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Multiple positive solutions to a $(2m)$ th-order boundary value problem

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ABSTRACT. The aim of the present paper is to study the existence, localization and multiplicity of positive solutions for a $(2m)$th-order boundary value problem subject to the Dirichlet conditions. Our approach is based on critical point theory in conical shells and Harnack type inequalities.

1. INTRODUCTION

The aim of this paper is to obtain existence, localization and multiplicity results for positive solutions of the following two-point boundary value problem

$$\begin{cases} (-1)^m u^{(2m)}(t) = f(t,u(t)), & t \in (0,1), \\ u^{(i)}(0) = u^{(i)}(1) = 0, & i = 0, \ldots, m-1, \end{cases} \tag{1.1}$$

where m is an integer, $m \geq 1$.

Throughout this paper, we assume that $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with $f([0,1] \times \mathbb{R}^+) \subset \mathbb{R}^+$, and that there exist $\theta > 0$ and two nonnegative functions $a,\, b \in L^2(0,1)$ such that

$$|f(t,s)| \leq a(t) + b(t)\,|s|^{\theta} \quad \text{for all } s \in \mathbb{R} \text{ and a.e. } t \in [0,1]. \tag{1.2}$$

Higher order two-point boundary value problems have been extensively studied in the literature by using fixed point techniques. For example, Graef and Henderson [6] have studied $(2m)$th-order focal boundary value problems and they have proved the existence of at least two positive solutions by using a fixed point theorem owed to Avery and Henderson [1]. Also, Shi and Chen [12] have studied a singular boundary value problem of Lidstone type where the nonlinearity is superlinear and depends on the derivatives of even order, proving the existence of a positive solution by using Krasnosel'skii's compression-expansion theorem. The same equation has been studied by Chyan and Henderson [5], but under different boundary conditions and for both superlinear and sublinear nonlinearities. Krasnosel'skii's fixed point theorem has been also used by Liu [8] for $(2m)$th-order equations with Dirichlet boundary conditions. Multiple symmetric positive solutions for a class of Lidstone-type higher order boundary value problems have been obtained by Graef, Qian and Yang [7], while concave solutions for a $(2m)$th-order problem with Dirichlet conditions have been obtained by Al Twaty and Eloe [13] using an extension of the Leggett-Williams fixed point theorem. Multiple positive solutions for $(p, n-p)$ focal boundary value problems has been presented in Chapter 10 of the book by O'Regan and Precup [9], using instead of fixed point index arguments, a more elementary approach based on essential and inessential mappings.

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A variational alternative to the fixed point approach for the localization and multiplicity of positive solutions has been recently proposed in [10], [11] and applied to several classes of nonlinear problems having a variational structure. Such an application is presented in [3] for a fourth order boundary value problem. We would like to emphasize the advantage of the variational approach compared to the fixed point one, which consists in underlining additional properties of the solutions with respect to the energy functional of the problem, for instance of being a local or a global minimum, or a saddle point of the energy functional.

The aim of this paper is to show that the variational technique of localization and multiplicity developed in [11] also applies to higher order boundary value problems. One of the main ingredients in this respect is a Harnack type inequality, which is helpful for establishing lower estimations of solutions.

The paper is organized as follows: In Section 2 we shortly present the abstract critical point theory for localization and multiplicity of solutions. Also we give the fixed point formulation of the boundary value problem in terms of Green's function, and the variational formulation of the problem. The main results are contains in Section 3. First we obtain a Harnack type inequality associated to the problem and then we state and prove the existence and localization in conical shells of two critical points (a minimum and a saddle point) of the energy functional, Theorem 3.3. As a consequence, we have the multiplicity result, Theorem 3.4. An example is given to illustrate the results.

2. Preliminaries

2.1. **Critical point theorems in conical shells.** We first recall some general critical point results from [11] which are the main tool of this paper.

For any real Hilbert space H with inner product $(\cdot,\cdot)_H$ and norm $|\cdot|_H$, let H' be its dual space. Denoting by $\langle\cdot,\cdot\rangle$ the duality between H and H', i.e. $\langle u^*, u\rangle = u^*(u)$ for $u^* \in H'$ and $u \in H$, according to the Riesz representation theorem, one can consider the canonical isomorphism $L_H : H \to H'$, given by

$$(u,v)_H = \langle L_H\, u, v\rangle \quad \text{for all } \; u,\; v \in H\,, \tag{2.3}$$

and its inverse $J_H : H' \to H$ for which

$$(J_H u, v)_H = \langle u, v\rangle \quad \text{for } \; u \in H'\,,\; v \in H\,.$$

Using this isomorphism, the spaces H with H', are identified by letting $L_H u \equiv u$, $J_H u \equiv u$, and so $L_H = J_H = I_H$ (identity map of H).

Consider now two real Hilbert spaces, X with inner product and norm $(\cdot,\cdot)_X, |\cdot|_X,$ and Y with inner product and norm $(\cdot,\cdot)_Y\,, |\cdot|_Y$; assume that X is continuously embedded into Y (with $|u|_Y \leq c_0\,|u|_X$, for $u \in X$) and that Y is identified with Y'. Then, from $X \subset Y$, one has $Y' \subset X'$, and therefore

$$X \subset Y \equiv Y' \subset X'.$$

Note that for every $u \in X$, the notation $J_X u$ is used to denote the element $J_X L_Y u$. Also, if $u, v \in Y$, then according to (2.3) and the identification $L_Y u = u$,

$$\langle u, v\rangle = (u,v)_Y\,.$$

This is the reason for using the symbol $\langle\cdot,\cdot\rangle$ instead of $(\cdot,\cdot)_Y$. In what follows, the inner product and norm will be denoted by $(\cdot,\cdot)$ and $|\cdot|$ for X, and by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ for Y. Also, the notations L and J are used instead of L_X and J_X.

Let K be a wedge in X, i.e. a convex closed nonempty set K, $K \neq \{0\}$, with $\lambda\, u \in K$ for every $u \in K$ and $\lambda \geq 0$. For any two positive numbers R_0 and R_1, denote by $K_{R_0 R_1}$

the *conical shell*

$$K_{R_0R_1} := \{u \in K \;:\; \|u\| \geq R_0 \text{ and } |u| \leq R_1\}.$$

Such a set may be empty (even if $R_0 < R_1$) and may be disconnected. Let $\phi \in K \setminus \{0\}$ be a fixed element with $|\phi| = 1$. If $R_0 < \|\phi\| R_1$, then $\mu\phi \in K_{R_0R_1}$ for every $\mu \in [R_0/\|\phi\|, R_1]$, and $\mu\phi$ is an interior point of $K_{R_0R_1}$, in the sense that $\|\mu\phi\| > R_0$ and $|\mu\phi| < R_1$, for $\mu \in (R_0/\|\phi\|, R_1)$. In particular, any two elements of $K_{R_0R_1}$ of the form $\mu\phi$, with $\mu \in [R_0/\|\phi\|, R_1]$, belong to the same connected component of $K_{R_0R_1}$.

Notice that if $Y = X$, the spaces X and X' are identified, so $L = J = I_X$. Also, in this case, the conical shell $K_{R_0R_1}$ is nonempty and simply connected for every R_0, R_1 with $0 < R_0 < R_1$.

Let E be a C^1 functional defined on X. The functional E is said to have a *mountain pass geometry* in $K_{R_0R_1}$ if there exist u_0 and u_1 in the same connected component of $K_{R_0R_1}$, and $r > 0$ such that $|u_0| < r < |u_1|$, and

$$\max\{E(u_0), E(u_1)\} < \inf\{E(u) : \; u \in K_{R_0R_1}, \; |u| = r\}.$$

In this case one considers the set

$$\Gamma = \{\gamma \in C([0,1]; K_{R_0R_1}) \;:\; \gamma(0) = u_0, \; \gamma(1) = u_1\}, \tag{2.4}$$

and the number

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} E(\gamma(t)). \tag{2.5}$$

Finally, the functional E is said to be *bounded from below* in $K_{R_0R_1}$ if

$$m := \inf_{u\in K_{R_0R_1}} E(u) > -\infty. \tag{2.6}$$

Consider the following

(c1): *Invariance condition*:

$$J(K) \subset K \text{ and } (I - JE')(K) \subset K, \quad (I \text{ is the identity map on } X).$$

(c2): *Boundedness condition on the shell boundary*: there exists a constant $\nu_0 > 0$ such that

$$(JE'(u), Ju) \leq \nu_0 \text{ for all } u \in K \text{ with } \|u\| = R_0;$$
$$(JE'(u), u) \geq -\nu_0 \text{ for all } u \in K \text{ with } |u| = R_1.$$

(c3): *Compactness condition*:

the maps J and $N = I - JE'$ are compact from X to itself.

(c4): *Compression condition*:

$$JE'(u) - \lambda Ju \neq 0 \text{ for } u \in K_{R_0R_1}, \; \|u\| = R_0, \; \lambda > 0; \tag{2.7}$$

$$JE'(u) + \lambda u \neq 0 \text{ for } u \in K_{R_0R_1}, \; |u| = R_1, \; \lambda > 0. \tag{2.8}$$

The following theorems give existence and localization of critical points in a conical shell.

Theorem 2.1. *Let the conditions (c1)-(c4) hold. If in addition E is bounded from below in $K_{R_0R_1}$ and that there is a $\rho > 0$ with*

$$E(u) \geq m + \rho$$

for all $u \in K_{R_0R_1}$ which simultaneously satisfy $|u| = R_1$, $\|u\| = R_0$, then there exists $u \in K_{R_0R_1}$ such that

$$E'(u) = 0 \quad \textit{and} \quad E(u) = m.$$

Theorem 2.2. *Let the conditions (c1)-(c4) hold. If in addition E has the mountain pass geometry in $K_{R_0R_1}$ and that there is a $\rho > 0$ with*

$$|E(u) - c| \geq \rho$$

for all $u \in K_{R_0R_1}$ which simultaneously satisfy $|u| = R_1$, $\|u\| = R_0$, then there exists $u \in K_{R_0R_1}$ such that

$$E'(u) = 0 \quad \text{and} \quad E(u) = c.$$

Remark 2.1. Since $m < c$, if the assumptions of both Theorems 2.1 and 2.2 are satisfied, then E has two distinct critical points in $K_{R_0R_1}$.

Repeated application of Theorems 2.1 and 2.2 to disjoint conical shells immediately gives the following multiplicity results.

Theorem 2.3. (1^0) *Let $(R_0^i)_{1\leq i\leq k}$, $(R_1^i)_{1\leq i\leq k}$ $(k \leq \infty)$ be increasing finite or infinite sequences with $0 < R_0^i < \|\phi\| R_1^i$ and $c_0R_1^i < R_0^{i+1}$ for all i. If the assumptions of Theorems 2.1 or 2.2 are satisfied in each $K_{R_0^iR_1^i}$, then E has k (or, when $k = \infty$, an infinite sequence of) distinct critical points u_i, with*

$$R_0^i \leq \|u_i\|, \quad |u_i| \leq R_1^i. \tag{2.9}$$

(2^0) *Let $(R_0^i)_{1\leq i\leq k}$, $(R_1^i)_{1\leq i\leq k}$ $(k \leq \infty)$ be decreasing finite or infinite sequences with $0 < R_0^i < \|\phi\| R_1^i$ and $c_0R_1^{i+1} < R_0^i$ for all i. If the assumptions of Theorems 2.1 or 2.2 are satisfied in each $K_{R_0^iR_1^i}$, then E has k (or, when $k = \infty$, an infinite sequence of) distinct critical points u_i satisfying (2.9).*

2.2. **Fixed point formulation.** According to [4] and [14], for each $v \in L^2(0,1)$, the problem

$$\begin{cases} (-1)^m u^{(2m)}(t) = v(t), & t \in (0,1), \\ u^{(i)}(0) = u^{(i)}(1) = 0, & i = 0,\dots,m-1, \end{cases} \tag{2.10}$$

has in $H^{2m}(0,1)$ a unique solution denoted by Sv, namely

$$(Sv)(t) = \int_0^1 G(t,s)v(s)ds, \quad t \in [0,1],$$

where $G : [0,1] \times [0,1] \to [0,1]$ is the corresponding Green's function

$$G(t,s) = \frac{1}{[(m-1)!]^2} \begin{cases} \int_0^{t(1-s)} \tau^{m-1}(\tau + s - t)^{m-1} d\tau, & \text{if } 0 \leq t \leq s \leq 1, \\ \int_0^{s(1-t)} \tau^{m-1}(\tau + t - s)^{m-1} d\tau, & \text{if } 0 \leq s \leq t \leq 1. \end{cases}$$

Then problem (1.1) is equivalent to the integral equation

$$u(t) = \int_0^1 G(t,s) f(s,u(s)) ds, \quad u \in C[0,1],$$

which is a fixed point equation in $C[0,1]$.

Note the following properties of Green's function:

Lemma 2.1. (a) $0 \leq G(t,s) \leq \frac{1}{[(m-1)!]^2} s^m(1-s)^m$, *for all* $t,\ s \in [0,1]$.

(b) $G(t,s) \geq \frac{1}{2m-1} t^m(1-t)^m \frac{1}{[(m-1)!]^2} s^m(1-s)^m$, *for all* $t,\ s \in [0,1]$.

2.3. **Variational formulation of the problem.** In order to apply Theorems 2.1 and 2.2 to our problem (1.1), let us take $X = H_0^m(0,1)$ endowed with the inner product and norm

$$(u,v) = \int_0^1 u^{(m)} v^{(m)} dt, \quad |u| = \left(\int_0^1 \left| u^{(m)} \right|^2 dt \right)^{\frac{1}{2}},$$

and $Y = L^2(0,1)$ with the inner product and norm

$$\langle u,v \rangle = \int_0^1 uvdt, \quad \|u\| = \left(\int_0^1 u^2 dt \right)^{\frac{1}{2}}.$$

Recall that (see [2])

$$H_0^m(0,1) \subset L^2(0,1) \subset H^{-m}(0,1),$$

with continuous embeddings, where $H^{-m}(0,1)$ is the dual space of $H_0^m(0,1)$. We denote by c_0 the embedding constant for $H_0^m(0,1) \subset L^2(0,1)$ and $L^2(0,1) \subset H^{-m}(0,1)$, i.e.,

$$\|u\| \le c_0 |u| \ \ (u \in H_0^m(0,1)) \ \text{ and } \ \|u\|_{H^{-m}(0,1)} \le c_0 \|u\| \ (u \in L^2(0,1)).$$

Also we consider the space $C[0,1]$ endowed with the norm

$$\|u\|_\infty = \max_{t \in [0,1]} |u(t)|,$$

and denote by c_∞ the embedding constant for $H_0^m(0,1) \subset C[0,1]$, i.e.,

$$\|u\|_\infty \le c_\infty |u| \ \ (u \in H_0^m(0,1)).$$

Remark 2.2. We can take for example

$$c_\infty = \frac{2^{m-1}}{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2m-1)}. \tag{2.11}$$

Indeed for every $u \in H_0^m(0,1)$ the Cauchy-Schwarz inequality gives

$$\begin{aligned} u(t) &= \int_0^t \int_0^{s_1} \cdots \int_0^{s_{m-1}} u^{(m)}(s_m)\, ds_m ds_{m-1} \ldots ds_1 \\ &\le |u| \int_0^t \int_0^{s_1} \cdots \int_0^{s_{m-2}} \sqrt{s_{m-1}} ds_{m-1} ds_{m-2} \ldots ds_1 \\ &\le \frac{2^{m-1}}{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2m-1)} |u|. \end{aligned} \tag{2.12}$$

In this specific case, $L : H_0^m(0,1) \to H^{-m}(0,1)$ is given by

$$\langle Lu, v \rangle = \langle (-1)^m u^{(2m)}, v \rangle = (u,v), \quad \text{for } u, v \in H_0^m(0,1), \tag{2.13}$$

and its inverse is the continuous operator $J : H^{-m}(0,1) \to H_0^m(0,1)$ defined by

$$(Jv, w) = \langle v, w \rangle, \quad \text{for } v \in H^{-m}(0,1), \ w \in H_0^m(0,1).$$

Notice that for each $v \in L^2(0,1)$ one has

$$(Jv)(t) = (Sv)(t), \ \ t \in (0,1).$$

The energy functional associated to boundary value problem (1.1) is given by

$$E(u) = \int_0^1 \left[\frac{1}{2} |u^{(m)}(t)|^2 - F(t, u(t)) \right] dt, \tag{2.14}$$

where $F(t,\cdot)$ is the primitive of $f(t,\cdot)$ which vanishes at zero.

Lemma 2.2. *Assume that the growth condition* (1.2) *holds. Then the Nemytskii operator $N_f(u)(t) = f(t,u(t))$ is well-defined, continuous and bounded from $C[0,1]$ into $L^2(0,1)$.*

Proof. Let $u \in C[0,1]$. By (1.2) and Young's inequality we have

$$\begin{aligned}\|N_f(u)\|^2 &= \int_0^1 f(t,u(t))^2 dt \\ &\leq \int_0^1 \left(a(t)+b(t)|u(t)|^\theta\right)^2 dt \\ &\leq 2\int_0^1 a(t)^2 dt + 2\int_0^1 b(t)^2|u(t)|^{2\theta} dt \\ &\leq 2\|a\|^2 + 2\|b\|^2\|u\|_\infty^{2\theta} < +\infty.\end{aligned}$$

This shows that N_f is well-defined and bounded from $C[0,1]$ into $L^2(0,1)$. To prove that N_f is continuous, let (u_n) be a sequence in $C[0,1]$ such that $u_n \to u$ in the $C[0,1]$-norm. Then there exists $R>0$ such that $|u_n(t)| \leq R$ for every $n \in \mathbb{N}$, $|u(t)| \leq R$ and $u_n(t) \to u(t)$ for every $t \in [0,1]$. Since $f(t,.)$ is continuous for a.e. $t \in [0,1]$, we have $f(t,u_n(t)) \to f(t,u(t))$ for a.e. $t \in [0,1]$. Then $N_f(u_n) \to N_f(u)$ pointwise a.e. Moreover, the growth condition (1.2) yields

$$\begin{aligned}|N_f(u_n)(t)-N_f(u)(t)|^2 &\leq 2\left(|N_f(u_n)(t)|^2+|N_f(u)(t)|^2\right) \\ &\leq 2\left\{\left(a(t)+b(t)|u_n(t)|^\theta\right)^2+\left(a(t)+b(t)|u(t)|^\theta\right)^2\right\} \\ &\leq 4\left\{a(t)^2+b(t)^2|u_n(t)|^{2\theta}+a(t)^2+b(t)^2|u(t)|^{2\theta}\right\} \\ &= 8a(t)^2+4b(t)^2\left(|u_n(t)|^{2\theta}+|u(t)|^{2\theta}\right) \\ &\leq 8a(t)^2+8b(t)^2R^{2\theta}.\end{aligned}$$

Since a and $b \in L^2(0,1)$, we have $8a(t)^2+8b(t)^2R^{2\theta} \in L^1(0,1)$, and the Lebesgue dominated convergence theorem guarantees that

$$N_f(u_n) \to N_f(u) \quad \text{in } L^2(0,1),$$

proving the continuity of N_f. □

Remark 2.3. Since the embedding $H_0^m(0,1) \subset C[0,1]$ is compact, the operator N_f is compact from $H_0^m(0,1)$ to $L^2(0,1)$.

Lemma 2.3. *Under the growth condition* (1.2), *the functional E is of class C^1, bounded from bellow on each bounded subset of $H_0^m(0,1)$, and*

$$E'(u) = Lu - N_f(u) \quad \textit{in } H^{-m}(0,1),$$

or equivalently

$$JE'(u) = u - JN_f(u) \quad \textit{in } H_0^m(0,1). \tag{2.15}$$

Proof. Step 1 : The functional E is well-defined. Indeed, for $u \in H_0^m(0,1)$, the growth condition (1.2) implies that

$$\begin{aligned}|F(t,u(t))| &\leq \left|\int_0^{u(t)} |f(t,s)|\, ds\right| \leq \left|\int_0^{u(t)} \left(a(t)+b(t)|s|^\theta\right) ds\right| \\ &\leq a(t)|u(t)| + \frac{b(t)}{\theta+1}|u(t)|^{\theta+1}.\end{aligned}$$

Then

$$\begin{aligned}\left|\int_0^1 F(t,u(t))dt\right| &\leq \int_0^1 |F(t,u(t))|\,dt \leq \int_0^1 \left(a(t)|u(t)| + \frac{b(t)}{\theta+1}|u(t)|^{\theta+1}\right)dt \\ &\leq \|u\|_\infty \int_0^1 a(t)dt + \frac{1}{\theta+1}\|u\|_\infty^{\theta+1}\int_0^1 b(t)dt \\ &\leq \|u\|_\infty \|a\| + \frac{1}{\theta+1}\|u\|_\infty^{\theta+1}\|b\|.\end{aligned}$$

Then since $H_0^m(0,1)$ embeds continuously in $C[0,1]$,

$$\left|\int_0^1 F(t,u(t))dt\right| \leq \left(c_\infty\|a\| + \frac{c_\infty^{\theta+1}}{\theta+1}\|b\||u|^\theta\right)|u|. \tag{2.16}$$

Hence E is well-defined.

Step 2 : We prove that E is Fréchet differentiable on $H_0^m(0,1)$ and

$$\langle E'(u), v\rangle = \int_0^1 u^{(m)}(t)v^{(m)}(t)dt - \int_0^1 f(t,u(t))v(t)dt, \quad \text{for all } u,v \in H_0^m(0,1).$$

Indeed, from the mean-value theorem we have for $u,v \in H_0^m(0,1)$

$$\begin{aligned}&\left|E(u+v) - E(u) - \int_0^1 u^{(m)}(t)v^{(m)}(t)dt + \int_0^1 f(t,u(t))v(t)dt\right| \\ \leq\;& \frac{1}{2}|v|^2 + \left|\int_0^1 f(t,u(t))v(t)dt - \int_0^1 [F(t,u(t)+v(t)) - F(t,u(t))]dt\right| \\ \leq\;& \frac{1}{2}|v|^2 + \int_0^1 |f(t,u(t)+\eta(t)v(t)) - f(t,u(t))|\,|v(t)|\,dt \\ \leq\;& \frac{1}{2}|v|^2 + \|v\|_\infty \int_0^1 |f(t,u(t)+\eta(t)v(t)) - f(t,u(t))|\,dt \\ \leq\;& \frac{1}{2}|v|^2 + c_\infty|v| \int_0^1 |f(t,u(t)+\eta(t)v(t)) - f(t,u(t))|\,dt,\end{aligned}$$

for some function $0 < \eta(t) \leq 1$. Since $a,\ b \in L^2(0,1) \subset L^1(0,1)$ by the Lebesgue dominated convergence theorem

$$\int_0^1 |f(t,u+\eta v) - f(t,u)|\,dt \to 0 \text{ as } |v| \to 0.$$

So E is Fréchet differentiable on $H_0^m(0,1)$ and

$$\langle E'(u), v\rangle = \int_0^1 u^{(m)}(t)v^{(m)}(t)dt - \int_0^1 f(t,u(t))v(t)dt = (u,v) - \langle N_f u, v\rangle. \tag{2.17}$$

Step 3 : We prove that $E'(u) = Lu - N_f(u)$ in $H^{-m}(0,1)$ and $E' : H_0^m(0,1) \to H^{-m}(0,1)$ is continuous. The first assertion follows directly from (2.13) and (2.17). Furthermore, since the embeddings $H_0^m(0,1) \subset C[0,1]$ and $L^2(0,1) \subset H^{-m}(0,1)$ are continuous, by Lemma 2.2, N_f is continuous from $H_0^m(0,1)$ to $H^{-m}(0,1)$. Finally the continuity of L from $H_0^m(0,1)$ to $H^{-m}(0,1)$ implies that E' is continuous. Therefore the functional E is of class C^1.

Step 4: The functional E is bounded from below on each bounded subset of $H_0^m(0,1)$. Indeed, if $u \in H_0^m(0,1)$, then by using (2.16), we have

$$E(u) = \frac{1}{2}|u|^2 - \int_0^1 F(t,u(t))dt \tag{2.18}$$

$$\geq -\int_0^1 F(t,u(t))dt \tag{2.19}$$

$$\geq -c_\infty \|a\| |u| - \frac{1}{\theta+1} c_\infty^{\theta+1} \|b\| |u|^{\theta+1}.$$

Hence, if $|u| \leq C$, then

$$E(u) \geq -c_\infty \|a\| C - \frac{1}{\theta+1} c_\infty^{\theta+1} \|b\| C^{\theta+1} > -\infty.$$

□

3. Main results

3.1. **A Harnack type inequality.** Our first result is a Harnack type inequality for the positive solutions of problem (2.10), whose proof is based on the properties of Green's function given by Lemma 2.1. The result is an essential tool for the existence and localization of critical points of the energy functional (2.14).

Lemma 3.4. *For any nonnegative function $v \in L^2(0,1)$ one has*

$$(Jv)(t) \geq M(t)\|Jv\|_\infty, \quad \text{for all } t \in [0,1], \tag{3.20}$$

and

$$(Jv)(t) \geq M(t)\|Jv\|, \quad \text{for all } t \in [0,1], \tag{3.21}$$

where $M(t) = t^m(1-t)^m/(2m-1)$ and $\|\cdot\|$ is the L^2-norm.

Proof. Let $v \in C[0,1]$ be any nonnegative function. For all $t, \ \tau \in [0,1]$, using the properties of Green's function, we have

$$(Jv)(t) = \int_0^1 G(t,s)v(s)ds \geq \frac{t^m(1-t)^m}{2m-1} \int_0^1 \frac{s^m(1-s)^m}{[(m-1)!]^2} v(s)ds$$

$$\geq \frac{t^m(1-t)^m}{2m-1} \int_0^1 G(\tau,s)v(s)ds = \frac{t^m(1-t)^m}{2m-1}(Jv)(\tau).$$

Then

$$(Jv)(t) \geq \frac{t^m(1-t)^m}{2m-1} \max_{\tau\in[0,1]} (Jv)(\tau).$$

Hence

$$(Jv)(t) \geq \frac{t^m(1-t)^m}{2m-1} \|Jv\|_\infty,$$

and (3.20) is proved. Inequality (3.21) is a consequence of (3.20) and of the obvious inequality $\|u\| \leq \|u\|_\infty$ for all $u \in C[0,1]$. □

3.2. **Existence and localization principles.** In connection with Harnack inequality (3.21), we consider a cone K in $H_0^m(0,1)$, defined by

$$K = \{u \in H_0^m(0,1) \ : \ u(t) \geq M(t) \|u\| \ \text{ for all } t \in [0,1]\}.$$

Obviously, all functions in K are nonnegative. In this case we may take the required element ϕ from Section 2.1 to be $\phi = Jv$, where v is any fixed nonnegative nonzero function in $L^2(0,1)$. Up to a positive multiplicative constant, we may assume that $|\phi| = 1$. Also, for any two positive numbers, we define the conical shell

$$K_{R_0 R_1} = \{u \in K : \quad \|u\| \geq R_0, \ |u| \leq R_1\}.$$

Denote

$$\Phi(t) = \inf_{\tau \in [M(t)R_0, c_\infty R_1]} f(t,\tau), \quad \Psi(t) = \sup_{\tau \in [M(t)R_0, c_\infty R_1]} f(t,\tau)$$

and consider the following hypotheses:

(H1): There exist R_0, R_1 with $0 < R_0 < \|\phi\| R_1$ such that

$$R_0 \leq \|J\Phi\|, \text{ where } (J\Phi)(t) = \int_0^1 G(t,s)\Phi(s)\,ds, \tag{3.22}$$

$$R_1 \geq c_\infty \|\Psi\|_{L^1(0,1)}. \tag{3.23}$$

(H2): There exists $\rho > 0$ such that

$$E(u) \geq m + \rho \quad \text{for all} \quad u \in K_{R_0 R_1} \text{ satisfying simultaneously } |u| = R_1, \ \|u\| = R_0.$$

(H3): There exist u_0, u_1 in the same connected component of $K_{R_0 R_1}$ and $r > 0$ such that $|u_0| < r < |u_1|$ and

$$\max\{E(u_0), E(u_1)\} < \inf\{E(u) : u \in K_{R_0 R_1}, \ |u| = r\}.$$

(H4): There exists $\rho > 0$ such that

$$|E(u) - c| \geq \rho \quad \text{for all} \quad u \in K_{R_0 R_1} \text{ satisfying simultaneously } |u| = R_1, \ \|u\| = R_0.$$

Note that the numbers m and c in (H2) and (H4) are defined by (2.6) and (2.5), respectively.

We have the following principles of existence and localization of solutions.

Theorem 3.4. (1^0) *If the conditions* (H1) *and* (H2) *hold, then problem* (1.1) *has at least one positive solution u_m in $K_{R_0 R_1}$ such that*

$$E(u_m) = m.$$

(2^0) *If the conditions* (H1), (H3) *and* (H4) *hold, then problem* (1.1) *has at least one positive solution u_c in $K_{R_0 R_1}$ such that*

$$E(u_c) = c.$$

(3^0) *If all conditions (H1)-(H4) hold, then problem* (1.1) *has in $K_{R_0 R_1}$ at least two distinct positive solutions, u_m and u_c.*

Proof. We apply Theorem 2.1 and 2.2 to the functional E defined on $H_0^m(0,1)$. Thus we need to check that conditions (c1)-(c4) are fulfilled. First note that from (2.15) one has

$$N(u) = u - JE'(u) = JN_f(u),$$

with

$$JN_f(u)(t) = \int_0^1 G(t,s) f(s,u(s))ds.$$

Check of the invariance condition (c1): In view of inequality (3.21), the condition $J(K) \subset K$ is trivially satisfied. To prove that $(I - JE')(K) \subset K$, let $u \in K$ be any element of K. Then $u \in H_0^m(0,1)$ and $u \geq 0$ on $[0,1]$. Next, the properties of f guarantee that $N_f(u)$ is a nonnegative function in $L^2(0,1)$, and then Harnack inequality (3.21) implies that $JN_f(u) \in K$. Thus $(I - JE')(u) = JN_f(u) \in K$, as wished.

Check of the boundedness condition (c2): A simple consequence of the fact that JE' maps bounded sets into bounded sets.

Check of the compactness condition (c3): The operators J and N are compact from $H_0^m(0,1)$ to itself. Indeed the compactness of J is a consequence of the compactness of the embedding $H_0^m(0,1) \subset L^2(0,1)$, while the compactness of $N = JN_f$ follows from Remark 2.3.

Check of the compression condition (c4): First we show that (2.8) holds. Indeed, if we assume the contrary, then

$$JE'(u) + \lambda u = 0$$

for some $u \in K_{R_0R_1}$ with $|u| = R_1$ and $\lambda > 0$. Since $N = I - JE'$, it follows that $N(u) = (1+\lambda)u$, and then

$$R_1^2 = |u|^2 = (1+\lambda)^{-1}(N(u), u) < (JN_f(u), u) = \langle N_f(u), u\rangle. \tag{3.24}$$

Since $u \in K_{R_0R_1}$, for every $t \in (0,1)$, we have

$$M(t)R_0 \leq M(t)\|u\| \leq u(t) \leq \|u\|_\infty \leq c_\infty |u| = c_\infty R_1.$$

Then

$$\begin{aligned}\langle N_f(u), u\rangle &= \int_0^1 f(t, u(t))\, u(t) dt \leq \|u\|_\infty \int_0^1 \sup_{\tau \in [M(t)R_0, c_\infty R_1]} f(t,\tau)\, dt \\ &\leq c_\infty |u| \int_0^1 \Psi(t)\, dt \leq c_\infty R_1 \|\Psi\|_{L^1(0,1)}.\end{aligned}$$

This together with (3.24) yields

$$R_1 < c_\infty \|\Psi\|_{L^1(0,1)},$$

a contradiction to (3.23). Thus (2.8) holds.

Now we check condition (2.7). Assume that (2.7) does not holds. Then

$$JE'(u) - \lambda Ju = 0,$$

for some $u \in K_{R_0R_1}$ with $\|u\| = R_0$ and $\lambda > 0$. Hence $N(u) + \lambda Ju = u$, or equivalently $J(N_f(u) + \lambda u) = u$. Since $u \in K_{R_0R_1}$ and $\|u\| = R_0$, for all $t \in (0,1)$, we have

$$M(t)R_0 = M(t)\|u\| \leq u(t) \leq \|u\|_\infty \leq c_\infty |u| \leq c_\infty R_1.$$

Then

$$\begin{aligned}u(t) &= \int_0^1 G(t,s)\left(f(s,u(s)) + \lambda u(s)\right) ds > \int_0^1 G(t,s)\, f(s,u(s))\, ds \\ &\geq \int_0^1 G(t,s) \inf_{\tau \in [M(s)R_0, c_\infty R_1]} f(s,\tau)\, ds = \int_0^1 G(t,s)\, \Phi(s)\, ds = (J\Phi)(t).\end{aligned}$$

Consequently

$$R_0 = \|u\| > \|J\Phi\|,$$

which contradicts (3.22). Thus (2.7) holds.

Notice that the solutions u_m and u_c are distinct since $m < c$. □

3.3. **Case of nonlinearities with separated variables.** It deserves to see how condition (H1) looks in the particular case of a nonlinearity with separated variables, i.e., when f has the form

$$f(t,s) = g(t)h(s),$$

where $h : \mathbb{R} \to \mathbb{R}$ is a continuous function with $h(\mathbb{R}^+) \subset \mathbb{R}^+$ and

$$|h(s)| \le a_0 + b_0 |s|^\theta \quad \text{for all } s \in \mathbb{R}, \tag{3.25}$$

and g is a nonnegative function from $L^2(0,1)$. It is clear that under these conditions, f is a Carathéodory function with $f([0,1] \times \mathbb{R}^+) \subset \mathbb{R}^+$ and satisfies the growth condition (1.2) with $a(t) = a_0 g(t)$ and $b(t) = b_0 g(t)$.

Assume in addition that h is nondecreasing. Then

$$\Phi(t) = g(t)h(M(t)R_0), \quad \Psi(t) = g(t)h(c_\infty R_1)$$

and the compression conditions (3.22) and (3.23) become

$$R_0 \le \|J(gh(MR_0))\|, \text{ where } J(gh(MR_0))(t) = \int_0^1 G(t,s)g(s)h(M(s)R_0)ds, \tag{3.26}$$

$$R_1 \ge c_\infty h(c_\infty R_1) \|g\|_{L^1(0,1)}. \tag{3.27}$$

A sufficient condition for (3.26) to hold is

$$R_0 \le h(\widetilde{M}R_0) \left\| J\left(g\chi_{[a_1,b_1]}\right)\right\|, \tag{3.28}$$

where a_1, b_1 are numbers from $(0,1)$, $\widetilde{M} = \min\{M(a_1), M(b_1)\}$ and $\chi_{[a_1,b_1]}$ is the characteristic function of the interval $[a_1,b_1]$. Also a sufficient condition for (3.27) to hold is

$$\frac{\psi(R_1)}{R_1} \le 1, \quad \text{where } \psi(\tau) = c_\infty a_0 \|g\|_{L^1(0,1)} + c_\infty^{\theta+1} b_0 \|g\|_{L^1(0,1)} \tau^\theta.$$

Notice that if we choose $[a_1,b_1] = [\sigma, 1-\sigma]$ with some $\sigma \in (0,1/2)$, then condition (3.28) reads as

$$R_0 \le h(M(\sigma)R_0) \left\| Jg\chi_{[\sigma,1-\sigma]}\right\|.$$

For nonlinearities with separated variables we have the following existence result.

Theorem 3.5 (Existence)**.** *Assume that $f(t,s) = g(t)h(s)$, with $g \in L^2(0,1;\mathbb{R}^+)$ and $h : \mathbb{R}^+ \to \mathbb{R}^+$, continuous, nondecreasing, and satisfying (3.25) for some $\theta \in (0,1)$. In addition assume that*

$$\lim_{\tau \to 0} \frac{h(\tau)}{\tau} = \infty. \tag{3.29}$$

Then (1.1) has at least one positive solution in K.

Proof. From (3.29) it follows that condition (3.26) holds for every R_0 sufficiently small. Next, since $0 < \theta < 1$, one has $\lim_{\tau\to+\infty} \frac{h(\tau)}{\tau} = 0$ and so condition (3.27) holds for every $R_1 > 0$ sufficiently large. Furthermore, from (2.18), one has

$$\begin{aligned} E(u) &\ge \frac{1}{2}|u|^2 - \frac{1}{\theta+1} c_\infty^{\theta+1} \|b\| |u|^{\theta+1} - c_\infty \|a\| |u| \\ &\ge \frac{1}{2}|u|^2 - \frac{1}{\theta+1} c_\infty^{\theta+1} b_0 \|g\| |u|^{\theta+1} - c_\infty a_0 \|g\| |u|, \end{aligned} \tag{3.30}$$

which in view of $\theta < 1$, implies that $E(u) \to +\infty$ as $|u| \to +\infty$. Then for a fixed $\rho > 0$, we can choose R_1 sufficiently large such that

$$E(u) \ge E(\mu\phi) + \rho \ge m + \rho, \quad \text{for all } u \in K \text{ with } |u| = R_1, \tag{3.31}$$

where $\mu > 0$ is such that $\|\mu\phi\| = R_0$. Hence conditions (H1) and (H2) are satisfied and Theorem 3.4 applies and yields the conclusion. $\square$

Remark 3.4. The conclusion of Theorem 3.5 remains true if

$$\theta = 1 \quad \text{and} \quad c_\infty^2 b_0 \|g\| < 1.$$

Indeed, for $\theta = 1$ one has

$$\begin{aligned} E(u) &\geq \frac{1}{2}|u|^2 - \frac{1}{2}c_\infty^2 b_0 \|g\| |u|^2 - c_\infty a_0 \|g\| |u| \\ &= \frac{1}{2}\left(1 - c_\infty^2 b_0 \|g\|\right)|u|^2 - a_0 \|g\| |u|. \end{aligned}$$

Then $E(u) \to +\infty$ as $|u| \to +\infty$ and (3.31) holds for R_1 sufficiently large. Furthermore, for $\theta = 1$ we have

$$\lim_{\tau \to +\infty} \frac{h(\tau)}{\tau} \leq b_0 < \frac{1}{c_\infty^2 b_0 \|g\|}.$$

Hence, since $\|g\| \geq \|g\|_{L^1(0,1)}$, (3.27) holds for R_1 sufficiently large.

Theorem 3.6 (Multiplicity). *Assume that $f(t,s) = g(t)h(s)$, with $g \in L^2(0,1;\mathbb{R}^+)$ and $h : \mathbb{R}^+ \to \mathbb{R}^+$, continuous, nondecreasing, and satisfying (3.25) for some $\theta \in (0,1)$. Let $(R_0^i)_{1\leq i\leq k}$, $(R_1^i)_{1\leq i\leq k}$ $(k \leq \infty)$ be increasing sequences of positive numbers satisfying the following conditions*

(3.32) $$R_0^i < \|\phi\| R_1^i, \quad \textit{for } i = 1,2,\ldots,k;$$

(3.33) $$c_0 R_1^i < R_0^{i+1}, \quad \textit{for } i = 1,2,\ldots,k-1;$$

(3.34) $$R_0 \leq h(\widetilde{M}R_0)\left\|J\left(g\chi_{[a_1,b_1]}\right)\right\|, \quad \textit{for } i = 1,2,\ldots,k.$$

Then, if R_1^1 is large enough, problem (1.1) has at least k distinct solutions u_i with $\|u_i\| \geq R_0^i$ and $|u_i| \leq R_1^i$, $i = 1,2,\ldots,k$.

Proof. First, condition (3.32) guarantees that the set $K_{R_0^i R_1^i}$ is nonempty. Also, (3.33) implies that the sets $K_{R_0^i R_1^i}$ are disjoint. Indeed, from (3.33) we have the inclusion

$$K_{R_0^i R_1^i} \subset \{u \in K : \|u\| \leq R_0^{i+1}\},$$

since for every $u \in K_{R_0^i R_1^i}$, $\|u\| \leq c_0|u| \leq c_0 R_1^i < R_0^{i+1}$. Let us show that conditions (H1) and (H2) of Theorem 3.4 are satisfied for each set $K_{R_0^i R_1^i}$. Since $\theta < 1$, $\lim_{\tau\to+\infty} \psi(\tau)/\tau = 0$ and the function $\psi(\tau)/\tau$ is decreasing on $(0,+\infty)$. So, there exists $\tau_0 > 0$ such that $\psi(\tau)/\tau \leq 1$ for all $R_1^1 \geq \tau_0$, and $\psi\left(R_1^i\right)/R_1^i \leq 1$ for $i = 2,3,\ldots,k$. According to (3.34), (H1) holds for each $K_{R_0^i R_1^i}$ for R_1^1 large enough. Next, let $u \in K_{R_0^i R_1^i}$ with $|u| = R_1^i$. From (3.30) we have

$$E(u) \geq \omega(|u|), \quad \text{where } \omega(\tau) = \frac{1}{2}\tau^2 - \frac{1}{\theta+1}c_\infty^{\theta+1} b_0 \|g\| \tau^{\theta+1} - c_\infty a_0 \|g\| \tau.$$

Note that since $\theta < 1$, then for each $B \in (0,1)$, there exists $\tau_1 > 0$ such that

$$\omega(\tau) \geq \frac{B^2}{2}\tau^2 \quad \text{for all } \tau \geq \tau_1.$$

Let $B = \max_{1\leq i\leq k} R_0^i / \left(\|\phi\| R_1^i\right)$. From (3.32), one has $B \in (0,1)$. Then, if $R_1^1 \geq \tau_1$,

$$E(u) \geq \omega\left(R_1^i\right) \geq \frac{B^2}{2}(R_1^i)^2 \geq \frac{(R_0^i)^2}{2\|\phi\|^2} = E(\mu_i\phi) + \rho_i$$

for all i, where $\|\mu_i\phi\| = R_0^i$ and $\rho_i = \int_0^1 F(t, \mu_i\phi)dt$. Hence conditions (H1) and (H2) hold for each set $K_{R_0^i R_1^i}$, $i = 1, 2, \ldots, k$ provided that $R_1^1 \geq \max\{\tau_0, \tau_1\}$, and Theorem 3.4 applies to each set $K_{R_0^i R_1^i}$, $i = 1, 2, \ldots, k$. □

3.4. **Example.** Let $g \in L^2(0, 1; \mathbb{R}^+)$ with $g(t) > 0$ for a.e. $t \in [0, 1]$, and let $h : \mathbb{R} \to \mathbb{R}$ be an even function with

$$h(s) = \begin{cases} \alpha\sqrt{s}, & \text{if } 0 \leq s \leq 1, \\ \alpha s^2, & \text{if } 1 < s \leq \beta, \\ \alpha\left(\sqrt{s-\beta} + \beta^2\right), & \text{if } s > \beta, \end{cases}$$

where $\alpha > 0$ and $\beta > 2$ are chosen below. Then the problem (1.1) with $f(t, s) = g(t)h(s)$, has at least two positives solutions.

It is clear that h is continuous, nondecreasing on $\mathbb{R}^+$ and $h(\mathbb{R}^+) \subset \mathbb{R}^+$. In addition we can find $a_0, b_0 > 0$ depending on α and β such that

$$|h(s)| \leq a_0 + b_0 s^2, \quad s \in \mathbb{R}.$$

Hence condition (1.2) holds with $\theta = 2$, $a(t) = a_0 g(t)$ and $b(t) = b_0 g(t)$.

Fulfillment of condition (H3): Choose $r = 2$. Then

$$\inf_{u \in K,\, |u| = r} E(u) > \frac{1}{2}.$$

Indeed, for $u \in K$ with $|u| = 2$, using (2.12) with c_∞ given by (2.11), and observing that $c_\infty \leq 1$, we find

$$0 \leq u(t) \leq \|u\|_\infty \leq c_\infty |u| = 2c_\infty \leq 2.$$

Then

$$F(t, u(t)) = g(t) \int_0^{u(t)} h(s)\, ds \leq g(t) \int_0^2 h(s)\, ds = 3\alpha g(t).$$

Hence

$$E(u) = \frac{1}{2}|u|^2 - \int_0^1 F(t, u(t))dt \geq 2 - 3\alpha \int_0^1 g(t)\, dt \geq 2 - 3\alpha \|g\|.$$

So, for $\alpha > 0$ sufficiently small, we have

$$E(u) \geq \frac{1}{2} \quad \text{for all } u \in K \text{ with } |u| = 2.$$

Next, let $u_0 \in K$ be any function satisfying $|u_0| = 1$. Clearly $|u_0| < r = 2$, and

$$E(u_0) = \frac{1}{2} - \int_0^1 F(t, u_0(t))dt < \frac{1}{2}.$$

Now we look for an element $u_1 \in K$ such that $|u_1| > 2$ and $E(u_1) < 1/2$. Let

$$u_1 = \beta \frac{u_0}{\|u_0\|_\infty}.$$

One has $|u_1| = \beta / \|u_0\|_\infty$, and if we choose $\beta > 2\|u_0\|_\infty$, then $|u_1| > 2$. Also, since $u_1(0) = u_1(1) = 0$ and $\|u_1\|_\infty = \beta > 2$, the level set $(u > 1)$ is nonempty and is a proper subset of $[0, 1]$. Hence

$$\begin{aligned} E(u_1) &= \frac{\beta^2}{2\|u_0\|_\infty^2} - \int_0^1 F(t, u_1(t))\, dt \\ &\leq \frac{\beta^2}{2\|u_0\|_\infty^2} - \int_{(u_1 > 1)} F(t, u_1(t))\, dt. \end{aligned}$$

In view of $0 \le u_1(t) \le \beta$ for all $t \in [0,1]$, on the level set $(u_1 > 1)$ we have

$$F\left(t, u_1(t)\right) = g\left(t\right)\left(\frac{2}{3}\alpha + \frac{\alpha}{3}\left(u_1\left(t\right)^3 - 1\right)\right) \ge \frac{\alpha}{3} g\left(t\right) u_1\left(t\right)^3.$$

Thus

$$\begin{aligned} E\left(u_1\right) &\le \frac{\beta^2}{2\left\|u_0\right\|_\infty^2} - \frac{\alpha}{3}\int_{(u_1>1)} g\left(t\right) u_1\left(t\right)^3 dt \\ &\le \frac{\beta^2}{2\left\|u_0\right\|_\infty^2} - \frac{\alpha}{3}\frac{\beta^3}{\left\|u_0\right\|_\infty^3}\int_{(u_1>1)} g\left(t\right) u_0\left(t\right)^3 dt. \end{aligned}$$

Since the level set $(u_1 > 1)$ enlarges as β increases, we deduce that

$$\frac{\beta^2}{2\left\|u_0\right\|_\infty^2} - \frac{\alpha}{3}\frac{\beta^3}{\left\|u_0\right\|_\infty^3}\int_{(u_1>1)} g\left(t\right) u_0\left(t\right)^3 dt \to -\infty \quad \text{as } \beta \to +\infty.$$

Thus we can find a β sufficiently large to have $E\left(u_1\right) < 1/2$. With this choice of α and β, condition (H3) is fulfilled.

Fulfillment of condition (H1): Since

$$\lim_{\tau \to 0}\frac{h(\tau)}{\tau} = +\infty \quad \text{and} \quad \lim_{\tau \to +\infty}\frac{h(\tau)}{\tau} = 0,$$

we may find $R_0, \overline{R}_1$ such that $u_0, u_1 \in K_{R_0 R_1}$ and (3.26), (3.27) (and consequently (H1)) hold for every $R_1 \ge \overline{R}_1$.

Fulfillment of conditions (H2) and (H4): Since $c > m$, it suffices to prove that there exists $\rho > 0$ such that

(3.35) $$E\left(u\right) \ge c + \rho, \quad \text{for all } u \in K_{R_0 R_1} \quad \text{with } |u| = R_1 \text{ and } \|u\| = R_0.$$

Let us fix R_0 and seek $R_1 \ge \overline{R}_1$ such that (3.35) holds. Denote by Γ_{R_1} and c_{R_1} the corresponding Γ and c as given by (2.4) and (2.5), respectively. It is clear that if $R_1 \ge \overline{R}_1$, then $\Gamma_{\overline{R}_1} \subset \Gamma_{R_1}$ and $c_{R_1} \le c_{\overline{R}_1}$. Hence to have (3.35) it suffices to find an $R_1 \ge \overline{R}_1$ such that

$$E\left(u\right) \ge c_{\overline{R}_1} + \rho \quad \text{for all } u \in K_{R_0 R_1}, \text{ with } |u| = R_1 \text{ and } \|u\| = R_0.$$

We can show even more, namely

(3.36) $$E\left(u\right) \ge c_{\overline{R}_1} + \rho \quad \text{for all } u \in K \quad \text{with } |u| = R_1.$$

Let $u \in K$ with $|u| = R_1$. We have

$$\begin{aligned} \int_{(u \le \beta)} F(t, u(t)) dt &= \int_{(u \le \beta)} g(t)\left(\frac{2\alpha}{3} + \frac{\alpha}{3}\left(u(t)^3 - 1\right)\right) dt \\ &\le \left(\frac{2\alpha}{3} + \frac{\alpha}{3}\left(\beta^3 - 1\right)\right)\|g\| \end{aligned}$$

and

$$\begin{aligned}
&\int_{(u\geq\beta)} F(t,u(t))dt \\
= &\int_{(u\geq\beta)} g(t)\left\{\frac{2\alpha}{3}+\frac{\alpha}{3}\left(\beta^3-1\right)+\frac{2\alpha}{3}\left(u(t)-\beta\right)^{3/2}+\alpha\beta^2u(t)-\alpha\beta^3\right\}dt \\
\leq &\left(\frac{2\alpha}{3}+\frac{\alpha}{3}\left(\beta^3-1\right)-\alpha\beta^3\right)\|g\|+\frac{2\alpha}{3}\int_0^1 g(t)u(t)^{3/2}dt+\alpha\beta^2\int_0^1 g(t)u(t)dt \\
\leq &\left(\frac{2\alpha}{3}+\frac{\alpha}{3}\left(\beta^3-1\right)-\alpha\beta^3\right)\|g\|+\frac{2\alpha}{3}\|u\|_\infty^{3/2}\|g\|+\alpha\beta^2\|u\|_\infty\|g\| \\
\leq &\left(\frac{2\alpha}{3}+\frac{\alpha}{3}\left(\beta^3-1\right)-\alpha\beta^3\right)\|g\|+\frac{2\alpha}{3}c_\infty^{3/2}|u|^{3/2}\|g\|+\alpha\beta^2c_\infty|u|\,\|g\|.
\end{aligned}$$

Then

$$\begin{aligned}
E(u) &= \frac{1}{2}|u|^2-\int_0^1 F(t,u(t))dt \\
&= \frac{1}{2}|u|^2-\int_{(u\leq\beta)} F(t,u(t))dt-\int_{(u\geq\beta)} F(t,u(t))dt \\
&\geq \frac{1}{2}|u|^2-\frac{\alpha}{3}\left(2-\beta^3\right)-\frac{2\alpha}{3}c_\infty^{3/2}|u|^{3/2}\|g\|-\alpha\beta^2c_\infty|u|\,\|g\| \\
&= \frac{1}{2}R_1^2-\frac{\alpha}{3}\left(2-\beta^3\right)-\frac{2\alpha}{3}c_\infty^{3/2}\|g\|\,R_1^{3/2}-\alpha\beta^2c_\infty\|g\|\,R_1.
\end{aligned}$$

Since the expression in the right side of the last inequality tends to $+\infty$ as $R_1\to+\infty$, we can find $R_1\geq\overline{R}_1$ such that (3.36) holds, as desired. Hence conditions (H2) and (H4) are fulfilled.

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