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A new approach to optimality in a class of nonconvex smooth optimization problems

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ABSTRACT. In this paper, a new approximation method for a characterization of optimal solutions in a class of nonconvex differentiable optimization problems is introduced. In this method, an auxiliary optimization problem is constructed for the considered nonconvex extremum problem. The equivalence between optimal solutions in the considered differentiable extremum problem and its approximated optimization problem is established under (Φ, ρ) -invexity hypotheses.

1. INTRODUCTION

In optimization, several methods give a characterization of optimality for a constrained extremum problem by the help of solving an auxiliary optimization problem associated to the original extremum problem. These methods rely on the relationship between the minimum point in the original mathematical programming problem and the optimal solution in its associated approximated optimization problem. In recent years, considerable attention has been given to devising new methods by the help of which solvability of the original mathematical programming problem is characterized by solvability of its associated optimization problem (see, for example, [1], [2], [3], [7], [9], [10], [11], [12], [14], and others).

The aim of the present work is to introduce a new method of a characterization of optimal solutions in the considered nonconvex differentiable optimization problem. By using the new approximation technique, we construct at the given feasible solution $\overline{x}$ an auxiliary optimization problem for the original nonconvex differentiable minimization problem. We prove that if the functions constituting the considered differentiable optimization problem are (Φ, ρ)-invex functions (not necessarily with respect to the same ρ), then there is a equivalence between an optimal solution $\overline{x}$ in the original optimization problem and a minimizer $\overline{x}$ in its associated optimization problem constructed in the introduced approximation method. This result is illustrated by an example of a nonconvex optimization problem involving (Φ, ρ)-invex functions. Further, it turns out that, in some cases, an auxiliary optimization problem constructed in the introduced approximation method is linear or convex although the original optimization problem is nonlinear or nonconvex. Also such a case is illustrated by an example of a nonconvex optimization problem involving (Φ, ρ)-invex functions.

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2. (Φ,ρ)-INVEXITY NOTION AND THE CONSIDERED CONSTRAINED OPTIMIZATION PROBLEM

In this section, we provide some definitions and some results that we shall use in the sequel. Let us assume that X is a nonempty open subset of R^n. We now give the definition of a differentiable (Φ,ρ)-invex function introduced by Caristi et al. [8].

Definition 2.1. Let $f: X \to R$ be a differentiable function on X and $u \in X$. If there exist a function $\Phi: X \times X \times R^{n+1} \to R$, where $\Phi(x,u,(\cdot,\cdot))$ is convex on R^{n+1}, $\Phi(x,u,(0,a)) \geq 0$ for every $x \in X$ and any $a \in R_+$, and a real number ρ such that the following inequality

$$f(x) - f(u) \geq \Phi(x,u,(\nabla f(u),\rho)) \quad (>), \tag{2.1}$$

holds for all $x \in X$, then f is said to be (Φ,ρ)-invex (strictly (Φ,ρ)-invex) at u on X. If inequality (2.1) is satisfied at any point u, then f is said to be a (Φ,ρ)-invex (strictly (Φ,ρ)-invex) function on X.

Remark 2.1. Note that the concept of (Φ,ρ)-invexity generalizes and extends several generalized convexity notions, earlier introduced in the literature (see Remark 2.1 [4] in the nondifferentiable case). For other properties of a class of scalar differentiable (Φ,ρ)-invex functions, the readers are advised to consult Caristi et al.[8].

In the paper, consider the following differentiable constrained optimization problem:

$$\begin{array}{c} f(x) \to \min \\ \text{s.t. } g_j(x) \leq 0, \;\; j \in J = \{1,...,m\}, \;\; x \in X, \end{array} \tag{P}$$

where $f: X \to R$ and $g_j: X \to R$, $j \in J$, are differentiable functions on a nonempty open convex set $X \subseteq R^n$. Let $D := \{x \in X : g_j(x) \leq 0, \; j \in J\}$ be the set of all feasible solutions for the problem (P). Further, we denote by $J(\overline{x}) = \{j \in J : g_j(\overline{x}) = 0\}$ the set of indices of inequality constraints that are active at the arbitrary feasible point $\overline{x} \in D$.

The generalized Slater constraint qualification. It is said that the generalized Slater constraint qualification is satisfied at $\overline{x} \in D$ for the considered optimization problem (P) if there exists a feasible solution $\widetilde{x}$ such that $g_j(\widetilde{x}) < 0$, $j \in J(\overline{x})$ and, moreover, g_j, $j \in J(\overline{x})$, are $\left(\Phi, \rho_{g_j}\right)$-invex at $\overline{x}$ on D.

Theorem 2.1. *Let $\overline{x}$ be an optimal solution of the considered optimization problem (P) and the generalized Slater constraint qualification be satisfied at $\overline{x}$. Then, there exist Lagrange multipliers $\overline{\xi}_j \geq 0, \; j = 1,...,m$, such that*

$$\nabla f(\overline{x}) + \sum_{j=1}^{m} \overline{\xi}_j \nabla g_j(\overline{x}) = 0, \tag{2.2}$$

$$\overline{\xi}_j g_j(\overline{x}) = 0, \quad j = 1,...,m. \tag{2.3}$$

3. AN AUXILIARY OPTIMIZATION PROBLEM AND OPTIMALITY

In this section, for the considered optimization problem (P), we introduce a definition of an auxiliary optimization problem $(\mathrm{P}_{(\Phi,\rho)}(\overline{x}))$ at the given feasible solution $\overline{x} \in D$. Further, we prove the equivalence between optimization problems (P) and $(\mathrm{P}_{(\Phi,\rho)}(\overline{x}))$ under assumption that the functions constituting the problem (P) are (Φ,ρ)-invex at $\overline{x}$ on D (with respect to, not necessarily, the same ρ).

Let $\overline{x}$ be the given feasible solution of the considered minimization problem (P). Further, assume that there exist $\Phi : X \times X \times R^{n+1} \to R$, where $\Phi(x,\overline{x},(\cdot,\cdot))$ is convex on R^{n+1}, $\Phi\left(x,\overline{x},(0,a)\right) \geq 0$ for all $x \in X$ and any $a \in R_+$ and, moreover, real numbers ρ_f, $\rho_{g_1},...,\rho_{g_m}$. We denote $\rho = (\rho_f, \rho_{g_1},...,\rho_{g_m}) \in R^{m+1}$. Then, for the considered minimization problem (P), we define the following approximation optimization problem $(\mathrm{P}_{(\Phi,\rho)}(\overline{x}))$ as follows:

$$\begin{gathered} f(\overline{x}) + \Phi\left(x,\overline{x},\left(\nabla f\left(\overline{x}\right),\rho_f\right)\right) \to \min \\ g_j\left(\overline{x}\right) + \Phi\left(x,\overline{x},\left(\nabla g_j\left(\overline{x}\right),\rho_{g_j}\right)\right) \leq 0, \;\; j \in J, \\ x \in X. \end{gathered} \tag{$\mathrm{P}_{(\Phi,\rho)}(\overline{x})$}$$

Further, let us denote by $\Omega_{\overline{x}}$ the set of all feasible solutions of the problem $(\mathrm{P}_{(\Phi,\rho)}(\overline{x}))$. First, we prove that, if the constraint function g is (Φ,ρ_g)-invex at $\overline{x}$ on D, then any feasible solution in the original constrained minimization problem is also feasible in its associated approximated optimization problem $(\mathrm{P}_{(\Phi,\rho)}(\overline{x}))$.

Lemma 3.1. *Let $\overline{x}$ be a given feasible solution of the considered optimization problem (P). Assume, furthermore, that each g_j, $j \in J$, is $\left(\Phi,\rho_{g_j}\right)$-invex at $\overline{x}$ on D. Then any feasible solution in the considered optimization problem (P) is also feasible in its associated approximated optimization problem $(\mathrm{P}_{(\Phi,\rho)}(\overline{x}))$.*

Proof. Proof of this lemma follows directly from Definition 2.1. □

Theorem 3.2. *Let $\overline{x}$ be a feasible solution in the considered minimization problem (P) and the Karush-Kuhn-Tucker optimality conditions (2.2)-(2.3) be satisfied at $\overline{x}$ with Lagrange multipliers $\overline{\xi}_j \geq 0,\ j \in J$. Further, assume that there exist $\Phi : X \times X \times R^{n+1} \to R$, where $\Phi\left(x,\overline{x},(\cdot,\cdot)\right)$ is convex on R^{n+1}, $\Phi\left(\overline{x},\overline{x},(\cdot,\cdot)\right) = 0$, $\Phi\left(x,\overline{x},(0,a)\right) \geq 0$ for every $x \in X$ and any $a \in R_+$ and real numbers ρ_f, $\rho_{g_1},...,\rho_{g_m}$ such that $\rho_f + \sum_{j=1}^m \overline{\xi}_j \rho_{g_j} \geq 0$. Then $\overline{x}$ is an optimal solution in an approximated optimization problem $(\mathrm{P}_{(\Phi,\rho)}(\overline{x}))$ associated to the problem (P).*

Proof. Let $\overline{x}$ be the given feasible solution of the considered minimization problem (P) and the Karush-Kuhn-Tucker optimality conditions (2.2)-(2.3) be satisfied at $\overline{x}$ with Lagrange multipliers $\overline{\xi}_j \geq 0,\ j \in J$,. Further, assume that there exist $\Phi : X \times X \times R^{n+1} \to R$, where $\Phi\left(x,\overline{x},\cdot\right)$ is convex on R^{n+1}, $\Phi\left(\overline{x},\overline{x},(\cdot,\cdot)\right) = 0$, $\Phi\left(x,\overline{x},(0,a)\right) \geq 0$ for every $x \in X$ and any $a \in R_+$ and real numbers ρ_f, $\rho_{g_1},...,\rho_{g_m}$ such that

$$\rho_f + \sum_{j=1}^m \overline{\xi}_j \rho_{g_j} \geq 0. \tag{3.4}$$

Let us denote

$$\overline{\mu}_0 = \frac{1}{1+\sum_{j=1}^m \overline{\xi}_j},\ \overline{\mu}_j = \frac{\overline{\xi}_j}{1+\sum_{j=1}^m \overline{\xi}_j},\ j = 1,...,m. \tag{3.5}$$

By (3.5), it follows that

$$\overline{\mu}_0 + \sum_{j=1}^m \overline{\mu}_j = 1. \tag{3.6}$$

Using (3.5) in the Karush-Kuhn-Tucker optimality condition (2.2), we get

$$\overline{\mu}_0 \nabla f(\overline{x}) + \sum_{j=1}^m \overline{\mu}_j \nabla g_j(\overline{x}) = 0. \tag{3.7}$$

Combining (3.4) and (3.5), we obtain

$$\overline{\mu}_0 \rho_f + \sum_{j=1}^m \overline{\mu}_j \rho_{g_j} \geq 0. \tag{3.8}$$

By assumption, $\Phi(x, \overline{x}, (0, a)) \geq 0$ for every $x \in X$. Hence, by (3.7) and (3.8), it follows, therefore, that the inequality

$$\Phi\left(x, \overline{x}, \left(\overline{\mu}_0 \nabla f(\overline{x}) + \sum_{j=1}^{m} \overline{\mu}_j \nabla g_j(\overline{x}), \overline{\mu}_0 \rho_f + \sum_{j=1}^{m} \overline{\mu}_j \rho_{g_j}\right)\right) \geq 0 \tag{3.9}$$

is satisfied for all $x \in X$. By assumption, $\Phi(x, \overline{x}, (\cdot, \cdot))$ is convex on R^{n+1}. Thus, by (3.6) and the definition of a convex function, it follows that

$$\begin{aligned} &\Phi\left(x, \overline{x}, \left(\overline{\mu}_0 \nabla f(\overline{x}) + \textstyle\sum_{j=1}^{m} \overline{\mu}_j \nabla g_j(\overline{x}), \overline{\mu}_0 \rho_f + \sum_{j=1}^{m} \overline{\mu}_j \rho_{g_j}\right)\right) \leq \\ &\overline{\mu}_0 \Phi\left(x, \overline{x}, (\nabla f(\overline{x}), \rho_f)\right) + \textstyle\sum_{j=1}^{m} \overline{\mu}_j \Phi\left(x, \overline{x}, \left(\nabla g_j(\overline{x}), \rho_{g_j}\right)\right). \end{aligned} \tag{3.10}$$

By (3.9) and (3.10), it follows that the inequality

$$\overline{\mu}_0 \Phi\left(x, \overline{x}, (\nabla f(\overline{x}), \rho_f)\right) + \sum_{j=1}^{m} \overline{\mu}_j \Phi\left(x, \overline{x}, \left(\nabla g_j(\overline{x}), \rho_{g_j}\right)\right) \geq 0 \tag{3.11}$$

holds for all $x \in X$. Therefore, by (3.5), (3.11) implies that the following inequality

$$\Phi\left(x, \overline{x}, (\nabla f(\overline{x}), \rho_f)\right) + \sum_{j=1}^{m} \overline{\xi}_j \Phi\left(x, \overline{x}, \left(\nabla g_j(\overline{x}), \rho_{g_j}\right)\right) \geq 0 \tag{3.12}$$

holds for all $x \in X$. We proceed by contradiction. Suppose, contrary to the result, that $\overline{x}$ is not optimal of problem ($P_{(\Phi,\rho)}(\overline{x})$). Then, there exists $\widetilde{x} \in X$ satisfying the constraint of ($P_{(\Phi,\rho)}(\overline{x})$), that is,

$$g_j(\overline{x}) + \Phi\left(\widetilde{x}, \overline{x}, \left(\nabla g_j(\overline{x}), \rho_{g_j}\right)\right) \leq 0, \quad j \in J, \tag{3.13}$$

and, moreover,

$$f(\overline{x}) + \Phi\left(\widetilde{x}, \overline{x}, (\nabla f(\overline{x}), \rho_f)\right) < f(\overline{x}) + \Phi\left(\overline{x}, \overline{x}, (\nabla f(\overline{x}), \rho_f)\right). \tag{3.14}$$

By $\Phi(\overline{x}, \overline{x}, (\cdot, \cdot)) = 0$, (3.14) yields

$$\Phi\left(\widetilde{x}, \overline{x}, (\nabla f(\overline{x}), \rho_f)\right) < 0. \tag{3.15}$$

Multiplying (3.13) by $\overline{\xi}_j \geq 0$, $j \in J$, we get

$$\overline{\xi}_j g_j(\overline{x}) + \overline{\xi}_j \Phi\left(\widetilde{x}, \overline{x}, \left(\nabla g_j(\overline{x}), \rho_{g_j}\right)\right) \leq 0, \quad j \in J.$$

By the Karush-Kuhn-Tucker necessary optimality condition (2.3), it follows that

$$\overline{\xi}_j \Phi\left(\widetilde{x}, \overline{x}, \left(\nabla g_j(\overline{x}), \rho_{g_j}\right)\right) \leq 0, \quad j \in J. \tag{3.16}$$

Adding both sides of (3.15) and (3.16), we obtain that the following inequality

$$\Phi\left(\widetilde{x}, \overline{x}, (\nabla f(\overline{x}), \rho_f)\right) + \sum_{j=1}^{m} \overline{\xi}_j \Phi\left(\widetilde{x}, \overline{x}, \left(\nabla g_j(\overline{x}), \rho_{g_j}\right)\right) < 0,$$

holds, which contradicts (3.12). This means that $\overline{x}$ is optimal in the approximated optimization problem ($P_{(\Phi,\rho)}(\overline{x})$) associated to the problem (P) and completes the proof of this theorem. □

Remark 3.2. Note that we prove Theorem 3.2 without any (Φ, ρ)-invexity hypothesis imposed on the functions constituting the considered optimization problem (P).

The following result follows directly from Theorem 3.2.

Corollary 3.1. *Let $\overline{x}$ be an optimal solution of the considered extremum problem (P). Further, assume that all hypotheses of Theorem 3.2 are fulfilled. Then $\overline{x}$ is also optimal of the approximated optimization problem ($P_{(\Phi,\rho)}(\overline{x})$) associated to the problem (P).*

In order to prove the converse result, some (Φ, ρ)-invexity hypotheses are imposed on the functions constituting the considered optimization problem (P).

Theorem 3.3. *Let $\overline{x}$ be optimal in the approximated optimization problem ($P_{(\Phi,\rho)}(\overline{x})$) associated to the considered extremum problem (P). Further, assume that the objective function f is (Φ, ρ_f)-invex at $\overline{x}$ on D and each constraint function g_j, $j \in J$, is $\left(\Phi, \rho_{g_j}\right)$-invex at $\overline{x}$ on D. If $\Phi\left(\overline{x}, \overline{x}, (\cdot, \cdot)\right) = 0$, then $\overline{x}$ is optimal in the considered optimization problem (P).*

Proof. We proceed by contradiction. Suppose, contrary to the result, that $\overline{x}$ is not optimal in the considered optimization problem (P). Then there exists $\widetilde{x} \in D$ such that

$$f\left(\widetilde{x}\right) < f(\overline{x}). \tag{3.17}$$

By assumption, $\overline{x}$ is an optimal solution in an approximation optimization problem ($P_{(\Phi,\rho)}(\overline{x})$) associated to the problem (P). Then, the inequality

$$f\left(\overline{x}\right) + \Phi\left(x, \overline{x}, \left(\nabla f(\overline{x}), \rho_f\right)\right) \geq f\left(\overline{x}\right) + \Phi\left(\overline{x}, \overline{x}, \left(\nabla f(\overline{x}), \rho_f\right)\right) \tag{3.18}$$

is satisfied for all $x \in D$. By assumption, each constraint function g_j, $j \in J$, is $\left(\Phi, \rho_{g_j}\right)$-invex at $\overline{x}$ on D. Hence, by Lemma 3.1, it follows that $D \subseteq \Omega_{\overline{x}}$. Therefore, (3.18) is also satisfied for $x = \widetilde{x} \in D$. Thus, (3.18) yields

$$f\left(\overline{x}\right) + \Phi\left(\widetilde{x}, \overline{x}, \left(\nabla f(\overline{x}), \rho_f\right)\right) \geq f\left(\overline{x}\right) + \Phi\left(\overline{x}, \overline{x}, \left(\nabla f(\overline{x}), \rho_f\right)\right). \tag{3.19}$$

Since $\Phi\left(\overline{x}, \overline{x}, (\cdot, \cdot)\right) = 0$, the inequality (3.19) gives

$$\Phi\left(\widetilde{x}, \overline{x}, \left(\nabla f(\overline{x}), \rho_f\right)\right) \geq 0. \tag{3.20}$$

By assumption, f is (Φ, ρ_f)-invex at $\overline{x}$ on X. Thus, by Definition 2.1, it follows that

$$f(\widetilde{x}) - f(\overline{x}) \geq \Phi\left(\widetilde{x}, \overline{x}, \left(\nabla f\left(\overline{x}\right), \rho_f\right)\right). \tag{3.21}$$

By (3.20) and (3.21), it follows that the inequality $f\left(\widetilde{x}\right) \geq f(\overline{x})$ holds, contradicting (3.17). This means that $\overline{x}$ is optimal in the considered optimization problem (P). □

Example 3.1. Consider the following nonconvex optimization problem

$$\begin{aligned} f(x) = \ln\left(x_1^2 + x_2^2 + 1\right) + e^{x_1^2} + x_1 + e^{-x_1}\arctan\left(x_2^2\right) + x_2^2 + e^{x_2} &\to \min \\ g_1(x) = -x_1 x_2 &\leq 0, \\ g_2(x) = -\arctan\left(x_1\right) &\leq 0, \\ g_3(x) = -\arctan\left(x_2\right) &\leq 0. \end{aligned} \tag{P1}$$

Note that $D = \left\{x = (x_1, x_2) \in R^2 : -\arctan\left(x_1\right) \leq 0 \wedge -\arctan\left(x_2\right) \leq 0 \wedge -x_1 x_2 \leq 0\right\}$ and $\overline{x} = (0, 0)$ is optimal in the problem (P1). Further, it can be proved by Definition 2.1 that the objective function f is (Φ, ρ_f)-invex at $\overline{x}$ on D and the constraint functions g_j, $j = 1, 2, 3$, are $\left(\Phi, \rho_{g_j}\right)$-invex at $\overline{x}$ on D, where

$$\Phi\left(x, \overline{x}, (\vartheta, \rho)\right) = \vartheta_1\left(x_1 - \overline{x}_1\right) + \vartheta_2\left(x_2 - \overline{x}_2\right) + \rho\left[\left(x_1 - \overline{x}_1\right)^2 + \left(x_2 - \overline{x}_2\right)^2\right],$$

$$\rho_f = 1, \ \rho_{g_1} = -\frac{1}{2}, \ \rho_{g_2} = -\frac{1}{2}, \ \rho_{g_3} = -\frac{1}{2}.$$

We use the approximation method presented in the paper for solving the considered optimization problem (P1). Therefore, as it follows from its definition, we construct the following approximation optimization problem ($\text{P1}_{(\Phi,\rho)}\left(\overline{x}\right)$) associated to problem (P1)

$$\begin{aligned} &2 + x_1 + x_2 + x_1^2 + x_2^2 \to \min \\ &-x_1 - \tfrac{1}{2}\left(x_1^2 + x_2^2\right) \leq 0, \\ &-x_2 - \tfrac{1}{2}\left(x_1^2 + x_2^2\right) \leq 0. \end{aligned} \qquad (\text{P1}_{(\Phi,\rho)}\left(\overline{x}\right))$$

Note that the Karush-Kuhn-Tucker necessary optimality conditions are satisfied at $\overline{x} = (0,0)$with the following Lagrange multipliers $\overline{\xi}_1 = 0$, $\overline{\xi}_2 = 1$, $\overline{\xi}_3 = 1$. Thus, $\rho_f + \sum_{j=1}^{3} \overline{\xi}_j \rho_{g_j} \geq 0$. Hence, all hypotheses of Theorem 3.2 are satisfied and, therefore, $\overline{x} = (0,0)$ is also optimal in the approximated optimization problem ($\text{P1}_{(\Phi,\rho)}\left(\overline{x}\right)$) defined above. Now, we consider the converse case. Namely, note that all hypotheses of Theorem 3.3 are also fulfilled. Therefore, $\overline{x} = (0,0)$, which is optimal of the problem ($\text{P1}_{(\Phi,\rho)}\left(\overline{x}\right)$), is also optimal in the considered optimization problem (P). Furthermore, by Theorem 1 [5], the constraint function g_1 is not invex at $\overline{x} = (0,0)$ on R^2 with respect to any function η defined by $\eta : R^2 \times R^2 \to R^2$. Therefore, it is not possible to use the η-approximation method introduced by Antczak [1] for solving invex optimization problems. Since the constraint function g_1 is not r-invex at $\overline{x} = (0,0)$ on R^2 with respect to any real number r, the η-approximation method can not be used also for solving such nonconvex optimization problems in which the involved functions are r-invex (see [2]).

In some cases, the approximated optimization problem constructed in the introduced approximation method for a nonlinear nonconvex extremum problem is linear or convex.

Example 3.2. Consider the following nonconvex optimization problem

$$\begin{aligned} f(x) &= \ln\left(x^2+1\right) + \arctan^2 x + x \to \min \\ g(x) &= x^2 + \arctan^2 x - \arctan x \leq 0. \end{aligned} \qquad \text{(P2)}$$

Note that $D = \left\{x \in R : x^2 + \arctan^2 x - \arctan x \leq 0\right\}$ and $\overline{x} = 0$ is a feasible solution of the problem (P2). Let us define

$$\Phi\left(x, \overline{x}, (\vartheta, \rho)\right) = \vartheta\left(x - \overline{x}\right) + \rho\left(x - \overline{x}\right)^2,$$

$$\rho_f = \frac{1}{2}, \ \rho_g = 1.$$

Note that the Karush-Kuhn-Tucker necessary optimality conditions are satisfied at $\overline{x} = 0$ with the Lagrange multiplier $\overline{\xi} = 1$. Thus, $\rho_f + \overline{\xi}\rho_g \geq 0$. Further, it can be shown by Definition 2.1 that the involved functions are (Φ, ρ)-invex at $\overline{x}$ on D. We use the approximation method introduced in the paper for solving the considered extremum problem (P2) and, therefore, we construct the following approximation optimization problem ($\text{P2}_{(\Phi,\rho)}\left(\overline{x}\right)$) associated to the considered minimization problem (P2)

$$\begin{aligned} f(x) &= \tfrac{1}{2}x^2 + x \to \min \\ g(x) &= x^2 - x \leq 0, \end{aligned} \qquad (\text{P2}_{(\Phi,\rho)}\left(\overline{x}\right))$$

which is convex. The property of the introduced approximation method illustrated in this example is important form the practical point of view since a complex nonconvex extremum problem can be replaced and, therefore, solved by the help of a convex (or linear) optimization problem.

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