(J_{B_1}(x_n) - \gamma A^* J_{B_2}(I-S)Ax_n)) \\
&= \|p\|^2 - 2\langle p, J_{B_1}(x_n) - \gamma A^* J_{B_2}(I-S)Ax_n\rangle + \|J_{B_1}(x_n) - \gamma A^* J_{B_2}(I-S)Ax_n\|^2 \\
&= \|p\|^2 - 2\langle p, J_{B_1}(x_n) - \gamma A^* J_{B_2}(I-S)Ax_n\rangle + \|x_n - \gamma J_{B_1}^{-1} A^* J_{B_2}(I-S)Ax_n\|^2 \\
&\leq \|p\|^2 - 2\langle p, J_{B_1}(x_n)\rangle + 2\gamma\langle Ap, J_{B_2}(I-S)Ax_n\rangle \\
&\quad + \|\gamma J_{B_1}^{-1} A^* J_{B_2}(I-S)Ax_n\|^2 - 2\langle x_n, \gamma A^* J_{B_2}(I-S)Ax_n\rangle + 2\|\kappa x_n\|^2 \\
&\leq \|p\|^2 - 2\langle p, J_{B_1}(x_n)\rangle + 2\gamma\langle Ap, J_{B_2}(I-S)Ax_n\rangle \\
&\quad + \gamma^2\|A\|^2\|(I-S)Ax_n\|^2 - 2\gamma\langle Ax_n, J_{B_2}(I-S)Ax_n\rangle + \|x_n\|^2 \\
&\leq \phi(p, x_n) + \gamma^2\|A\|^2\|(I-S)Ax_n\|^2 + 2\gamma\langle Ap - Ax_n, J_{B_2}(I-S)Ax_n\rangle.
\end{aligned} \tag{3.17}$$

From nonexpansiveness of S and Lemma 2.3, we have

$$\begin{aligned}
&\langle Ap - Ax_n, J_{B_2}(I-S)Ax_n\rangle \\
&= \langle Ap - S(Ax_n), J_{B_2}(I-S)Ax_n\rangle - \|(I-S)Ax_n\|^2 \\
&\leq \frac{1}{2}\|(I-S)Ax_n\|^2 + \frac{1}{2}\|Ap - S(Ax_n)\|^2 - \frac{1}{2}\|Ax_n - Ap\|^2 - \|(I-S)Ax_n\|^2 \\
&= -\frac{1}{2}\|(I-S)Ax_n\|^2 + \frac{1}{2}\|S(Ax_n) - Ap\|^2 - \frac{1}{2}\|Ax_n - Ap\|^2 \\
&= -\frac{1}{2}\|(I-S)Ax_n\|^2,
\end{aligned} \tag{3.18}$$

that is,

$$2\gamma\langle Ap - Ax_n, J_{B_2}(I-S)Ax_n\rangle \leq -\gamma\|(I-S)Ax_n\|^2. \tag{3.19}$$

Notice that $\gamma \in \left(0, 1/\|A\|^2\right)$ and making use of inequality (3.19) in (3.17), we have

$$\begin{aligned}
\phi(p, y_n) &\leq \phi(p, x_n) + \gamma^2\|A\|^2\|(I-S)Ax_n\|^2 - \gamma\|(I-S)Ax_n\|^2 \\
&= \phi(p, x_n) - \gamma(1 - \gamma\|A\|^2)\|(I-S)Ax_n\|^2.
\end{aligned} \tag{3.20}$$

In view of relative nonexpansiveness of J_λ^K and (3.20), we have

$$\begin{aligned}\phi(p, x_{n+1}) &= \phi(p, J_\lambda^K y_n) \le \phi(p, y_n)\\ &\le \phi(p, x_n) - \gamma(1-\gamma\|A\|^2)\|(I-S)Ax_n\|^2 \quad (3.21)\\ &\le \phi(p, x_n). \quad (3.22)\end{aligned}$$

Hence, from (3.22), the sequence $\phi(p, x_n)$ is a decreasing sequence and from (2.6), it is bounded below by 0. Consequently, it converges to some finite limit, so $\lim_{n\to\infty}\phi(p, x_n)$ exists and, in particular, $\phi(p, x_n)$ is bounded. Then by (2.6), $\{x_n\}$ is also bounded. Again by the fact that $\gamma \in \left(0, \frac{1}{\|A\|^2}\right)$ and by passing to the limit in (3.21), we obtain

$$\gamma(1-\gamma\|A\|^2)\lim_{n\to\infty}\left(\|(I-S)Ax_n\|^2\right) \le \lim_{n\to\infty}\left(\phi(p, x_n) - \phi(p, x_{n+1})\right),$$

so, we have

$$\lim_{n\to\infty}\|(I-S)Ax_n\| = 0. \quad (3.23)$$

Now, consider

$$\begin{aligned}\phi(x_n, y_n) &= \phi(x_n, J_{B_1}^{-1}(J_{B_1}(x_n) - \gamma A^* J_{B_2}(I-S)Ax_n)\\ &= \|x_n\|^2 - 2\langle x_n, J_{B_1}(x_n) - \gamma A^* J_{B_2}(I-S)Ax_n\rangle\\ &\quad + \|J_{B_1}(x_n) - \gamma A^* J_{B_2}(I-S)Ax_n\|^2\\ &\le \|x_n\|^2 - 2\langle x_n, J_{B_1}(x_n)\rangle + 2\gamma\langle Ax_n, J_{B_2}(I-S)Ax_n\rangle\\ &\quad + \|x_n\|^2 - 2\gamma\langle Ax_n, J_{B_2}(I-S)Ax_n\rangle + \gamma^2\|A\|^2\|(I-S)Ax_n\|^2\\ &\le \phi(x_n, x_n) + \gamma^2\|A\|^2\|(I-S)Ax_n\|^2. \quad (3.24)\end{aligned}$$

In view of (2.7), (3.23) and (3.24), we have

$$\lim_{n\to\infty}\phi(x_n, y_n) = 0. \quad (3.25)$$

By Lemma 2.6, we have

$$\lim_{n\to\infty}\|x_n - y_n\| = 0. \quad (3.26)$$

In view of relative nonexpansiveness of J_λ^K and (3.22), we have

$$\begin{aligned}0 &\le \phi(p, y_n) - \phi(p, J_\lambda^K y_n)\\ &\le \phi(p, x_n) - \phi(p, x_{n+1}) \to 0 \text{ as } n\to\infty. \quad (3.27)\end{aligned}$$

From (3.20), we obtain the boundedness of $\phi(p, y_n)$. Again in view of (2.6), we have the boundedness of $\{y_n\}$. Thus by strongly relative nonexpansiveness of J_λ^K, and from (3.27), we have

$$\lim_{n\to\infty}\phi(J_\lambda^K y_n, y_n) = 0, \quad (3.28)$$

so by Lemma 2.6, we have

$$\lim_{n\to\infty}\|J_\lambda^K y_n - y_n\| = 0. \quad (3.29)$$

Since B_1 is uniformly convex, it is reflexive [13, Milman-Pettis'theorem, Theorem 1.17]. Therefore, B_1 is reflexive and by the boundedness of $\{x_n\}$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup z \in B_1$ (see [14, property 1.8]).

Now we show that $x_n \rightharpoonup z$. In order to show this, we have to show that every subsequence of x_n converges weakly to z. Assume to the contrary that there exists another subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup y \in B_1$ where $z \neq y$. Since J_{B_1} is weakly sequentially continuous, so $J_{B_1}x_{n_i} \overset{*}{\rightharpoonup} J_{B_1}z$ and $J_{B_1}x_{n_j} \overset{*}{\rightharpoonup} J_{B_1}y$,

$$\begin{aligned}\langle y-z, J_{B_1}z\rangle &= \lim_{i\to\infty}\langle y-z, J_{B_1}x_{n_i}\rangle = \lim_{n\to\infty}\langle y-z, J_{B_1}x_n\rangle \\ &= \lim_{j\to\infty}\langle y-z, J_{B_1}x_{n_j}\rangle = \langle y-z, J_{B_1}y\rangle.\end{aligned}$$

Thus we obtain $\langle z-y, J_{B_1}z - J_{B_1}y\rangle = 0$. Since B_1 is uniformly convex, by [2, Theorem 2.14] it is strictly convex. Then by Lemma 2.2, we have $z = y$. Thus we have shown that every subsequence of $\{x_n\}$ converges weakly to z. This implies that $x_n \rightharpoonup z$. Since A is bounded linear operator, so $Ax_n \rightharpoonup Az$. Thus by (3.23) and using the fact that S is demiclosed at 0, we have $S(Az) = Az$. From (3.26), we have $y_n \rightharpoonup z$ as follow: For all $f \in B_1^*$,

$$\begin{aligned}\|f(y_n) - f(z)\| &= \|f(y_n) - f(x_n) + f(x_n) - f(z)\| \\ &\leq \|f(y_n) - f(x_n)\| + \|f(x_n) - f(z)\| \\ &\leq \|f\|\|y_n - x_n\| + \|f(x_n) - f(z)\| \to 0 \text{ as } n \to \infty. \end{aligned} \tag{3.30}$$

Thus $y_n \rightharpoonup z$. By (3.29) and by the relative nonexpansiveness of J_λ^K, we have $J_\lambda^K z = z$. Thus we have shown that $x_n \rightharpoonup z$ such that $z \in K^{-1}0$ and $Az \in \mathrm{Fix}(S)$. This completes the proof. □

Theorem 3.2. *Let C be a closed convex subset of B_1, $K : B_1 \rightrightarrows B_1^*$ be a maximal monotone operator such that $D(K) \subseteq C$ and $K^{-1}0 \neq \emptyset$. Let $V : C \to C$ be a generalized nonspreading mapping such that $\mathrm{Fix}(V) \neq \emptyset$. If J_{B_1} is weakly sequentially continuous, then the sequence $\{x_n\}$ generated by Algorithm 3.1 converges weakly to an element $z \in \Lambda$, which is identified as the strong limit of the orthogonal projection of $\{x_n\}$ onto Λ, that is, $z = \lim_{n\to\infty} \Pi_\Lambda x_n$.*

Proof. Let $y_n = J_{B_1}^{-1}(J_{B_1}(x_n) - \gamma A^* J_{B_2}(I-S)Ax_n))$ and $z_n = J_\lambda^K y_n$. Then (3.15) takes the following form

$$x_{n+1} = J_{B_1}^{-1}(\beta_n J_{B_1}(x_n) + (1-\beta_n)J_{B_1}Vz_n).$$

Let $p \in \Lambda$. Then $J_\lambda^K p = p$, $Vp = p$ and $S(Ap) = Ap$. It follows that

$$\begin{aligned}\phi(p, x_{n+1}) &= \phi(p, J_{B_1}^{-1}(\beta_n J_{B_1}(x_n) + (1-\beta_n)J_{B_1}(Vz_n))) \\ &= \|p\|^2 - 2\langle p, \beta_n J_{B_1}x_n + (1-\beta_n)J_{B_1}(Vz_n)\rangle \\ &\quad + \|\beta_n J_{B_1}(x_n) + (1-\beta_n)J_{B_1}(V(z_n))\|^2 \\ &\leq \|p\|^2 - 2\beta_n\langle p, J_{B_1}x_n\rangle - 2(1-\beta_n)\langle p, J_{B_1}(Vz_n)\rangle \\ &\quad + \beta_n\|x_n\|^2 + (1-\beta_n)\|Vz_n\|^2 \\ &\leq \beta_n(\|p\|^2 - 2\langle p, J_{B_1}x_n\rangle + \|x_n\|^2) + (1-\beta_n)(\|p\|^2 - 2\langle p, J_{B_1}Vz_n\rangle + \|Vz_n\|^2) \\ &= \beta_n\phi(p, x_n) + (1-\beta_n)\phi(p, Vz_n) \\ &\leq \beta_n\phi(p, x_n) + (1-\beta_n)\phi(p, z_n) \\ &= \beta_n\phi(p, x_n) + (1-\beta_n)\phi(p, J_\lambda^K y_n) \\ &\leq \beta_n\phi(p, x_n) + (1-\beta_n)\phi(p, y_n) \\ &\leq \beta_n\phi(p, x_n) + (1-\beta_n)\phi(p, x_n) - (1-\beta_n)^2\gamma(1-\gamma\|A\|^2)\|(I-S)Ax_n\|^2 \\ &= \phi(p, x_n) - (1-\beta_n)^2\gamma(1-\gamma\|A\|^2)\|(I-S)Ax_n\|^2 \qquad (3.31) \\ &\leq \phi(p, x_n). \qquad (3.32)\end{aligned}$$

Hence, from (3.32), the sequence $\phi(p, x_n)$ is decreasing, and from (2.6) it is bounded below by 0. Consequently, it converges to some finite limit, so $\lim\limits_{n\to\infty} \phi(p, x_n)$ exists and, in particular $\phi(p, x_n)$ is bounded. Then by (2.6), $\{x_n\}$ is also bounded. Again by the fact that $\gamma \in \left(0, \frac{1}{\|A\|^2}\right)$, $0 < c \leq \beta_n \leq d < 1$, and by passing to the limit in (3.31), we obtain

$$(1-\beta_n)^2\gamma(1-\gamma\|A\|^2) \lim_{n\to\infty} \left(\|(I-S)Ax_n\|^2\right) \leq \lim_{n\to\infty} \left(\phi(p,x_n) - \phi(p,x_{n+1})\right),$$

so, we have

$$\lim_{n\to\infty} \|(I-S)Ax_n\| = 0. \tag{3.33}$$

Since J_λ^K and V are relative nonexpansive, from (3.20), we have

$$\phi(p, Vz_n) \leq \phi(p, z_n) = \phi(p, J_\lambda^K y_n) \leq \phi(p, y_n) \leq \phi(p, x_n).$$

Hence boundedness of $\phi(p, x_n)$ implies the boundedness of $\phi(p, J_\lambda^K y_n)$ and $\phi(p, Vz_n)$. Thus from (2.6), $\{J_\lambda^K y_n\}$ and $\{Vz_n\}$ are bounded. Put

$$r = \sup_{n\in\mathbb{N}\cup\{0\}} \{\|J_{B_1}(x_n)\|, \|J_{B_1}(J_\lambda^K(y_n))\|, \|J_{B_1}(Vz_n)\|\}.$$

Since B_1 is uniformly smooth Banach space, B_1^* is a uniformly convex Banach space [13]. So by Lemma 2.1, we have

$$\begin{aligned}
&\phi(p, x_{n+1})\\
&= \phi(p, J_{B_1}^{-1}(\beta_n J_{B_1}(x_n) + (1-\beta_n)J_{B_1}(Vz_n)))\\
&= \|p\|^2 - 2\langle p, \beta_n J_{B_1}x_n + (1-\beta_n)J_{B_1}(Vz_n)\rangle + \|\beta_n J_{B_1}(x_n) + (1-\beta_n)J_{B_1}(V(z_n))\|^2\\
&\leq \|p\|^2 - 2\beta_n\langle p, J_{B_1}x_n\rangle - 2(1-\beta_n)\langle p, J_{B_1}(Vz_n)\rangle + \beta_n\|x_n\|^2\\
&\quad + (1-\beta_n)\|Vz_n\|^2 - \beta_n(1-\beta_n)g(\|J_{B_1}(x_n) - J_{B_1}Vz_n\|)\\
&\leq \beta_n(\|p\|^2 - 2\langle p, J_{B_1}x_n\rangle + \|x_n\|^2) + (1-\beta_n)(\|p\|^2 - 2\langle p, J_{B_1}Vz_n\rangle + \|Vz_n\|^2)\\
&\quad - \beta_n(1-\beta_n)g(\|J_{B_1}(x_n) - J_{B_1}Vz_n\|)\\
&= \beta_n\phi(p, x_n) + (1-\beta_n)\phi(p, Vz_n) - \beta_n(1-\beta_n)g(\|J_{B_1}(x_n) - J_{B_1}(Vz_n)\|)\\
&\leq \beta_n\phi(p, x_n) + (1-\beta_n)\phi(p, z_n) - \beta_n(1-\beta_n)g(\|J_{B_1}(x_n) - J_{B_1}Vz_n\|)\\
&= \beta_n\phi(p, x_n) + (1-\beta_n)\phi(p, J_\lambda^K y_n) - \beta_n(1-\beta_n)g(\|J_{B_1}(x_n) - J_{B_1}Vz_n\|)\\
&\leq \beta_n\phi(p, x_n) + (1-\beta_n)\phi(p, y_n) - \beta_n(1-\beta_n)g(\|J_{B_1}(x_n) - J_{B_1}Vz_n\|)\\
&\leq \beta_n\phi(p, x_n) + (1-\beta_n)\phi(p, x_n) - (1-\beta_n)^2\gamma(1-\gamma\|A\|^2)\|(I-S)Ax_n\|^2\\
&- \beta_n(1-\beta_n)g(\|J_{B_1}(x_n) - J_{B_1}Vz_n\|)\\
&= \phi(p, x_n) - (1-\beta_n)^2\gamma(1-\gamma\|A\|^2)\|(I-S)Ax_n\|^2 - \beta_n(1-\beta_n)g(\|J_{B_1}(x_n) - J_{B_1}Vz_n\|),
\end{aligned}$$

and

$$\begin{aligned}(1-\beta_n)^2\gamma(1-\gamma\|A\|^2)\|(I-S)Ax_n\|^2 + \beta_n(1-\beta_n)g(\|J_{B_1}(x_n) - J_{B_1}(Vz_n)\|)\\
\leq \phi(p, x_n) - \phi(p, x_{n+1}),\end{aligned} \tag{3.34}$$

using the fact $0 < c \leq \beta_n \leq d < 1$, $\gamma \in \left(0, \frac{1}{\|A\|^2}\right)$ and by (3.33). Passing to the limit in (3.34), we have

$$\lim_{n\to\infty} g(\|J_{B_1}(x_n) - J_{B_1}(Vz_n)\|) = 0. \tag{3.35}$$

Since $g : [0, \infty) \to [0, \infty)$ is a continuous, strictly increasing, and convex function with $g(0) = 0$, therefore

$$\lim_{n\to\infty} \|J_{B_1}(x_n) - J_{B_1}(Vz_n)\| = 0. \tag{3.36}$$

Since B_1 is uniformly convex and uniformly smooth, it is a smooth, strictly convex and reflexive Banach space. Then J_{B_1} is a single-valued bijection. In this case, the duality mapping $J_{B_1^*}$ from B_1^* onto $B_1^{**} = B_1$ coincides with the inverse of the duality mapping J_{B_1} from B_1 onto B_1^*, that is, $J_{B_1^*} = J_{B_1}^{-1}$. Since B_1 is uniformly convex, B_1^* is uniformly smooth (see [13]). Therefore, by uniformly smoothness of B_1^*, $J_{B_1}^{-1}$ is uniformly norm-to-norm continuous on bounded sets (see [13, 23]). Thus, we obtain

$$\lim_{n\to\infty} \|x_n - Vz_n\| = \lim_{n\to\infty} \left\| J_{B_1}^{-1}(J_{B_1}(x_n)) - J_{B_1}^{-1}(J_{B_1}(Vz_n)) \right\| = 0. \tag{3.37}$$

Also, as $y_n = J_{B_1}^{-1}(J_{B_1}(x_n) - \gamma A^* J_{B_2}(I-S)Ax_n))$, we have

$$\begin{aligned}
\phi(x_n, y_n) &= \phi(x_n, J_{B_1}^{-1}(J_{B_1}(x_n) - \gamma A^* J_{B_2}(I-S)Ax_n) \\
&= \|x_n\|^2 - 2\langle x_n, J_{B_1}(x_n) - \gamma A^* J_{B_2}(I-S)Ax_n\rangle \\
&\quad + \|J_{B_1}(x_n) - \gamma A^* J_{B_2}(I-S)Ax_n\|^2 \\
&\le \|x_n\|^2 - 2\langle x_n, J_{B_1}(x_n)\rangle + 2\gamma\langle Ax_n, J_{B_2}(I-S)Ax_n\rangle \\
&\quad + \|x_n\|^2 - 2\gamma\langle Ax_n, J_{B_2}(I-S)Ax_n\rangle + \gamma^2\|A\|^2\|(I-S)Ax_n\|^2 \\
&\le \phi(x_n, x_n) + \gamma^2\|A\|^2\|(I-S)Ax_n\|^2. \qquad (3.38)
\end{aligned}$$

In view of (2.7), (3.33) and (3.38), we have

$$\lim_{n\to\infty} \phi(x_n, y_n) = 0. \tag{3.39}$$

Thus in view of (3.39), boundedness of $\{x_n\}$ and Lemma 2.6, we have

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.40}$$

Consequently, we have

$$\lim_{n\to\infty} \|y_n - Vz_n\| = 0, \tag{3.41}$$

that is,

$$\lim_{n\to\infty} \|y_n - VJ_\lambda^K y_n\| = 0. \tag{3.42}$$

Since B_1 is uniformly convex, it is reflexive. By the boundedness of $\{x_n\}$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup z \in B_1$. Since J_{B_1} is weakly sequentially continuous, as in the proof of Theorem 3.1, $x_n \rightharpoonup z$, and so $Ax_n \rightharpoonup Az$. Thus from (3.33) and knowing the fact that S is demiclosed at 0, we have $S(Az) = Az$. In view of (3.40), we have $y_n \rightharpoonup z$. Further note that $V : C \to C$ is relative nonexpansive mapping and $J_\lambda^K : B_1 \to D(K)$ is strongly relative nonexpansive such that $D(K) \subseteq C$. Hence in view of [5, Lemma 3.2 and 3.3], we have $VJ_\lambda^K : B_1 \to C$ is relative nonexpansive mapping such that $\text{Fix}(VJ_\lambda^K) = \text{Fix}(V) \cap \text{Fix}(J_\lambda^K)$. Since $y_n \rightharpoonup z$, from (3.42) and by relative nonexpansiveness of VJ_λ^K, we have $z \in \text{Fix}(VJ_\lambda^K)$. Thus we have $J_\lambda^K z = z$ and $Vz = z$. Thus we have shown that $z \in \Lambda := \text{Fix}(V) \cap K^{-1}0 \cap A^{-1}\text{Fix}(S)$. In view of Lemma 2.8, $\text{Fix}(V)$ is closed and convex. Since K is maximal monotone set-valued map, so $K^{-1}0$ is closed and convex (see Remark 2.4). Since S is nonexpansive, so $\text{Fix}(S)$ is closed and convex. By the continuity and linearity of A, we have that $A^{-1}(\text{Fix}(S))$ is closed and convex. Thus Λ is closed convex subspace of B_1. Now we have to show that $z = \lim_{n\to\infty} \Pi_\Lambda x_n$. Let $u_n = \Pi_\Lambda x_n$, for each $n \in \mathbb{N} \cup \{0\}$. Then $u_n \in \Lambda$ and $u_{n+1} = \Pi_\Lambda x_{n+1}$. Since inequality (3.32) holds for each $p \in \Lambda$, we have

$$\phi(u_n, x_{n+1}) \le \phi(u_n, x_n). \tag{3.43}$$

From Lemma 2.5 (ii), we have

$$\phi(u_n, \Pi_\Lambda x_{n+1}) + \phi(\Pi_\Lambda x_{n+1}, x_{n+1}) \leq \phi(u_n, x_{n+1}),$$

which implies that

$$\phi(\Pi_\Lambda x_{n+1}, x_{n+1}) \leq \phi(u_n, x_{n+1}) - \phi(u_n, \Pi_\Lambda x_{n+1}). \tag{3.44}$$

Since $\phi(u_n, \Pi_\Lambda x_{n+1}) \geq 0$, we have

$$\phi(u_{n+1}, x_{n+1}) \leq \phi(u_n, x_{n+1})$$

and hence from (3.43), we have

$$\phi(u_{n+1}, x_{n+1}) \leq \phi(u_n, x_n).$$

So, $\phi(u_n, x_n)$ is a decreasing sequence. Since $\phi(u_n, x_n)$ is bounded below by 0, it is convergent. Also, in view of (3.43) and (3.44), we have

$$\phi(u_n, u_{n+1}) \leq \phi(u_n, x_{n+1}) - \phi(u_{n+1}, x_{n+1}) \leq \phi(u_n, x_n) - \phi(u_{n+1}, x_{n+1}).$$

By induction, we have

$$\phi(u_n, u_{n+m}) \leq \phi(u_n, x_n) - \phi(u_{n+m}, x_{n+m}), \quad \text{for each } m \in \mathbb{N}.$$

Using Lemma 2.7, we have, for $m,\ n$ with $n > m$,

$$g(\|u_m - u_n\|) \leq \phi(u_m, u_n) \leq \phi(u_m, x_m) - \phi(u_n, x_n),$$

and hence

$$\lim_{n\to\infty} g(\|u_n - u_m\|) = 0.$$

Then the properties of g yield that

$$\lim_{n\to\infty} \|u_n - u_m\| = 0.$$

This implies that $\{u_n\}$ is a Cauchy sequence in Λ. Since B_1 is complete and Λ is closed, therefore Λ is complete. Hence $\{u_n\}$ converges strongly to some point $u \in \Lambda$. Now we will show that $u = z$. Since $u_n = \Pi_\Lambda x_n$, so by Lemma 2.5 (i), we have

$$\langle u_n - z, J_{B_1} x_n - J_{B_1} u_n\rangle \geq 0, \quad \text{for each } z \in \Lambda. \tag{3.45}$$

Also, we know that $\{u_n\}$ converges strongly to some $u \in \Lambda$ and J_{B_1} is weakly sequentially continuous. Letting $n \to \infty$ in (3.45), we have

$$\langle u - z, J_{B_1} z - J_{B_1} u\rangle \geq 0, \text{ that is, } \langle u - z, J_{B_1} u - J_{B_1} z\rangle \leq 0.$$

Also, the monotonicity of J_{B_1} implies that $\langle u - z, J_{B_1} u - J_{B_1} z\rangle \geq 0$. Thus, $\langle u - z, J_{B_1} u - J_{B_1} z\rangle = 0$. By using the strict convexity of B_1 and Lemma 2.3, we obtain that $u = z$. Therefore, $\{x_n\}$ converges weakly to $z = \lim\limits_{n\to\infty} \Pi_\Lambda x_n$. This completes the proof. □

When $V \equiv I$ the identity operator in Theorem 3.2, we have the following Corollary.

Corollary 3.1. *If J_{B_1} is weakly sequentially continuous, then the sequence $\{x_n\}$ generated by the following algorithm, for any $x_1 \in B_1$*

$$x_{n+1} = J_{B_1}^{-1}\left(\beta_n J_{B_1}(x_n) + (1-\beta_n) J_{B_1} J_\lambda^K \left(J_{B_1}^{-1}\left(J_{B_1}(x_n) - \gamma A^* J_{B_2}(I - S)Ax_n\right)\right)\right), \tag{3.46}$$

for all $n \in \mathbb{N}$, where $0 < c \leq \beta_n \leq d < 1$ and $\gamma \in \left(0, \frac{1}{\|A\|^2}\right)$ converges weakly to an element $z \in \Gamma$, which is identified as the strong limit of the orthogonal projection of $\{x_n\}$ onto Γ, that is, $z = \lim_{n\to\infty} \Pi_\Gamma x_n$.

Remark 3.5. Theorems 3.1 and 3.2, and Corollary 3.1 are the extension of Theorems 4.2, 4.4 and 4.3 in [27] from Hilbert space setting to Banach space setting.

4. Application

Let C be a nonempty closed convex subset of a smooth strictly convex and reflexive Banach space B. Let i_C be the indicator function for $C \subseteq B$, that is, $i_C(x) = 0$ if $x \in C$ and ∞ otherwise. Then $i_C : B \to (-\infty, \infty]$ is a proper lower semicontinuous convex function. Rockafellar's maximal monotonicity theorem [21] ensures that the subdifferential $\partial i_C \subset B \times B^*$ of i_C is maximal monotone. In this case, it is known that ∂i_C is reduced to the normality operator N_C for C, that is,

$$N_C(x) = \{x^* \in B^* : \langle y - x, x^* \rangle \leq 0 \text{ for all } y \in C\}.$$

Indeed, for any $x \in C$,

$$\begin{aligned} \partial i_C(x) &= \{x^* \in B^* : i_C(x) + \langle y - x, x^* \rangle \leq i_C(y) \text{ for all } y \in B\} \\ &= \{x^* \in B^* : \langle y - x, x^* \rangle \leq 0 \text{ for all } y \in C\} = N_C(x). \end{aligned}$$

We also know that Π_C is the resolvent of N_C. In fact, $\Pi_C = (J + 2^{-1} N_C)^{-1} J$ (see [18]).

Remark 4.6. If P_C is a metric projection of B onto C, then (see [5, 24, 25]), we have

$$\langle P_C x - P_C y, J(x - P_C x) - J(y - P_C y) \rangle \geq 0, \quad \text{for all } x, y \in C.$$

We also have that if x_n is a sequence in B such that $x_n \rightharpoonup p$ and $\|x_n - P_C x_n\| \to 0$, then $p = P_C p$, that is, $p \in C$.

Indeed, assume that $x_n \rightharpoonup p$ and $\|x_n - P_C x_n\| \to 0$. It is clear that $P_C x_n \rightharpoonup p$ and $\|J(x_n - P_C x_n)\| = \|x_n - P_C x_n\| \to 0$. Since P_C is the metric projection of B onto C, we have

$$\langle P_C x_n - P_C p, J(x_n - P_C x_n) - J(p - P_C p) \rangle \geq 0, \quad \text{for all } x, y \in C.$$

Then,

$$-\|p - P_C p\|^2 = \langle p - P_C p, -J(p - P_C p) \rangle \geq 0, \quad \text{for all } x, y \in C,$$

and hence, $p = P_C p$.

4.1. **Split Feasibility Problem.** Let C be a nonempty closed convex subset of B_1. Consider $K = \partial i_C$ and $S = P_Q$, where P_Q is the metric projection onto a nonempty closed convex subset Q of B_2. Then, we have $J_\lambda^K = \Pi_C$ and $\text{Fix}(S) = Q$. Now we recover the split feasibility problem in the setting of Banach spaces as follow:

$$\text{Find } x^* \in C \text{ such that } Ax^* \in Q, \tag{4.47}$$

and the algorithm (3.16) reduces to the following algorithm: For any $x_1 = x \in B_1$,

$$x_{n+1} = \Pi_C \left(J_{B_1}^{-1} \left(J_{B_1}(x_n) - \gamma A^* J_{B_2}(I - P_Q) A x_n \right) \right), \quad \text{for all } n \in \mathbb{N}. \tag{4.48}$$

Let Ω denote the solution set of (4.47), that is, $\Omega = \{x \in C : Ax \in Q\}$.

The iterative scheme (4.48) is studied by Xu [31] in the setting of Hilbert spaces.

Theorem 4.3. *Let B_1 and B_2 be uniformly convex and 2-uniformly smooth real Banach spaces having smoothness constant κ satisfying $0 < \kappa \leq \frac{1}{\sqrt{2}}$. Let C and Q be nonempty closed convex subsets of B_1 and B_2, respectively, $A : B_1 \to B_2$ be a bounded linear operator, and $\gamma \in \left(0, \frac{2}{\|A\|^2}\right)$. If $\Omega \neq \emptyset$ and J_{B_1} is weakly sequentially continuous, then the sequence $\{x_n\}$ generated by (4.48) converges weakly to an element $z \in \Omega$.*

Proof. Let $p \in \Omega$. This implies that $x_n \rightharpoonup p$, $\Pi_C p = p$ and $P_Q(Ap) = Ap$. Let

$$y_n = J_{B_1}^{-1}(J_{B_1}(x_n) - \gamma A^* J_{B_2}(I - P_Q) A x_n).$$

In view of equation (2.5) and Lemma 2.3, we have

$$\begin{aligned}\phi(p,y_n) &= \phi(p, J_{B_1}^{-1}(J_{B_1}(x_n) - \gamma A^* J_{B_2}(I-P_Q)Ax_n))\\ &= \|p\|^2 - 2\langle p, J_{B_1}(x_n) - \gamma A^* J_{B_2}(I-P_Q)Ax_n\rangle\\ &\quad + \|J_{B_1}(x_n) - \gamma A^* J_{B_2}(I-P_Q)Ax_n\|^2\\ &= \|p\|^2 - 2\langle p, J_{B_1}(x_n) + \gamma A^* J_{B_2}(I-P_Q)Ax_n\rangle\\ &\quad + \|x_n - \gamma J_{B_1}^{-1} A^* J_{B_2}(I-P_Q)Ax_n\|^2\\ &\le \|p\|^2 - 2\langle p, J_{B_1}(x_n)\rangle + 2\gamma\langle Ap, J_{B_2}(I-P_Q)Ax_n\rangle \qquad (4.49)\\ &\quad + \|\gamma J_{B_1}^{-1} A^* J_{B_2}(I-P_Q)Ax_n\|^2 - 2\langle x_n, \gamma A^* J_{B_2}(I-P_Q)Ax_n\rangle + 2\|\kappa x_n\|^2\\ &\le \|p\|^2 - 2\langle p, J_{B_1}(x_n)\rangle + 2\gamma\langle Ap, J_{B_2}(I-P_Q)Ax_n\rangle\\ &\quad + \gamma^2\|A\|^2\|(I-P_Q)Ax_n\|^2 - 2\gamma\langle Ax_n, J_{B_2}(I-P_Q)Ax_n\rangle + \|x_n\|^2\\ &\le \phi(p,x_n) + \gamma^2\|A\|^2\|(I-P_Q)Ax_n\|^2 + 2\gamma\langle Ap - Ax_n, J_{B_2}(I-P_Q)Ax_n\rangle.\end{aligned}$$

From Remark 4.6, we have

$$\begin{aligned}&\langle Ap - Ax_n, J_{B_2}(I-P_Q)Ax_n\rangle\\ &= \langle Ap - P_Q(Ax_n), J_{B_2}(I-P_Q)Ax_n\rangle - \|(I-P_Q)Ax_n\|^2\\ &\le -\|(I-P_Q)Ax_n\|^2, \qquad (4.50)\end{aligned}$$

that is,

$$2\gamma\langle Ap - Ax_n, J_{B_2}(I-P_Q)Ax_n\rangle \le -2\gamma\|(I-P_Q)Ax_n\|^2. \qquad (4.51)$$

Notice that $\gamma \in \left(0, 2/\|A\|^2\right)$ and making use of inequality (4.51) in (4.49), we have

$$\begin{aligned}\phi(p,y_n) &\le \phi(p,x_n) + \gamma^2\|A\|^2\|(I-P_Q)Ax_n\|^2 - 2\gamma\|(I-P_Q)Ax_n\|^2\\ &= \phi(p,x_n) - \gamma(2-\gamma\|A\|^2)\|(I-P_Q)Ax_n\|^2. \qquad (4.52)\end{aligned}$$

In view of relative nonexpansiveness of Π_C and (4.52), we have

$$\begin{aligned}\phi(p,x_{n+1}) = \phi(p,\Pi_C y_n) &\le \phi(p,y_n)\\ &\le \phi(p,x_n) - \gamma(2-\gamma\|A\|^2)\|(I-P_Q)Ax_n\|^2 \qquad (4.53)\\ &\le \phi(p,x_n). \qquad (4.54)\end{aligned}$$

Hence, from (4.54), the sequence $\phi(p,x_n)$ is a decreasing sequence and from (2.6), it is bounded below by 0. Consequently, it converges to some finite limit, so $\lim_{n\to\infty}\phi(p,x_n)$ exists and, in particular $\phi(p,x_n)$ is bounded. Then by (2.6), $\{x_n\}$ is also bounded. Again by the fact that $\gamma \in \left(0, \frac{2}{\|A\|^2}\right)$ and by passing to the limit in (4.53), we obtain

$$\gamma(2-\gamma\|A\|^2)\lim_{n\to\infty}\left(\|(I-P_Q)Ax_n\|^2\right) \le \lim_{n\to\infty}\left(\phi(p,x_n) - \phi(p,x_{n+1})\right),$$

so, we have

$$\lim_{n\to\infty}\|(I-P_Q)Ax_n\| = 0. \qquad (4.55)$$

Now, consider

$$\begin{aligned}\phi(x_n, y_n) &= \phi(x_n, J_{B_1}^{-1}(J_{B_1}(x_n) - \gamma A^* J_{B_2}(I - P_Q)Ax_n)\\ &= \|x_n\|^2 - 2\langle x_n, J_{B_1}(x_n) - \gamma A^* J_{B_2}(I - P_Q)Ax_n\rangle\\ &\quad + \|J_{B_1}(x_n) - \gamma A^* J_{B_2}(I - P_Q)Ax_n\|^2\\ &\le \|x_n\|^2 - 2\langle x_n, J_{B_1}(x_n)\rangle + 2\gamma\langle Ax_n, J_{B_2}(I - P_Q)Ax_n\rangle\\ &\quad + \|x_n\|^2 - 2\gamma\langle Ax_n, J_{B_2}(I - P_Q)Ax_n\rangle + \gamma^2\|A\|^2\|(I - P_Q)Ax_n\|^2\\ &\le \phi(x_n, x_n) + \gamma^2\|A\|^2\|(I - P_Q)Ax_n\|^2. \end{aligned} \tag{4.56}$$

In view of (2.7), (4.55) and (4.56), we have

$$\lim_{n\to\infty} \phi(x_n, y_n) = 0. \tag{4.57}$$

By Lemma 2.6, we have

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{4.58}$$

In view of relative nonexpansiveness of Π_C and (4.54), we have

$$\begin{aligned} 0 &\le \phi(p, y_n) - \phi(p, \Pi_C y_n)\\ &\le \phi(p, x_n) - \phi(p, x_{n+1}) \to 0 \text{ as } n \to \infty. \end{aligned} \tag{4.59}$$

From (4.52), we obtain the boundedness of $\phi(p, y_n)$. Again in view of (2.6), we have the boundedness of $\{y_n\}$. Thus by strongly relative nonexpansiveness of Π_C, and from (4.59), we have

$$\lim_{n\to\infty} \phi(\Pi_C y_n, y_n) = 0, \tag{4.60}$$

so by Lemma 2.6, we have

$$\lim_{n\to\infty} \|\Pi_C y_n - y_n\| = 0. \tag{4.61}$$

Since B_1 is uniformly convex, it is reflexive [13, Milman-Pettis'theorem, Theorem 1.17]. Therefore, B_1 is reflexive and by the boundedness of $\{x_n\}$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup z \in B_1$ (see [14, property 1.8]).

Now we show that $x_n \rightharpoonup z$. In order to show this, we have to show that every subsequence of x_n converges weakly to z. Assume to the contrary that there exists another subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup y \in B_1$ where $z \neq y$. Since J_{B_1} is weakly sequentially continuous, so $J_{B_1}x_{n_i} \overset{*}{\rightharpoonup} J_{B_1}z$ and $J_{B_1}x_{n_j} \overset{*}{\rightharpoonup} J_{B_1}y$,

$$\begin{aligned}\langle y - z, J_{B_1}z\rangle &= \lim_{i\to\infty}\langle y - z, J_{B_1}x_{n_i}\rangle = \lim_{n\to\infty}\langle y - z, J_{B_1}x_n\rangle\\ &= \lim_{j\to\infty}\langle y - z, J_{B_1}x_{n_j}\rangle = \langle y - z, J_{B_1}y\rangle.\end{aligned}$$

Thus we obtain $\langle z - y, J_{B_1}z - J_{B_1}y\rangle = 0$. Since B_1 is uniformly convex, by [2, Theorem 2.14] it is strictly convex. Then by Lemma 2.2, we have $z = y$. Thus we have shown that every subsequence of $\{x_n\}$ converges weakly to z. This implies that $x_n \rightharpoonup z$. Since A is bounded linear operator, so $Ax_n \rightharpoonup Az$. Thus by (4.55) and using Remark 4.6, we have $P_Q(Az) = Az$. From (4.58), we have $y_n \rightharpoonup z$ as follow: For all $f \in B_1^*$

$$\begin{aligned}\|f(y_n) - f(z)\| &= \|f(y_n) - f(x_n) + f(x_n) - f(z)\|\\ &\le \|f(y_n) - f(x_n)\| + \|f(x_n) - f(z)\|\\ &\le \|f\|\|y_n - x_n\| + \|f(x_n) - f(z)\| \to 0 \text{ as } n \to \infty.\end{aligned} \tag{4.62}$$

Thus $y_n \rightharpoonup z$. Notice (4.61) and by the relative nonexpansiveness of Π_C, we have $\Pi_C z = z$. Thus we have shown that $x_n \rightharpoonup z$ such that $z \in C$ and $Az \in Q$. This completes the proof. □

As a consequence of Theorem 3.2, we have the following result.

Theorem 4.4. *Let B_1 and B_2 be uniformly convex and 2-uniformly smooth real Banach spaces having smoothness constant κ satisfying $0 < \kappa \leq \frac{1}{\sqrt{2}}$. Let C be a nonempty closed convex subset of B_1, $A : B_1 \to B_2$ be a bounded linear operator, and $S : B_2 \to B_2$ be a given nonexpansive mapping such that $\mathrm{Fix}(S) \neq \emptyset$. Let $V : C \to C$ be a nonspreading mapping such that $\mathrm{Fix}(V) \neq \emptyset$. For any $x_1 = x \in C$, define*

(4.63)

$$x_{n+1} = J_{B_1}^{-1}\left(\beta_n J_{B_1}(x_n) + (1-\beta_n) J_{B_1} V \Pi_C \left(J_{B_1}^{-1}(J_{B_1}(x_n) - \gamma A^* J_{B_2}(I-S)Ax_n)\right)\right)$$

for all $n \in \mathbb{N}$, where $\beta_n \in (0,1)$ such that $0 < c \leq \beta_n \leq d < 1$ and $\gamma \in \left(0, \frac{1}{\|A\|^2}\right)$. If J_{B_1} is weakly sequentially continuous, then the sequence $\{x_n\}$ generated by (4.63) converges weakly to an element $z \in \Phi$, where $\Phi = \{z \in \mathrm{Fix(V)} : \mathrm{Az} \in \mathrm{Fix(S)}\}$.

Proof. A generalized nonspreading mapping $V : C \to C$ is nonspreading by Remark 2.1. Also, the set of fixed points of nonspreading mapping T is closed and convex [19]. Furthermore, putting $K = partiali_C$ in Theorem 3.2, we have that $J_\lambda^K = \Pi_C$ for all $\lambda > 0$. Since Π_C is strongly relative nonexpansive [18, Lemma 2.4 and Theorem 5.2], therefore the desired result follows from the arguments given in the proof of Theorem 3.2. □

Now, we apply our results to the equilibrium problems.

Let C be a nonempty closed convex subset of a uniformly smooth, strictly convex and reflexive Banach space B, and $f : C \times C \to \mathbb{R}$ be a bifunction. The equilibrium problem is to find $x \in C$ such that

$$f(x,y) \geq 0, \quad \text{for all } y \in C. \tag{4.64}$$

The set of solutions of (4.64) is denoted by $\mathrm{EP}(f)$. For solving the equilibrium problem, let us assume that the bifunction f satisfies the following conditions

- (A1) $f(x,x) = 0$ for all $x \in C$;
- (A2) f is monotone, i.e., $f(x,y) + f(y,x) \leq 0$ for all $x, y \in C$;
- (A3) $\lim_{t \downarrow 0} f(tz + (1-t)x, y) \leq f(x,y)$ for all $x, y, z \in C$;
- (A4) for each $x \in B$, $y \mapsto f(x,y)$ is convex and lower semicontinuous.

Takahashi and Zembayashi [29] obtained the following result.

Lemma 4.11. *Let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)–(A4). For $r > 0$, define a resolvent operator of f by $T_r : B \to C$ by*

$$T_r x = \left\{ z \in C : f(z,y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0 \, \textit{for all } y \in C \right\},$$

for all $x \in B$. Then the following assertions hold:

- (a) *T_r is single-valued;*
- (b) *T_r is a firmly nonexpansive-type mapping;*
- (c) $\mathrm{Fix}(T_r) = \mathrm{EP}(f)$;
- (d) $\mathrm{EP}(f)$ is closed and convex.

The following result is a special case of a result by Aoyama et al. [4, Theorem 3.5].

Lemma 4.12. *Let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)–(A4). Let $A_f : B \rightrightarrows B^*$ be a set-valued mapping defined by*

$$A_f(x) = \begin{cases} x^* \in B^* : f(x,y) \geq \langle y - x, x^* \rangle \text{ for all } y \in C, & \text{if } x \in C, \\ \emptyset, & \text{if } x \notin C. \end{cases} \tag{4.65}$$

Then, A_f is a maximal monotone operator with $D(A_f) \subseteq C$ and $EP(f) = A_f^{-1}0$. Furthermore, for $r > 0$, the resolvent T_r of f coincides with the resolvent $(J + rA_f)^{-1}J$ of A_f, that is,

$$T_r(x) = (J + rA_f)^{-1}J(x) \tag{4.66}$$

As a consequence of Theorem 3.2, we have the following results.

Theorem 4.5. *Let B_1 and B_2 be uniformly convex and 2-uniformly smooth real Banach spaces having smoothness constant κ satisfying $0 < \kappa \leq \frac{1}{\sqrt{2}}$. Let C be a nonempty closed convex subset of B_1, $f : C \times C \to \mathbb{R}$ satisfy the conditions (A1)–(A4), and T_λ denote the resolvent of A_f (as defined in (4.66)) of index $\lambda > 0$. Let $A : B_1 \to B_2$ be a bounded linear operator and $S : B_2 \to B_2$ be a given nonexpansive mapping such that* $\mathrm{Fix}(S) \neq \emptyset$. *For any $x_1 = x \in B_1$, define*

$$x_{n+1} = J_{B_1}^{-1}\left(\beta_n J_{B_1}(x_n) + (1-\beta_n) J_{B_1} T_\lambda \left(J_{B_1}^{-1}(J_{B_1}(x_n) - \gamma A^* J_{B_2}(I - S)Ax_n)\right)\right) \tag{4.67}$$

for all $n \in \mathbb{N}$, where $\beta_n \in (0,1)$ such that $0 < c \leq \beta_n \leq d < 1$ and $\gamma \in \left(0, \frac{1}{\|A\|^2}\right)$. If J_{B_1} is weakly sequentially continuous, then the sequence $\{x_n\}$ generated by (4.67) converges weakly to an element $z \in \Xi$, where $\Xi = \{z \in \mathrm{EP}(f) : Az \in \mathrm{Fix}(S)\}$.

Proof. Putting $V \equiv I$ and $K \equiv A_f$ in Theorem 3.2, we have that $J_\lambda^K \equiv T_\lambda$ for all $\lambda > 0$. Since T_λ is firmly nonexpansive type, so by [18, Theorem 5.2], it is strongly relative nonexpansive. Thus the result follows from the arguments given in the proof of Theorem 3.2. □

Theorem 4.6. *Let B_1 and B_2 be uniformly convex and 2-uniformly smooth real Banach spaces having smoothness constant κ satisfying $0 < \kappa \leq \frac{1}{\sqrt{2}}$. Let C be a nonempty closed convex subset of B_1. Let $f : C \times C \to \mathbb{R}$ satisfy the conditions (A1)–(A4), and T_λ denote the resolvent of A_f (as defined in (4.66)) of index $\lambda > 0$. Let $A : B_1 \to B_2$ be a bounded linear operator, $S : B_2 \to B_2$ be a given nonexpansive mapping such that* $\mathrm{Fix}(S) \neq \emptyset$, *and $V : C \to C$ be a generalized nonspreading mapping such that* $\mathrm{Fix}(V) \neq \emptyset$. *For any $x_1 = x \in C$, define*

$$x_{n+1} = J_{B_1}^{-1}\left(\beta_n J_{B_1}(x_n) + (1-\beta_n) J_{B_1} V T_\lambda \left(J_{B_1}^{-1}(J_{B_1}(x_n) - \gamma A^* J_{B_2}(I - S)Ax_n)\right)\right) \tag{4.68}$$

for all $n \in \mathbb{N}$, where $\beta_n \in (0,1)$ such that $0 < c \leq \beta_n \leq d < 1$ and $\gamma \in \left(0, \frac{1}{\|A\|^2}\right)$. If J_{B_1} is weakly sequentially continuous, then the sequence $\{x_n\}$ generated by (4.68) converges weakly to an element $z \in \{z \in \mathrm{EP}(f) \cap \mathrm{Fix}(V) : Az \in \mathrm{Fix}(S)\}$.

Proof. Putting $K \equiv A_f$ in Theorem 3.2, we have that $J_\lambda^K \equiv T_\lambda$ for all $\lambda > 0$. Hence the conclusion follows from Theorem 3.2. □

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