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A new class of metric f-manifolds

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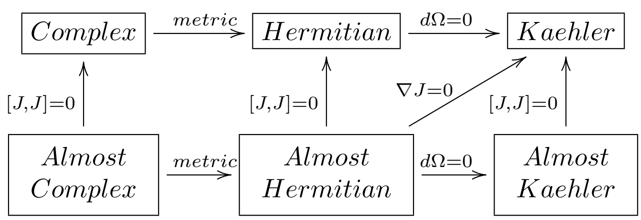
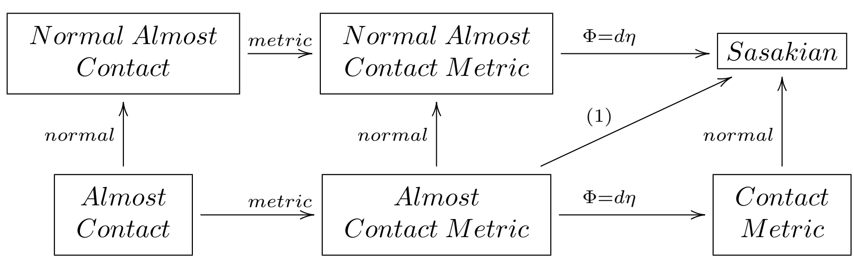
ABSTRACT. We introduce a new general class of metric f-manifolds which we call (almost) trans-S-manifolds and includes S-manifolds, C-manifolds, s-th Sasakian manifolds and generalized Kenmotsu manifolds studied previously. We prove their main properties and we present many examples which justify their study.

1. INTRODUCTION

In complex geometry, the relationships between the different classes of manifolds can be summarized in the well known diagram by Blair [3]:



In the case of contact geometry we have the diagram:



In the above diagram the almost contact structure (ϕ, η, ξ) is said to be normal if $[\phi, \phi] + 2d\eta \otimes \xi = 0$ and condition (1) is

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X,$$

for any tangent vector fields X and Y.

Moreover, an almost contact metric manifold (M, ϕ, ξ, η, g) is said to have an (α, β) trans-Sasakian structure if (see [11] for more details)

$$(\nabla_X \phi)Y = \alpha\{g(X, Y)\xi - \eta(Y)X\} + \beta\{g(\phi X, Y)\xi - \eta(Y)\phi X\}, \tag{1.1}$$

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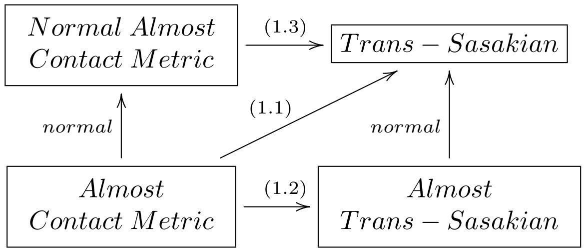
where α, β are differentiable functions (called characteristic functions) on M. Particular cases of trans-Sasakian manifolds are Sasakian ($\alpha = 1, \beta = 0$), cosymplectic ($\alpha = \beta = 0$) or Kenmotsu ($\alpha = 0, \beta = 1$) manifolds. In fact, we can extend the above diagram to



where

$$d\Phi = \Phi \wedge (\phi^*(\delta\Phi) - (\delta\eta)\eta), \quad d\eta = \frac{1}{2n}\{\delta\Phi(\xi)\Phi - 2\eta \wedge \phi^*(\delta\Phi)\}. \tag{1.2}$$

and:

$$d\Phi = \frac{-1}{n}\delta\eta(\Phi \wedge \eta), \quad d\eta = \frac{1}{2n}\delta\Phi(\xi)\Phi, \quad \phi^*(\delta\Phi) = 0. \tag{1.3}$$

More generally, K. Yano [15] introduced the notion of f-structure on a $(2n+s)$-dimensional manifold as a tensor field f of type (1,1) and rank $2n$ satisfying $f^3 + f = 0$. Almost complex ($s = 0$) and almost contact ($s = 1$) structures are well-known examples of f-structures. In this context, D. E. Blair [2] defined K-manifolds (and particular cases of S-manifolds and C-manifolds). Then, K-manifolds are the analogue of Kaehlerian manifolds in the almost complex geometry and S-manifolds (resp., C-manifolds) of Sasakian manifolds (resp., cosymplectic manifolds) in the almost contact geometry. Consequently, one can obtain a similar diagram for metric f-manifolds, that is, manifolds endowed with an f-structure and a compatible metric.

The purpose of the present paper is to introduce a new class of metric f-manifolds which generalizes the one of trans-Sasakian manifolds. In this context, we notice that there has been a previous generalization of $(\alpha, 0)$-trans-Sasakian manifolds for metric f-manifolds. It was due to I. Hasegawa, Y. Okuyama and T. Abe who introduced the so-called *homothetic s-contact Riemannian manifolds* in [8] as metric f-manifolds such that $2c_i g(fX, Y) = d\eta_i(X, Y)$ for certain nonzero constants c_i, $i = 1, \dots, s$ (actually, they use p instead of s). In particular, if the structure vector fields ξ_i are Killing vector fields and the f-structure is also normal, the manifold is called a *homothetic s-th Sasakian manifold*. They proved that a homothetic s-contact Riemannian manifold is a homothetic s-th Sasakian manifold if and only if

$$(\nabla_X f)Y = -\sum_{i=1}^{s} c_i\{g(fX, fY)\xi_i + \eta_i(Y)f^2X\},$$

and

$$\nabla_X \xi_i = c_i fX,$$

for any tangent vector fields X and Y and any $i = 1, \dots, s$.

More recently, M. Falcitelli and A. M. Pastore have introduced f-structures of Kenmotsu type as those normal f-manifolds with $dF = 2\eta^1 \wedge F$ and $d\eta^i = 0$ for $i = 1, \dots, s$ [5]. In this context, L. Bhatt and K. K. Dube [1] and A. Turgut Vanli and R. Sari [14] have studied a more general type of Kenmotsu f-manifolds for which all the structure 1-forms

η_i are closed and:

$$dF = 2\sum_{i=1}^{s}\eta_i \wedge F.$$

These examples motivate the idea of introducing the mentioned new more general class of metric f-manifolds, including the above ones, which we shall call *trans-S-manifolds* because trans-Sasakian manifolds become to be a particular case of them.

2. Metric f-manifolds

A $(2n+s)$-dimensional Riemannian manifold (M,g) endowed with an f-structure f (that is, a tensor field of type (1,1) and rank $2n$ satisfying $f^3+f=0$ [15]) is said to be a *metric f-manifold* if, moreover, there exist s global vector fields $\xi_1,\dots,\xi_s$ on M (called *structure vector fields*) such that, if $\eta_1,\dots,\eta_s$ are the dual 1-forms of $\xi_1,\dots,\xi_s$, then

$$f\xi_\alpha = 0;\ \eta_\alpha \circ f = 0;\ f^2 = -I + \sum_{\alpha=1}^{s}\eta_\alpha \otimes \xi_\alpha;$$

(2.4) $$g(X,Y) = g(fX,fY) + \sum_{i=1}^{s}\eta_i(X)\eta_i(Y),$$

for any $X,Y \in \mathcal{X}(M)$ and $i=1,\dots,s$. The distribution on M spanned by the structure vector fields is denoted by $\mathcal{M}$ and its complementary orthogonal distribution is denoted by $\mathcal{L}$. Consequently, $TM = \mathcal{L} \oplus \mathcal{M}$. Moreover, if $X \in \mathcal{L}$, then $\eta_\alpha(X) = 0$, for any $\alpha = 1,\dots,s$ and if $X \in \mathcal{M}$, then $fX = 0$.

For a metric f-manifold M we can construct very useful local orthonormal basis of tangent vector fields. To this end, let U be a coordinate neighborhood on M and X_1 any unit vector field on U, orthogonal to the structure vector fields. Then, fX_1 is another unit vector field orthogonal to X_1 and to the structure vector fields too. Now, if it is possible, we choose a unit vector field X_2 orthogonal to the structure vector fields, to X_1 and to fX_1. Then, fX_2 is also a unit vector field orthogonal to the structure vector fields, to X_1, to fX_1 and to X_2. Proceeding in this way, we obtain a local orthonormal basis $\{X_i, fX_i, \xi_j\}$, $i=1,\dots,n$ and $j=1,\dots,s$, called an *f-basis*.

Let F be the 2-form on M defined by $F(X,Y) = g(X,fY)$, for any $X,Y \in \mathcal{X}(M)$. Since f is of rank $2n$, then

$$\eta_1 \wedge \cdots \wedge \eta_s \wedge F^n \neq 0$$

and, in particular, M is orientable. A metric f-manifold is said to be a *metric f-contact manifold* if $F = d\eta_i$, for any $i=1,\dots,s$.

The f-structure f is said to be *normal* if

$$[f,f] + 2\sum_{i=1}^{s}\xi_i \otimes d\eta_i = 0,$$

where $[f,f]$ denotes the Nijenhuis tensor of f. If f is normal, then [7]

(2.5) $$[\xi_i,\xi_j] = 0,$$

for any $i,j=1,\dots,s$.

A metric f-manifold is said to be a K-manifold [2] if it is normal and $dF=0$. In a K-manifold M, the structure vector fields are Killing vector fields [2]. A K-manifold is called an S-manifold if $F = d\eta_i$, for any i and a C-manifold if $d\eta_i = 0$, for any i. Note that, for $s=0$, a K-manifold is a Kaehlerian manifold and, for $s=1$, a K-manifold is a quasi-Sasakian manifold, an S-manifold is a Sasakian manifold and a C-manifold is a cosymplectic manifold. When $s \geq 2$, non-trivial examples can be found in [2, 8]. Moreover, a K-manifold M is an S-manifold if and only if

$$\nabla_X \xi_i = -fX,\ X \in \mathcal{X}(M),\ i = 1, \dots, s, \tag{2.6}$$

and it is a C-manifold if and only if

$$\nabla_X \xi_i = 0,\ X \in \mathcal{X}(M),\ i = 1, \dots, s. \tag{2.7}$$

It is easy to show that in an S-manifold,

$$(\nabla_X f)Y = \sum_{i=1}^{s} \left\{ g(fX, fY)\xi_i + \eta_i(Y) f^2 X \right\}, \tag{2.8}$$

for any $X, Y \in \mathcal{X}(M)$ and in a C-manifold,

$$\nabla f = 0. \tag{2.9}$$

3. Definition of trans-S-manifolds and main properties

The original idea to define (α, β) trans-Sasakian manifolds is to generalize cosymplectic, Kenmotsu and Sasakian manifolds.

Kenmotsu: $d\eta = 0, normal$		
Cosymplectic: $d\Phi = 0, \quad d\eta = 0,$ normal	Quasi-Sasakian: $d\Phi = 0,$ normal	Trans-Sasakian: $d\Phi = 2\beta(\Phi \wedge \eta),$ $d\eta = \alpha\Phi,$ $\phi^*(\delta\Phi) = 0,$ normal
Sasakian: $\Phi = d\eta,$ normal		

In the same way, our idea is to define *trans-S-manifolds* generalizing C-manifolds, f-Kenmotsu and S-manifolds.

As we said in the Introduction, an almost contact manifold is trans-Sasakian if and only if it (1.1) holds. Then, we can introduce the notion of trans-S-manifold in a similar way.

Definition 3.1. A $(2n+s)$-dimensional metric f-manifold M is said to be a **almost trans-S-manifold** if it satisfies

$$\begin{aligned}(\nabla_X f)Y = \sum_{i=1}^{s} &\left[\alpha_i \{ g(fX, fY)\xi_i + \eta_i(Y) f^2 X \}\right. \\ &\left. + \ \beta_i \{ g(fX, Y)\xi_i - \eta_i(Y) fX \} \right],\end{aligned} \tag{3.10}$$

for certain smooth functions (called the **characteristic functions**) α_i, β_i, $i = 1....s$, on M and any $X, Y \in \mathcal{X}(M)$. If, moreover, M is normal, then it is said to be a **trans-S-manifold**.

So, if $s = 1$, a trans-S-manifold is actually a trans-Sasakian manifold. Furthermore, in this case, condition (3.10) implies normality. However, for $s \geq 2$, this does not hold. In fact, it is straightforward to prove that, for any $X, Y \in \mathcal{X}(M)$,

$$[f, f](X, Y) + 2\sum_{i=1}^{s} d\eta_i(X, Y)\xi_i = \sum_{i,j=1}^{s} \left[\eta_j(\nabla_X \xi_i)\eta_j(Y) - \eta_j(\nabla_Y \xi_i)\eta_j(X) \right] \xi_i, \tag{3.11}$$

which is not zero in general. But, in a trans-S-manifold, (3.11) implies that, for any $X \in \mathcal{X}(M)$ and any $i = 1, \ldots, s$:

$$\sum_{j=1}^{s} \eta_j(\nabla_X \xi_i)\eta_j(Y) - \sum_{j=1}^{s} \eta_j(\nabla_Y \xi_i)\eta_j(X) = 0.$$

If we put $Y = \xi_k$, from (2.5) we get that

$$\eta_k(\nabla_X \xi_i) = 0, \tag{3.12}$$

for any $i, k = 1, \ldots, s$. Using this fact, from (3.10), we deduce that

$$\nabla_X \xi_i = -\alpha_i f X - \beta_i f^2 X, \tag{3.13}$$

for any $X \in \mathcal{X}(M)$ and any $i = 1, \ldots, s$.

Now, we can prove:

Theorem 3.1. *A almost trans-S-manifold M is a trans-S-manifold if and only if (3.13) holds for any $X \in \mathcal{X}(M)$ and any $i = 1, \ldots, s$.*

Proof. From (3.10) we have that, for any $X \in \mathcal{X}(M)$ and any $i = 1, \ldots, s$:

$$\nabla_X \xi_i = -\alpha_i f X - \beta_i f^2 X + \sum_{j=1}^{s} \eta_j(\nabla_X \xi_i)\xi_j.$$

Comparing this equality and (3.13) we have that $\eta_j(\nabla_X \xi_i) = 0$, for any $i, j = 1, \ldots, s$. So, from (3.11), the metric f-manifold M is normal and, consequently, a trans-S-manifold. The converse is obvious. □

Observe that (3.10) can be re-written as

$$(\nabla_X F)(Y, Z) = \sum_{i=1}^{s} [\alpha_i \{g(fX, fZ)\eta_i(Y) - g(fX, fY)\eta_i(Z)\} + \beta_i \{g(X, fY)\eta_i(Z) - g(X, fZ)\eta_i(Y)\}],$$

for any $X, Y, Z \in \mathcal{X}(M)$. Then, if $X \in \mathcal{L}$ is a unit vector field, we have:

$$(\nabla_X F)(X, \xi_i) = -\alpha_i, \ (\nabla_X F)(\xi_i, fX) = \beta_i, \ i = 1, \ldots, s.$$

Moreover, from (3.13), we deduce

$$(\nabla_X \eta_i)Y = \alpha_i g(X, fY) + \beta_i g(fX, fY),$$

for any $X, Y \in \mathcal{X}(M)$ and any $i = 1, \ldots, s$. Again, if $X \in \mathcal{L}$ is a unit vector field, we get:

$$(\nabla_X \eta_i) fX = -\alpha_i, \ (\nabla_X \eta_i)X = \beta_i, \ i = 1, \ldots, s.$$

For trans-S-manifolds, we can prove the following theorem.

Theorem 3.2. *Let M be a trans-S-manifold. Then, $(\delta F)\xi_i = 2n\alpha_i$ and $\delta\eta_i = -2n\beta_i$, for any $i = 1, \ldots, s$.*

Proof. Taking a f-basis $\{X_1, \ldots, X_n, fX_1, \ldots, fX_n, \xi_1, \ldots, \xi_s\}$, since

$$\begin{aligned}(\delta F)X &= -\sum_{k=1}^{n} \{(\nabla_{X_k} F)(X_k, X) + (\nabla_{fX_k} F)(fX_k, X)\} \\ &\quad - \sum_{j=1}^{s} (\nabla_{\xi_j} F)(\xi_j, X) \\ &= \sum_{k=1}^{n} \{g(X_k, (\nabla_{X_k} \phi)X) + g(fX_k, (\nabla_{fX_k} \phi)X)\},\end{aligned}$$

for any $X \in \mathcal{X}(M)$, by using (3.10) it is straightforward to obtain

$$(\delta F)X = 2n\sum_{j=1}^{s}\alpha_j\eta_j(X) \tag{3.14}$$

and, putting $X = \xi_i$, it follows that $(\delta F)\xi_i = 2n\alpha_i$.

Moreover,

$$\delta\eta_i = -\sum_{k=1}^{n}\{(\nabla_{X_k}\eta_i)X_k + (\nabla_{fX_k}\eta_i)fX_k\} - \sum_{j=1}^{s}(\nabla_{\xi_j}\eta_i)\xi_j,$$

for any $i = 1,\dots,s$. But, from (3.12) we get that $(\nabla_{\xi_j}\eta_i)\xi_j = 0$, for any $j = 1,\dots,s$. Consequently, by using (3.13)

$$\begin{aligned}\delta\eta_i &= -\sum_{k=1}^{n}\{g(X_k,\nabla_{X_k}\xi_i) + g(fX_k,\nabla_{fX_k}\xi_i)\}\\ &= -\sum_{k=1}^{n}\beta_i\{g(X_k,X_k) + g(fX_k,fX_k)\} = -2n\beta_i.\end{aligned}$$

which concludes the proof. □

The above theorem generalizes the result given by D. E. Blair and J. A. Oubiña in [4] for trans-Sasakian manifolds. Moreover, trans-S-manifolds verify certain desirable conditions.

Proposition 3.1. *Let M be a trans-S-manifold. The following equations are verified:*

(i) $dF = 2F\wedge\sum_{i=1}^{s}\beta_i\eta_i$;
(ii) $d\eta_i = \alpha_iF$, $i = 1,\dots,s$;
(iii) $f^*(\delta F) = 0$.

Proof. From (3.10), a direct computation gives, for any $X, Y, Z \in \mathcal{X}(M)$:

$$\begin{aligned}dF(X,Y,Z) =& -g((\nabla_X f)Y,Z) + g((\nabla_Y f)X,Z) - g((\nabla_Z f)X,Y)\\ =&2\sum_{i=1}^{s}\{-\beta_i\eta_i(Z)g(fX,Y) + \beta_i\eta_i(Y)g(fX,Z) - \beta_i\eta_i(X)g(fY,Z)\}\\ =&2(F\wedge\sum_{i=1}^{s}\beta_i\eta_i)(X,Y,Z).\end{aligned}$$

Next, from (3.13) it is obtained the second statement. Finally, from (3.14) we get (iii). □

From (ii) of the above proposition we observe that if one of the functions α_i is a non-zero constant function, then the 2-form F is closed and the trans-S-manifold M is a K-manifold. Moreover we can prove:

Theorem 3.3. *A trans-S-manifold M is a K-manifold if and only if $\beta_1 = \cdots = \beta_s = 0$.*

Proof. Firstly, if all the functions β_i are equal to zero, from (i) of Proposition 3.1 we get $dF = 0$ and M is a K-manifold.

Conversely, it is known (see [6]) that, for K-manifolds, the following formula holds, for any $X, Y, Z \in \mathcal{X}(M)$:

$$g((\nabla_X f)Y,Z) = \sum_{i=1}^{s}\{d\eta_i(fY,X)\eta_i(Z) - d\eta_i(fZ,X)\eta_i(Y)\}.$$

Consequently, from (ii) of Proposition 3.1 and (3.10) we conclude that $\beta_i = 0$, for any $i = 1, \dots, s$. □

From Theorem 3.2 we deduce:

Corollary 3.1. *A trans-S-manifold M is a K-manifold if and only if $\delta\eta_i = 0$, for any $i = 1, \dots, s$.*

Furthermore, taking into account (2.6) and (2.7), we have:

Corollary 3.2. *Any trans-S-manifold is an S-manifold if and only if $\alpha_i = 1$, $\beta_i = 0$ and it is a C-manifold if and only if $\alpha_i = \beta_i = 0$, in both cases for any $i = 1, \dots, s$.*

In next section, we shall present some examples of trans-S-manifolds which are not K-manifolds due to the fact that not all their characteristic functions β_i are zero. Now, the natural question is if any K-manifold is a trans-S-manifold. In general, the answer is negative and to this end, we can consider the following example.

Let (N, J, G) be a Kaehler manifold, $(M, f, \xi_1, \dots, \xi_s, \eta_1, \dots, \eta_s, g)$ be an S-manifold and $\widetilde{M} = N \times M$.

If $\widetilde{X} = U + X, \widetilde{Y} = V + Y \in \mathcal{X}(\widetilde{M})$, where $U, V \in \mathcal{X}(N)$ and $X, Y \in \mathcal{X}(M)$, respectively, we can define a metric f-structure on $\widetilde{M}$ by the following structure elements:

$$\widetilde{f}(U + X) = JU + fX,\ \widetilde{\xi}_i = 0 + \xi_i,\ \widetilde{\eta}_i(U + X) = \eta_i(X),\ i = 1, \dots, s,$$

$$\widetilde{g}(U + X, V + Y) = G(U, V) + g(X, Y).$$

It is straightforward to check that $\widetilde{M}$ with this structure is a K-manifold. However, it is not a trans-S-manifold. In fact, since N is a Kaehler manifold and so, J is parallel, if ∇ and $\widetilde{\nabla}$ denote the Riemannian connections of M and $\widetilde{M}$, respectively, then

$$(\widetilde{\nabla}_{\widetilde{X}} \widetilde{f})\widetilde{Y} = 0 + (\nabla_X f)Y$$

and, consequently, (3.10) does not hold for $\widetilde{M}$.

However, we can observe that, from (2.8) and (2.9), the particular cases of S-manifolds and C-manifolds are trans-S-manifolds.

On the other hand, it is known [2] that, in a K-manifold, all the structure vector fields are Killing vector fields. For trans-S-manifolds we can prove:

Proposition 3.2. *Let M be a trans-S-manifold. Then, the structure vector field ξ_i is a Killing vector field if and only if the corresponding characteristic function $\beta_i = 0$.*

Proof. A direct computation by using (3.13) gives

$$(L_{\xi_i} g)(X, Y) = 2\beta_i g(fX, fY),$$

for any $X, Y \in \mathcal{X}(M)$. This completes the proof. □

4. Examples of trans-S-manifolds

As we have mentioned above, it is obvious that, from (2.8) and (2.9), S-manifolds and C-manifolds are trans-S-manifolds. Moreover, the homothetic s-th Sasakian manifolds of [8] are also trans-S-manifolds with the function α_i constant and $\beta_i = 0$, for any i.

From Propositions 2.2 and 2.5 of [5], we see that f-manifolds of Kenmotsu type, introduced by M. Falcitelli and A.M. Pastore, actually are trans-S-manifolds with functions $\alpha_1 = \cdots = \alpha_s = \beta_2 = \cdots = \beta_s = 0$ and $\beta_1 = 1$.

Also, from Theorem 2.4 in [14], we see that generalized Kenmotsu manifolds studied by L. Bhatt and K.K. Dube [1] and A. Turgut Vanli and R. Sari [14] are trans-S-manifolds with functions $\alpha_1 = \cdots = \alpha_s = 0$ and $\beta_1 = \cdots = \beta_s = 1$.

Then, we are going to look for examples with different non-constant functions α_i and β_i. We shall obtain these examples by using D-conformal deformations and warped products.

Firstly, given a metric f-manifold $(M, f, \xi_1, \dots, \xi_s, \eta_1, \dots, \eta_s, g)$, let us consider the *generalized D-conformal deformation* given by

$$\widetilde{f} = f, \quad \widetilde{\xi}_i = \frac{1}{a}\xi_i, \quad \widetilde{\eta}_i = a\eta_i, \quad \widetilde{g} = bg + (a^2 - b)\sum_{i=1}^{s} \eta_i \otimes \eta_i, \tag{4.15}$$

for any $i = 1, \dots, s$, where a, b are two positive differentiable functions on M. Then, it is easy to see that $(M, \widetilde{f}, \widetilde{\xi}_1, \dots, \widetilde{\xi}_s, \widetilde{\eta}_1, \dots, \widetilde{\eta}_s, \widetilde{g})$ is also a metric f-manifold. Let us notice that we can obtain conformal, D-homothetic (see [13]) or D-conformal (in the sense of S. Suguri and S. Nakayama [12]) deformations, by putting $a^2 = b$, $a = b = constant$ or $a = b$ in (4.15), respectively. In [9] Z. Olszack considered a and b constants, $a \neq 0$, $b > 0$ but not necesarily equal and he also called the resulting transformation a D-homothetic deformation.

Moreover, let us suppose that M is a trans-S-manifold and that a, b depend only on the directions of the structure vector fields ξ_i, $i = 1, \dots, s$. Therefore, we can calculate $\widetilde{\nabla}$ from ∇ and $\widetilde{g}$ by using Koszul's formula and (3.13). It follows that the Riemannian connection $\widetilde{\nabla}$ of $\widetilde{g}$ is given by

$$\begin{aligned}\widetilde{\nabla}_X Y = \nabla_X Y &+ \sum_{i=1}^{s} \frac{2(a^2 - b)\beta_i - \xi_i b}{2a^2} g(fX, fY)\xi_i \\ &- \frac{1}{2b}\{(Xb)f^2 Y + (Yb)f^2 X\} \\ &+ \frac{1}{2a^2}\sum_{i=1}^{s} \{(Xa^2)\eta_i(Y) + (Ya^2)\eta_i(X) \\ &- (\xi_i a^2)\sum_{j=1}^{s} \eta_j(X)\eta_j(Y)\}\,\xi_i \\ &- \frac{a^2 - b}{b}\sum_{i=1}^{s} \alpha_i\{\eta_i(Y)fX + \eta_i(X)fY\},\end{aligned} \tag{4.16}$$

for any vector fields $X, Y \in \mathcal{X}(M)$.

Theorem 4.4. *Let $(M, f, \xi_1, \dots, \xi_s, \eta_1, \dots, \eta_s, g)$ be a trans-S-manifold and consider a generalized D-conformal deformation on M, with a, b positive functions depending only on the directions of the structure vector fields. Then $(M, \widetilde{f}, \widetilde{\xi}_1, \dots, \widetilde{\xi}_s, \widetilde{\eta}_1, \dots, \widetilde{\eta}_s, \widetilde{g})$ is also a trans-S-manifold with functions:*

$$\widetilde{\alpha}_i = \frac{\alpha_i a}{b}, \ \widetilde{\beta}_i = \frac{\xi_i b}{2ab} + \frac{\beta_i}{a}, \ i = 1, \dots, s.$$

Proof. By using (4.16) and taking into account that b only depends on the directions of the structure vector fields, we have

$$\begin{aligned}(\widetilde{\nabla}_X \widetilde{f})Y = (\nabla_X f)Y &- \sum_{i=1}^{s} \frac{2(a^2 - b)\beta_i - \xi_i b}{2a^2} g(fX, Y)\xi_i \\ &- \frac{1}{2b}\sum_{i=1}^{s}(\xi_i b)\eta_i(Y)fX + \frac{a^2 - b}{b}\sum_{i=1}^{s} \alpha_i\eta_i(Y)f^2 X,\end{aligned}$$

for any $X, Y \in \mathcal{X}(M)$. Now, since M is trans-S-manifold, from (3.10) and (4.15) we obtain

$$(\widetilde{\nabla}_X \widetilde{f})Y = \sum_{i=1}^{s}\{\frac{\alpha_i a}{b}(\widetilde{g}(\widetilde{f}X, \widetilde{f}Y)\widetilde{\xi}_i + \widetilde{\eta}_i(Y)X) + \left(\frac{\xi_i b}{2ab} + \frac{\beta_i}{a}\right)(\widetilde{g}(\widetilde{f}X, Y)\widetilde{\xi}_i - \widetilde{\eta}_i(Y)\widetilde{f}X)\},$$

and this completes the proof. □

Note that if M is a Sasakian manifold, that is, if $s = 1$, $\alpha = 1$ and $\beta = 0$, this method does not produce an (α, β) trans-Sasakian manifold but an $(\alpha, 0)$ one because, by Darboux's theorem, if a, b only depend of the direction of ξ, they should be constants.

Corollary 4.3. *Let $(M, f, \xi_1, \ldots, \xi_s, \eta_1, \ldots, \eta_s, g)$ be an S-manifold and consider a generalized D-conformal deformation on M, with a, b positive functions depending only on the directions of the structure vector fields. Then $(M, \widetilde{f}, \widetilde{\xi}_1, \ldots, \widetilde{\xi}_s, \widetilde{\eta}_1, \ldots, \widetilde{\eta}_s, \widetilde{g})$ is a trans-S-manifold with functions:*

$$\widetilde{\alpha}_i = \frac{a}{b},\ \widetilde{\beta}_i = \frac{\xi_i b}{2ab},\ i = 1, \ldots, s.$$

Corollary 4.4. *Let $(M, f, \xi_1, \ldots, \xi_s, \eta_1, \ldots, \eta_s, g)$ be an C-manifold and consider a generalized D-conformal deformation on M, with a, b positive functions depending only on the directions of the structure vector fields. Then $(M, \widetilde{f}, \widetilde{\xi}_1, \ldots, \widetilde{\xi}_s, \widetilde{\eta}_1, \ldots, \widetilde{\eta}_s, \widetilde{g})$ is a trans-S-manifold with functions:*

$$\widetilde{\alpha}_i = 0,\ \widetilde{\beta}_i = \frac{\xi_i b}{2ab},\ i = 1, \ldots, s.$$

Corollary 4.5. *Let $(M, f, \xi_1, \ldots, \xi_s, \eta_1, \ldots, \eta_s, g)$ be a generalized Kenmotsu manifold and consider a generalized D-conformal deformation on M, with a, b positive functions depending only on the directions of the structure vector fields. Then $(M, \widetilde{f}, \widetilde{\xi}_1, \ldots, \widetilde{\xi}_s, \widetilde{\eta}_1, \ldots, \widetilde{\eta}_s, \widetilde{g})$ is a trans-S-manifold with functions:*

$$\widetilde{\alpha}_i = 0,\ \widetilde{\beta}_i = \frac{\xi_i b}{2ab} + \frac{1}{a},\ i = 1, \ldots, s.$$

Next, we are going to construct more examples of trans-S-manifolds by using warped products. For later use, we need the following lemma from [10] to compute the Riemannian connection of a warped product:

Lemma 4.1. *Let us consider $M = B \times_h F$ and denote by ∇, ∇^B and ∇^F the Riemannian connections on M, B and F. If X, Y are tangent vector fields on B and V, W are tangent vector fields on F, then:*

(i) $\nabla_X Y$ *is the lift of* $\nabla^B_X Y$.

(ii) $\nabla_X V = \nabla_V X = (Xh/h)V$.

(iii) The component of $\nabla_V W$ *normal to the fibers is:*

$$-(g_h(V, W)/h)\text{grad } h.$$

(iv) The component of $\nabla_V W$ *tangent to the fibers is the lift of* $\nabla^F_V W$.

In this context, if (N, J, G) is an almost Hermitian manifold, the warped product $\widetilde{M} = \mathbb{R}^s \times_h N$ can be endowed with a metric f-structure

$$(\widetilde{f}, \widetilde{\xi}_1, \ldots, \widetilde{\xi}_s, \widetilde{\eta}_1, \ldots, \widetilde{\eta}_s, g_h),$$

with the warped metric $g_h = -\pi^*(g_{\mathbb{R}^s}) + (h \circ \pi)^2\sigma^*(G)$, where $h > 0$ is a differentiable function on $\mathbb{R}^s$ and π and σ are the projections from $\mathbb{R}^s \times N$ on $\mathbb{R}^s$ and N, respectively. In fact, $\widetilde{f}(\widetilde{X}) = (J\sigma_*\widetilde{X})^*$, for any vector field $\widetilde{X} \in \mathcal{X}(\widetilde{M})$ and $\widetilde{\xi}_i = \partial/\partial t_i$, $i = 1, \ldots, s$, where t_i denotes the coordinates of $\mathbb{R}^s$. Note that this metric is the one used to construct the Robertson-Walker spaces (see [10]).

Now, we study the structure of this warped product.

Theorem 4.5. *Let N be an almost Hermitian manifold. Then, the warped product $(\widetilde{M} = \mathbb{R}^s \times_h N, \widetilde{f}, \widetilde{\xi}_1, \ldots, \widetilde{\xi}_s, \widetilde{\eta}_1, \ldots, \widetilde{\eta}_s, g_h)$ is a trans-S-manifold with functions $\widetilde{\alpha}_1 = \cdots = \widetilde{\alpha}_s = 0$ and $\widetilde{\beta}_i = h^{i)}/h$, $i = 1, \ldots, s$, if and only if N is a Kaehlerian manifold, where $h^{i)}$ are denoting the components of the gradient of the function h, for $i = 1, \ldots, s$.*

Proof. Consider $\widetilde{X} = U + X$ and $\widetilde{Y} = V + Y$, where U, V and X, Y are tangent vector fields on $\mathbb{R}^s$ and N, respectively. Then, taking into account Lemma 4.1, if $\widetilde{\nabla}$ and ∇^N denote the Riemannian connections of $\widetilde{M}$ and N, respectively, we have:

$$\begin{aligned}
(\widetilde{\nabla}_{\widetilde{X}} \widetilde{f})\widetilde{Y} =& \widetilde{\nabla}_U JX + \nabla_X JY \\
& - \widetilde{f}(\widetilde{\nabla}_U V + \widetilde{\nabla}_X V + \widetilde{\nabla}_U Y + \widetilde{\nabla}_X Y) \\
=& \frac{U(h)}{h} JY - \frac{g_h(X, JY)}{h} \operatorname{grad}(h) + \nabla^N_X JY \\
& - f(\nabla_U V + \frac{V(h)}{h} X + \frac{U(h)}{h} Y - \frac{g_h(X, Y)}{h} \operatorname{grad}(h) + \nabla_X Y) \\
=& - \frac{g_h(X, JY)}{h} \operatorname{grad}(h) - \frac{V(h)}{h} JX + (\nabla^N_X J)Y \\
=& \frac{g_h(JX, Y)}{h} \sum_{i=1}^s h^{i)} \widetilde{\xi}_i - \sum_{i=1}^s \widetilde{\eta}_i(V) \frac{h^{i)}}{h} JX + (\nabla^N_X J)Y \\
=& \frac{g_h(\widetilde{f}\widetilde{X}, \widetilde{Y})}{h} \sum_{i=1}^s h^{i)} \widetilde{\xi}_i - \sum_{i=1}^s \widetilde{\eta}_i(V) \frac{h^{i)}}{h} \widetilde{f}\widetilde{X} + (\nabla^N_X J)Y.
\end{aligned}$$

Therefore, (3.10) holds if and only if $(\nabla^N_U J)V = 0$, that is, if and only if N is a Kaehlerian manifold. Moreover, for any $i = 1, \ldots, s$,

$$\widetilde{\nabla}_{\widetilde{X}} \widetilde{\xi}_i = \nabla_U \widetilde{\xi}_i + \nabla_X \widetilde{\xi}_i = \frac{h^{i)}}{h} X = \frac{h^{i)}}{h} (\widetilde{X} - \sum_{i=1}^s \widetilde{\eta}(\widetilde{X}) \widetilde{\xi}_i) = -\frac{h^{i)}}{h} \widetilde{f}^2 \widetilde{X}$$

and then, Theorem 3.1 gives the result. □

Corollary 4.6. *The warped product $\mathbb{R}^s \times_h N$, being N a Kaehlerian manifold and h a constant function, is a C-manifold. In particular, if $h = 1$, the Riemannian product $\mathbb{R}^s \times N$ is a C-manifold.*

Combining these examples with a generalized D-conformal deformation, a great variety of non-trivial trans-S-manifolds can be presented.

Moreover, if we do the warped product of $\mathbb{R}^s$ with a $(2n + s_1)$-dimensional (almost) trans-S-manifold $(M, f, \xi_1, \ldots, \xi_{s_1}, \eta_1, \ldots, \eta_{s_1}, g)$, we obtain a new metric f-manifold

$$(\widetilde{M} = \mathbb{R}^s \times_h M, \widetilde{f}, \widetilde{\xi}_1, \ldots, \widetilde{\xi}_{s+s_1}, \widetilde{\eta}_1, \ldots, \widetilde{\eta}_{s+s_1}, g_h),$$

where $\widetilde{f}(\widetilde{X}) = (f\sigma_* \widetilde{X})^*$ and:

$$\widetilde{\xi}_i = \begin{cases} \dfrac{\partial}{\partial t_i} & \text{if} \quad 1 \leq i \leq s, \\ \\ \dfrac{1}{h} \xi_{i-s} & \text{if} \quad s + 1 \leq i \leq s + s_1. \end{cases}$$

These manifolds, under certain hypothesis about the function h, verify (3.10) but not (3.13), so from Theorem 3.1 they are not normal. Consequently, they are examples of almost trans-S-manifolds not trans-S-manifolds.

Theorem 4.6. *Let M be a $(2n+s_1)$-dimensional (almost) trans-S-manifold with characteristics functions (α_i, β_i), $i = 1, \ldots, s_1$. Then, the warped product $\widetilde{M} = \mathbb{R}^s \times_h M$, with the metric f-structure defined above, is a $(2n+s+s_1)$-dimensional almost trans-S-manifold with fuctions*

$$\widetilde{\alpha}_i = \begin{cases} 0 & \text{for} \quad i = 1, \ldots, s, \\ \dfrac{\alpha_{i-s}}{h} & \text{for} \quad i = s+1, \ldots, s+s_1. \end{cases}$$

and:

$$\widetilde{\beta}_i = \begin{cases} \dfrac{h^{i)}}{h} & \text{for} \quad i = 1, \ldots, s, \\ \dfrac{\beta_{i-s}}{h} & \text{for} \quad i = s+1 \ldots, s+s_1. \end{cases}$$

Proof. Consider $\widetilde{X} = U + X$ and $\widetilde{Y} = V + Y$, where U, V and X, Y are tangent vector fields on $\mathbb{R}^s$ and M, respectively. Then, taking into account Lemma 4.1, if ∇ is the Riemannian connection of M, we deduce:

$$\begin{aligned}(\nabla_{\widetilde{X}}\widetilde{f})\widetilde{Y} &= -\frac{g_h(X, fY)}{h}\text{grad}(h) - \frac{V(h)}{h} fX + (\nabla_X f)Y \\ &= \frac{g_h(fX, Y)}{h} \sum_{i=1}^{s} h^{i)} \frac{\partial}{\partial t_i} - \sum_{i=1}^{s} V(t_i) \frac{h^{i)}}{h} fX \\ &\quad + \sum_{i=s+1}^{s+s_1} \left[\alpha_{i-s}\left\{g(fX, fY)\xi_{i-s} + \eta_{i-s}(Y) f^2 X\right\}\right. \\ &\quad \left. + \beta_{i-s}\left\{g(fX, Y)\xi_{i-s} - \eta_{i-s}(Y) fX\right\}\right] \\ &= \sum_{i=1}^{s} \frac{h^{i)}}{h}\{g_h(\widetilde{f}\widetilde{X}, \widetilde{Y})\widetilde{\xi}_i - \widetilde{\eta}_i(\widetilde{Y})\widetilde{f}\widetilde{X}\} \\ &\quad + \sum_{i=s+1}^{s+s_1} \left[\frac{\alpha_{i-s}}{h}\{g_h(\widetilde{f}\widetilde{X}, \widetilde{f}\widetilde{Y})\widetilde{\xi}_{i-s} + \widetilde{\eta}_{i-s}(\widetilde{Y})\widetilde{f}^2\widetilde{X}\}\right. \\ &\quad \left. + \frac{\beta_{i-s}}{h}\{g_h(\widetilde{f}\widetilde{X}, \widetilde{Y})\widetilde{\xi}_{i-s} - \widetilde{\eta}_{i-s}(\widetilde{Y})\widetilde{f}\widetilde{X}\}\right].\end{aligned}$$

Joining the addends appropriately, it takes the form of (3.10) with the desired functions. Therefore, $\widetilde{M}$ is a almost trans-S-manifold. □

Observe that, in the above conditions, (3.13) is not verified in general. In fact, consider $\widetilde{\xi}_i$ with $1 \leq i \leq s$. Then, for any $\widetilde{X} \in \mathcal{X}(\widetilde{M})$,

$$\widetilde{\nabla}_{\widetilde{X}}\widetilde{\xi}_i = \frac{h^{i)}}{h} U = \frac{h^{i)}}{h}(\widetilde{X} - \sum_{j=1}^{s} \widetilde{\eta}_j(\widetilde{X})\xi_j)$$

and so, if h is not a constant function, from Theorem 3.1, we get that $\widetilde{M}$ is not a trans-S-manifold.

Corollary 4.7. *The warped product $\mathbb{R}^s \times_h M$, being M a trans-S-manifold, is a trans-S-manifold if and only if h is constant. In particular, the Riemannian product $\mathbb{R}^s \times M$ is a trans-S-manifold with functions*

$$(0, \overset{s)}{\ldots}, 0, \alpha_1, \ldots, \alpha_{s_1}, 0, \overset{s)}{\ldots}, 0, \beta_1, \ldots, \beta_{s_1}),$$

where (α_i, β_i), $i = 1, \ldots, s_1$, denote the characteristic functions of M.

Corollary 4.8. *Let M be a Sasakian manifold. Then, the warped product $\mathbb{R} \times_h M$ is a almost trans-S-manifold with functions:*

$$\alpha_1 = 0,\ \alpha_2 = \frac{1}{h},\ \beta_1 = \frac{h'}{h} \text{ and } \beta_2 = 0.$$

Corollary 4.9. *Let M be a three dimensional trans-Sasakian, with non-constant characteristic functions α and β. Then, the warped product $\mathbb{R} \times_h M$ is a four dimensional almost trans-S-manifold not trans-S-manifold with functions:*

$$\alpha_1 = 0,\ \alpha_2 = \frac{\alpha}{h},\ \beta_1 = \frac{h'}{h} \text{ and } \beta_2 = \frac{\beta}{h}.$$

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