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Random equations and its application towards best random proximity point theorems

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ABSTRACT. In this paper best random proximity points equations have been proved. As a result best random proximity points and best random p-proximity points have been proposed with the help of new generalized notions. These are generalizations of random fixed point theorems. Also, the concept of random best p-proximity points has also be proposed in this work and corresponding theorems are also defined here.

1. INTRODUCTION

It is well-known that the study of the random equations involving the random mappings in view of their need for dealing with probabilistic models in applied sciences is very important. Motivated and inspired by the recent research works in these fascinating areas, the random equations, random variational inequality problems, random variational inclusion problems, random proximity points and random fixed point problems have been introduced and studied by many researchers. It is the fact that Banach Contraction Principle [5] is very helpful and fruitful tool in fixed point theory to find the a solution to non-linear equations of the type $Fx = x$ if given mapping F is a self-mapping defined on any non-empty subset of metric space or any other relevant framework. If the mapping under consideration is non-self then it is not necessary that given mapping has solution to the equation $Fx = x$ where F is non-self mapping. Then in such type of cases one can approach to find those points for which non-self mapping F gives us the approximate solution to the equation $Fx = x$, with this solution we get the solution which is optimal that is $d(x, Fx) = d(A, B)$ and the point x is called best proximity point for given mapping which is non-self. Random best proximity points are further generalization of best proximity points.
Motivated by papers [6] and [10] the strong theme of this paper is to construct some new notions of random F_p-contractions and to introduce some theorems with new notions, we will discuss existence of the random best and random p-best proximity points for given mappings in metric spaces as well as other relevant spaces and will propose applications to best random proximity points. We have collected some definitions and mathematical symbols in this section which will be useful for the later sections.

Definition 1.1. [1] Let X be a metric space, A and B two nonempty subsets of X. Define

$$\begin{aligned} d(A,B) &= \inf\{d(a,b) : a \in A, b \in B\}, \\ A_0 &= \{a \in A : there\ exists\ some\ b \in B\ such\ that\ d(a,b) = d(A,B)\}, \\ B_0 &= \{b \in B : there\ exists\ some\ a \in A\ such\ that\ d(a,b) = d(A,B)\}. \end{aligned}$$

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Definition 1.2. [2] Let (A, B) be a pair of non-empty subsets of a metric space (X,d) with $A_0 \neq \emptyset$.Then the pair (A, B) is said to have the P-property if and only if for any $x_1, x_2, x_3, x_4 \in A_0$,

$$\left.\begin{aligned} d(x_1, fx_3) &= d(A, B) \\ d(x_2, fx_4) &= d(A, B) \end{aligned}\right\} \Rightarrow d(x_1, x_2) = d(fx_3, fx_4).$$

Definition 1.3. [6] Let $(X, \preceq, d)$ be a partially ordered metric space. A mapping $\mathcal{A} : A \times A \to \mathbb{R}$ where $A \subseteq X$, is said to be a generalized $\mathcal{P}$-function ($\mathcal{G}_{\mathcal{P}}$-function) w.r.t. $\preceq$ in X if it satisfies the following conditions:

(1) $\mathcal{A}(x, y) \geq 0$ for every comparable $x, y \in A$;
(2) for any sequences $\{x_n\}, \{y_n\}$ in A such that x_n and y_n are comparable at each $n \in \mathbb{N}$, if $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then $\lim_{n\to\infty} \mathcal{A}(x_n, y_n) = \mathcal{A}(x, y)$;
(3) for any sequences $\{x_n\}, \{y_n\}$ in A such that x_n and y_n are comparable at each $n \in \mathbb{N}$, if $\lim_{n\to\infty} \mathcal{A}(x_n, y_n) = 0$, then $\lim_{n\to\infty} d(x_n, y_n) = 0$.
(4) for any sequences $\{x_n\}, \{y_n\}$ in A such that x_n and y_n are comparable at each $n \in \mathbb{N}$, if $\lim_{n\to\infty} d(x_n, y_n)$ exists then $\lim_{n\to\infty} \mathcal{A}(x_n, y_n)$ also exists.

Definition 1.4. [6] Let $(X, \preceq, d)$ be a partially ordered metric space, a mapping $f : A \to B$ is called $\mathcal{G}_{\mathcal{P}}$-contraction w.r.t. $\preceq$ if there is $\mathcal{G}_{\mathcal{P}}$-function as $\mathcal{A} : A \times A \to \mathbb{R}$ where $A \subseteq X$, w.r.t. $\preceq$ in X such that

$$d(fx, fy) \leq d(x, y) - \mathcal{A}(x, y),$$

for any $x, y \in A$.

Theorem 1.1. [6] *Let $A, B \neq \phi$ be closed subsets of a complete partially ordered metric space $(X, \preceq, d)$ such that A_0 is nonempty. Define a map $f : A \to B$ with the following conditions:*

(1) f is continuous generalized $\mathcal{P}$-contraction w.r.t. $\preceq$ with $f(A_0) \subseteq B_0$;
(2) the pair (A, B) has the P-property.

Then there exists a unique x^* in A such that $d(x^*, fx^*) = d(A, B)$.

Definition 1.5. [10] Let A, B be the closed subsets of a Polish space X (i.e. X is separable and complete metric space) and $f : \Omega \times A \to B$ be random operator, Ω is just a set with elements w. It is called sample space. A measurable mapping $\xi : \Omega \to X$ is called best random proximity point of f if

$$d(\xi(w), f(w, \xi(w))) = d(A, B),$$

for any $w \in \Omega$.

Definition 1.6. [2] Given a non-self mapping $f : A \to B$, then an element x^* is called best proximity point of the mappings if this condition holds:

$$d(x^*, fx^*) = d(A, B),$$

where $BPP(f)$ denotes the set of best proximity points of f.

Definition 1.7. [4] Let (X, d) be a metric space. Then a function $p : X \times X \to [0, \infty)$ is called w-distance on X if the following are satisfied:

(1) $p(x, z) \leq p(x, y) + p(y, z)$, for any $x, y, z \in X$;
(2) for any $x \in X$, $p(x, .) : X \to [0, \infty)$ is lower semi continuous;
(3) for any $\epsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ $d(x, y) \leq \epsilon$.

Definition 1.8. [3]A is said to be approximatively compact with respect to B if every sequence $\{x_n\}$ of A satisfying the condition that $d(y, x_n) \to d(y, A)$ for some y in B has a convergent subsequence. It is easy to see that every set is approximatively compact with respect to itself.

Definition 1.9. [3] Given $T : A \to B$ and an isometry $g : A \to A$, the mapping T is said to preserve isometric distance with respect to g if

$$d(Tgx_1, Tgx_2) = d(Tx_1, Tx_2)$$

for all $x_1, x_2 \in A$.

Definition 1.10. [8] Let $F : \mathbb{R}^+ \to \mathbb{R}$ be a mapping satisfying

(1) F is strictly increasing, i.e. for all $a, b \in \mathbb{R}^+$ such that $\alpha < \beta \Rightarrow F(\alpha) < F(\beta)$;
(2) For each sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of positive numbers $lim_{n\to\infty}\alpha_n = 0$ if and only if $lim_{n\to\infty}F(\alpha_n) = -\infty$;
(3) there exists $k \in (0,1)$ such that $lim_{\alpha\to 0^+}\alpha^k F(\alpha) = 0$.

A mapping $T : A \to B$ is said to be an F-contraction if there exists a $\tau > 0$ such that for all $x, y \in A, d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y))$.

2. Main results

Motivated from [6], we define here the random $\mathcal{G}_\mathcal{P}$-functions, contractions and furthermore, we prove the existence of best random proximity points.

Definition 2.11. Let $(X, \preceq, d)$ be a partially ordered metric space. A mapping $\mathcal{A} : A \times A \to \mathbb{R}$ where $A \subseteq X$, is said to be a generalized $\mathcal{P}$-function ($\mathcal{G}_\mathcal{P}$-function) w.r.t. $\preceq$ in X if it satisfies given conditions:

(1) $\mathcal{A}(\xi(\omega), \eta(\omega)) \geq 0$ for every comparable $\xi(\omega), \eta(\omega) \in A$;
(2) for any sequences $\{\xi_n(\omega)\}, \{\eta_n(\omega)\}$ in A such that $\xi_n(\omega)$ and $\eta_n(\omega)$ are comparable at each $n \in \mathbb{N}$, if $\lim_{n\to\infty} \xi_n(\omega) = \xi(\omega)$ and $\lim_{n\to\infty} \eta_n(\omega) = \eta(\omega)$, then $\lim_{n\to\infty} \mathcal{A}(\xi_n(\omega), \eta_n(\omega)) = \mathcal{A}(\xi(\omega), \eta(\omega))$;
(3) for any sequences $\{\xi_n(\omega)\}, \{\eta_n(\omega)\}$ in A such that $\xi_n(\omega)$ and $\eta_n(\omega)$ are comparable at each $n \in \mathbb{N}$, if $\lim_{n\to\infty} \mathcal{A}(\xi_n(\omega), \eta_n(\omega)) = 0$, then $\lim_{n\to\infty} d(\xi_n(\omega), \eta_n(\omega)) = 0$.
(4) for any sequences $\{\xi_n(\omega)\}, \{\eta_n(\omega)\}$ in A such that $\xi_n(\omega)$ and $\eta_n(\omega)$ are comparable at each $n \in \mathbb{N}$, if $\lim_{n\to\infty} d(\xi_n(\omega), \eta_n(\omega))$ exists then $\lim_{n\to\infty} \mathcal{A}(\xi_n(\omega), \eta_n(\omega))$ also exists.

where $\xi, \eta, \xi_n, \eta_n : \Omega \to X$ are X-valued random variables.

Definition 2.12. Let $(X, \preceq, d)$ be a partially ordered metric space, Ω be a separable measure space and $\xi, \eta : \Omega \to A$. A mapping $f : \Omega \times A \to B$ is called random $\mathcal{G}_\mathcal{P}$-contraction w.r.t. $\preceq$ if there is a random $\mathcal{G}_\mathcal{P}$-function as $\mathcal{A} : A \times A \to \mathbb{R}$ where $A \subseteq X$, w.r.t. $\preceq$ in X such that

$$d(f(\omega, \xi(\omega)), f(\omega, \eta(\omega))) \leq d(\xi(\omega), \eta(\omega)) - \mathcal{A}(x, y),$$

for any $x, y \in A,\ \omega \in \Omega, \eta(\omega), \xi(\omega) \in A$.

Theorem 2.2. Let $A, B \neq \phi$ be compact subsets of a Cauchy partially ordered space (i.e. complete partially ordered metric space) $(X, \preceq, d)$ such that A_0 is nonempty. Suppose Ω is a probability space and $\xi : \Omega \to A$. Let $f : \Omega \times A \to B$ be a mapping satisfying the following conditions:

(1) f is continuous random $\mathcal{G}_\mathcal{P}$-contraction w.r.t. $\preceq$ with $f(A_0) \subseteq B_0$;
(2) the pair (A, B) has the P-property.

Then there exists a unique $\xi^*(\omega)$ in A such that $d(\xi^*(\omega), f(\omega, \xi^*(\omega))) = d(A, B)$.

Proof. Since for any $\xi_n(\omega) \in A$, $\xi_n(\omega) \preceq \xi_{n+1}$ for all $n \in \mathbb{N}$ and A_0 is nonempty so we take $\xi_0(\omega) \in A_0$, since $f(A_0) \subseteq B_0$, there exists $\xi_1(\omega) \in A_0$ as

$$d(\xi_1(\omega), f(\omega, \xi_0(\omega))) = d(A, B),$$

where $\omega \in \Omega,\ \xi_n(\omega) \in A$. Again, since $f(A_0) \subseteq B_0$, there exists $\xi_2(\omega) \in A_0$ such that

$$d(\xi_2(\omega), f(\omega, \xi_1(\omega))) = d(A, B).$$

Repeating this technique, we have a sequence $\{\xi_n(\omega)\}$ in A_0 satisfying $d(\xi_{n+1}(\omega), f(\omega, \xi_n(\omega)))$ $= d(A, B)$, for any $n \in \mathbb{N}$.
Since the pair (A, B) has the P-property, we have

$$\begin{aligned} d(\xi_n(\omega), \xi_{n+1}(\omega)) &= d(f(\omega, \xi_{n-1}(\omega)), f(\omega, \xi_n(\omega))) && (2.1)\\ &\leq d(\xi_{n-1}(\omega), \xi_n(\omega)) - \mathcal{A}(\xi_{n-1}(\omega), \xi_n(\omega)) && (2.2)\\ &\leq d(\xi_{n-1}(\omega), \xi_n(\omega)), && (2.3)\end{aligned}$$

for all $n \in \mathbb{N}$.
Therefore, $\{d(\xi_n(\omega), \xi_{n+1})(\omega)\}$ is a decreasing sequence, for any $n \in \mathbb{N}$.
Suppose that there exists $n_0 \in \mathbb{N}$ such that $0 = d(\xi_{n_0}(\omega), \xi_{n_0+1}(\omega)) = d(f(\omega, \xi_{n_0-1}(\omega)), f(\omega, \xi_{n_0}))$, and consequently

$$f(\omega, \xi_{n_0-1}(\omega)) = f(\omega, \xi_{n_0}(\omega)).$$

Therefore, we obtain

$$d(A, B) = d(\xi_{n_0}(\omega), f(\omega, \xi_{n_0-1}(\omega))) = d(\xi_{n_0}(\omega), f(\omega, \xi_{n_0}(\omega))),$$

note that $\xi_0(\omega) \in A_0, \xi_1(\omega) \in B_0$ and $\xi_0(\omega) = \xi_1(\omega)$, so $A \cap B$ is nonempty, then $d(A, B)$ $= 0$. Thus, in this case there exists random best proximity point, that is there exists unique $\xi^*(\omega) \in A$ such that $d(\xi^*, f(\omega, \xi^*(\omega)) = d(A, B)$.
In the contrary case, let $d(\xi_n(\omega), \xi_{n+1}(\omega)) > 0$ for any $n \in \mathbb{N}$. Since $\{d(\xi_n(\omega), \xi_{n+1}(\omega))\}$ is a bounded sequence of real numbers, so there exists $r \geq 0$ such that $\lim_{n\to\infty} d(\xi_n(\omega), \xi_{n+1}(\omega))$ $= r$. Thus, there is $s \geq 0$ such that $\lim_{n\to\infty} \mathcal{A}(\xi_n(\omega), \xi_{n+1}(\omega)) = s$. We shall prove that $r = 0$. Let $r \neq 0$ and $r > 0$, then by the generalized $\mathcal{P}$-contractivity of f, we have

$$r \leq r - s.$$

Thus, $s = 0$ so we get $r = 0$, a contradiction. Therefore, we have

$$\lim_{n\to\infty} d(\xi_n(\omega), \xi_{n+1}(\omega)) = 0.$$

Now, we claim that $\{\xi_n(\omega)\}$ is a Cauchy sequence. Assume that $\{\xi_n(\omega)\}$ is not Cauchy sequence. Then there must exists $\epsilon > 0$ and subsequences $\{\xi_{m_k}(\omega)\}, \{\xi_{n_k}(\omega)\}$ of $\{\xi_n(\omega)\}$ such that for any positive integers $n_k > m_k \geq k$

$$r_k := d(\xi_{m_k}(\omega), \xi_{n_k}(\omega)) \geq \epsilon,$$

$d(\xi_{m_k}(\omega), \xi_{n_k-1}(\omega)) < \epsilon$, for any $k \in \{1, 2, 3, ...\}$.
For each $n \geq 1$, let $\alpha_n := d(\xi_{n+1}(\omega), \xi_n(\omega))$. Then, we have

$$\begin{aligned} \epsilon \leq r_k &= d(\xi_{m_k}(\omega), \xi_{n_k}(\omega))\\ &\leq d(\xi_{m_k}(\omega), \xi_{n_k-1}(\omega)) + d(\xi_{n_k-1}(\omega), \xi_{n_k}(\omega))\\ &< \epsilon + \gamma_{n_k-1}. && (2.4)\end{aligned}$$

Taking limit as $k \to \infty$, we get

$$\begin{aligned} \epsilon &\leq \lim_{k\to\infty} r_k\\ &< \epsilon + \lim_{k\to\infty} \gamma_{n_k-1}\end{aligned}$$

(2.5)

$$\Rightarrow \epsilon \leq \lim_{k\to\infty} r_k < \epsilon + 0$$

$$\lim_{k\to\infty} d(\xi_{m_k}(\omega), \xi_{n_k}(\omega)) = \epsilon. \tag{2.6}$$

Notice also that

$$d(\xi_{m_k-1}(\omega), \xi_{n_k-1}(\omega)) \quad \leq \quad d(\xi_{m_k-1}(\omega), \xi_{m_k}(\omega)) + d(\xi_{n_k}(\omega), \xi_{m_k}(\omega)) + d(\xi_{n_k 1}(\omega), \xi_{n_k-1}(\omega)).$$

By taking limits as $n \to \infty$, we get $\lim_{n\to\infty} d(\xi_{m_k-1}(\omega)\xi_{n_k-1}(\omega)) = \epsilon$ which implies that $\lim_{n\to\infty} \mathcal{A}(\xi_{m_k-1}(\omega), \xi_{n_k-1}(\omega))$ also exists. Now, by generalized $\mathcal{P}$-contractivity, we have $d(\xi_{m_k}(\omega), \xi_{n_k}(\omega)) \leq d(\xi_{m_k-1}(\omega), \xi_{n_k-1}(\omega)) - \mathcal{A}(\xi_{m_k-1}(\omega), \xi_{n_k-1}(\omega))$. After taking limits, we get

$$0 \leq \lim_{k\to\infty} \mathcal{A}(\xi_{m_k-1}(\omega), \xi_{n_k-1}(\omega)),$$

implies that $\lim_{k\to\infty} \mathcal{A}(\xi_{m_k-1}(\omega), \xi_{n_k-1}(\omega)) = 0$. Thus

$$\lim_{n\to\infty} d(\xi_{m_k-1}(\omega), \xi_{n_k-1}(\omega)) = 0.$$

Hence $\epsilon = 0$, which is a contradiction. So, $\{\xi_n(\omega)\}$ is a Cauchy sequence in A and A is closed subset of X. There is $\xi^*(\omega) \in A$ such that $\xi_n(\omega) \to \xi^*(\omega)$ as $n \to \infty$. Since f is continuous, so we have

$$f(\omega, \xi_n(\omega)) \to f(\omega, \xi^*(\omega)).$$

$$\Rightarrow d(\xi_{n+1}(\omega), f(\omega, \xi_n(\omega))) \to d(\xi^*(\omega), f(\omega, \xi^*(\omega))).$$

Note that $\{d(\xi_{n+1}(\omega), f(\omega, \xi_n(\omega))\}$ is a constant sequence having a value $d(A, B)$, we may write as

$$d(\xi^*(\omega), f(\omega, \xi^*(\omega)) = d(A, B),$$

i.e., $\xi^*(\omega)$ is unique best random proximity point of f. □

Corollary 2.1. *Let us take a Cauchy partially ordered metric space $(X, \preceq, d)$. Ω is a probability space and $\xi : \Omega \to X$ .Define a map $f : \Omega \times X \to X$ with the following conditions:*

(1) f is continuous random self mapping $\mathcal{G}_{\mathcal{P}}$-contraction w.r.t. $\preceq$;
(2) the pair (A, B) has the P-property.

Then there exists a unique $\xi^*(\omega)$ in A such that $d(\xi^*(\omega), f(\omega, \xi^*(\omega))) = \xi^*(\omega)$.

Proof. If we consider self mapping as $A = B = X$ in above theorem then we obtain this result. □

3. Applications to best random proximity points

Definition 3.13. Let X be a metric space, A and B two nonempty subsets of X. Define

$$\begin{aligned} p(A, B) &= \inf\{p(a, b) : a \in A, b \in B\}, \\ A_{0,p} &= \{a \in A : there\ exists\ some\ b \in B\ such\ that\ p(a, b) = p(A, B)\}, \\ B_{0,p} &= \{b \in B : there\ exists\ some\ a \in A\ such\ that\ p(a, b) = p(A, B)\}. \end{aligned}$$

Definition 3.14. Let (X, d) be a metric space. Then a function $p : X \times X \to [0, \infty)$ is called w_s-distance on X if the following are satisfied:

(1) $p(x, z) \leq p(x, y) + p(y, z)$, for any $x, y, z \in X$;
(2) $p(x, y) \geq 0$, for any $x, y \in X$;
(3) if $\{x_m\}$ and $\{y_m\}$ be any sequences in X such that $x_n \to x,\ y_n \to y$ as $n \to \infty$, then $p(x_n, y_n) \to p(x, y)$ as $x \to \infty$;
(4) for any $\epsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ $d(x, y) \leq \epsilon$.

Definition 3.15. Let (A,B) be a part of nonempty subsets of a metric space (X,d) with $A_{0,p} \neq \emptyset$. Then the pair (A,B) is said to have P_p-property if and only if for any $x_1, x_2 \in A_{0,p}$ and $y_1, y_2 \in B_{0,p}$

$$\left.\begin{array}{ll} p(x_1,y_1) & = p(A,B) \\ p(x_2,y_2) & = p(A,B) \end{array}\right\} \Rightarrow p(x_1,x_2) = p(y_1,y_2).$$

Definition 3.16. Given non-self mappings $f : \Omega \times A \to B$ then an element $\xi^*(\omega) \in A$ is called best random p-proximity point of the mapping f if this condition satisfied:

$$p(\xi^*(\omega), f(\omega, \xi^*(\omega)))) = p(A,B).$$

By taking self mapping in above definition, we get random p-fixed point for self mapping $T : X \to X$.

Definition 3.17. Let $F : \mathbb{R}^+ \to \mathbb{R}$ be a mapping satisfying

(1) F is strictly increasing, i.e. for all $a, b \in \mathbb{R}^+$ such that $\alpha < \beta \Rightarrow F(\alpha) < F(\beta)$;
(2) For each sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of positive numbers $lim_{n\to\infty}\alpha_n = 0$ if and only if $lim_{n\to\infty}F(\alpha_n) = -\infty$;
(3) there exists $k \in (0,1)$ such that $lim_{\alpha\to 0^+}\alpha^k F(\alpha) = 0$.

A mapping $T : A \to B$ is said to be an F_p-contraction if there exists $\tau > 0$ such that for all $x, y \in A, p(T(\omega,\xi(\omega)), T(\omega,\eta(\omega))) > 0 \Rightarrow \tau + F(p(T(\omega,\xi(\omega))), T(\omega,\eta(\omega))) \leq F(p(\xi(\omega),\eta(\omega)))$, where p is w_s-distance.

Theorem 3.3. *Define a sequence $\{\xi_n\}$ such that $\xi : \Omega \to A$. Let A and B be non-empty, closed subsets of a complete metric space (X,d) such that $A_{0,p}$ is nonempty and Ω be a measure space. Let $T : \Omega \times A \to B$ be a random F_p-contraction such that $T(A_{0,p}) \subseteq B_{0,p}$ and $g : A \to A$ is an isometry such that $A_0 \subseteq g(A_0)$. Assume that the pair (A,B) has the P_p-property, where p is the w_s-distance. Then there exists a unique best random p-proximity point $\xi(\omega)$ in A such that $p(g\xi(\omega), T\xi(\omega)) = p(A,B)$.*

Proof. Let us consider $\xi_n : \Omega \to A$. Choose an element $\xi_0(\omega) \in A_{0,p}$. Since $T(\omega,\xi_0(\omega)) \subseteq T(A_{0,p}) \subseteq B_{0,p}$ so there exists $\xi_1(\omega) \in A_{0,p}$ such that

$$p(\xi_1(\omega), T(\omega,\xi_0(\omega))) = p(A,B). \tag{3.7}$$

Again, since $T(\omega,\xi_1(\omega)) \in T(A_{0,p}) \subseteq B_{0,p}$ and g is an isometry, we get $\xi_2(\omega) \in A_{0,p}$ such that

$$p(g\xi_2(\omega), T(\omega,\xi_1(\omega)) = p(A,B). \tag{3.8}$$

Continuing in similar fashion, we can find a sequence $\{\xi_n(\omega)\}$ in $A_{0,p}$ such that

$$p(g\xi_{n+1}(\omega), T\xi_n(\omega)) = p(A,B), \tag{3.9}$$

for all $n \in \mathbb{N}$. Also we know that (A,B) satisfies the P_p-property and g is an isometry, then we may write as

$$p(\xi_n(\omega), \xi_{n+1}(\omega)) = p(g\xi_n(\omega), g\xi_{n+1}(\omega)) = p(T(\omega,\xi_{n-1}(\omega), T(\omega,\xi_n(\omega))), \text{ for all } n \in \mathbb{N}.$$

Next, we will show the convergence of the sequence $\{\xi_n(\omega)\}$ in $A_{0,p}$. If there exists $n_0 \in \mathbb{N}$ such that $p(g\xi_{n_0-1}(\omega), T\xi_{n_0}(\omega))) = 0$, then by (3.9) we have $p(g\xi_{n_0}(\omega), g\xi_{n_0+1}(\omega)) = 0$ that implies $g\xi_{n_0}(\omega) = g\xi_{n_0+1}(\omega)$. Therefore

$$T(\omega,\xi_{n_0}(\omega)) = T(\omega,\xi_{n_0+1}(\omega))$$

implies that

$$p(T(\omega,\xi_{n_0}(\omega)), T(\omega,\xi_{n_0+1}(\omega))) = 0. \tag{3.10}$$

From (3.9) and (3.10) we have

$$p(g\xi_{n_0+2}, g\xi_{n_0+1}) = p(T(\omega, \xi_{n_0+1}(\omega), T(\omega, \xi_{n_0}(\omega))) = 0$$

implies that $g\xi_{n_0+2}(\omega) = g\xi_{n_0+1}(\omega)$. Therefore $g\xi_n(\omega) = g\xi_{n_0}(\omega)$, for all $n \geq n_0$ and $\{\xi_n(\omega)\}$ is convergent in $A_{0,p}$. Now let $p(T(\omega, \xi_{n-1}(\omega)), T(\omega, \xi_n(\omega))) = 0$, for all $n \in \mathbb{N}$. We know that T is a F_p-contraction, hence for any positive integer n we have

$$\tau + F(p(T(\omega, \xi_n(\omega)), T(\omega, \xi_{n-1}))) \leq F(p(\xi_n(\omega), \xi_{n-1}(\omega)))$$

implies that

$$F(p(\xi_{n+1}(\omega), \xi_n(\omega)) \leq F(p(\xi_n(\omega), \xi_{n-1}(\omega))) - \tau ... \leq F(p(\xi_1(\omega), \xi_0(\omega))) - n\tau. \tag{3.11}$$

By taking limit as $n \to \infty$ we get

$$\lim_{n\to\infty} F(p(\xi_{n+1}(\omega), \xi_n(\omega))) = -\infty$$

that together with axiom (2) of definition (3.17) gives

$$\lim_{n\to\infty} p(\xi_{n+1}(\omega), \xi_n(\omega)) = 0. \tag{3.12}$$

Also from third axiom of definition (3.17), we have there exists $k \in (0, 1)$ such that

$$p^k(\xi_{n+1}(\omega), \xi_n)(\omega)F(p(\xi_{n+1}(\omega), \xi_n(\omega))) = 0. \tag{3.13}$$

By (3.11) the following holds for all $n \in \mathbb{N}$.

$$F(p(\xi_{n+1}(\omega), \xi_n(\omega))) - F(p(\xi_1(\omega), \xi_0(\omega))) \leq -n\tau.$$

Therefore

$$p^k(\xi_{n+1}(\omega), \xi_n(\omega))F(p(\xi_{n+1}(\omega), \xi_n(\omega))) - p^k(\xi_{n+1}(\omega), \xi_n(\omega))F(p(\xi_1(\omega), \xi_0(\omega))) \leq$$

$$\leq -np^k(\xi_{n+1}(\omega), \xi_n(\omega))\tau \leq 0.$$

Considering $k \to \infty$ in the above inequality and using (3.12) and (3.13), we have

$$lim_{n\to\infty} np^k(\xi_{n+1}(\omega), \xi_n(\omega)) = 0.$$

Hence there exists $n_1 \in \mathbb{N}$ such that $np_k(\xi_{n+1}(\omega), \xi_n(\omega)) \leq 1$ for all $n \geq n_1$. Therefore for any $n \geq n_1$,

$$p(\xi_{n+1}(\omega), \xi_n(\omega)) \leq \frac{1}{n^{\frac{1}{k}}}, \tag{3.14}$$

which implies that the series $\Sigma_{i=1}^{\infty} p_i(\xi_{n+1}(\omega), \xi_n(\omega))$ is convergent. Now let $m \geq n \geq n_1$. Then by the triangular inequality and (3.11), we have

$$p(\xi_m(\omega), \xi_n(\omega)) \leq p_{m-1}(\xi_{n+1}(\omega), \xi_n(\omega)) + p_{m-2}(\xi_{n+1}(\omega), \xi_n(\omega)) +$$

$$... + p_n(\xi_{n+1}(\omega), \xi_n(\omega)) \leq \Sigma_{i=1}^{\infty} p_i(\xi_{n+1}(\omega), \xi_n(\omega)).$$

Therefore $\{\xi_n(\omega)\}$ is a Cauchy sequence in A. Since (X, d) is complete and A is a closed subset of X, there exist $\xi^*(\omega) \in A$ such that $\lim_{n\to\infty} \xi_n(\omega) = \xi^*(\omega)$. Since T is continuous, we have $\lim_{n\to\infty} T\xi_n(\omega) = T\xi^*(\omega)$. Hence $p(g\xi_{n+1}(\omega), T(\omega, \xi_n(\omega))) \to p(g\xi^*(\omega), T(\omega, \xi^*(\omega))$. From (3.9), $p(g\xi^*(\omega), T(\omega, \xi^*(\omega)) = p(A, B)$. So we show that $\xi^*(\omega)$ is a best random p-proximity coincidence point of (g, T).

The uniqueness of the p- optimal best proximity coincidence points can be proved as since T is random F_p- contraction and suppose that $\xi_1(\omega), \xi_2(\omega) \in A$ such that $\xi_1(\omega) \neq \xi_2(\omega)$ and

$$p(\xi_1(\omega), T(\omega, \xi_1(\omega))) = p(\xi_2(\omega), T(\omega, \xi_2(\omega))) = p(A, B).$$

Then by the P_p-property of (A, B), we have

$$p(g\xi_1(\omega), g\xi_2(\omega)) = p(T(\omega, \xi_1(\omega)), T(\omega, \xi_2(\omega))).$$

Also $\xi_1(\omega) \neq \xi_2(\omega) \Rightarrow p(\xi_1(\omega), \xi_2(\omega)) \neq 0$. Thus

$$\begin{aligned} F(p(g\xi_1(\omega), g\xi_2(\omega))) &= F(p(T(\omega, \xi_1(\omega), T(\omega, \xi_2(\omega)))) \\ &. \\ &. \\ &. \\ &\leq F(p(\xi_1(\omega), \xi_2(\omega))) - \tau \\ &< p(\xi_1(\omega), \xi_2(\omega)), \end{aligned}$$

which is a contraction. Hence the best random p-proximity coincidence point is unique. □

By setting $A = B = X$ in Theorem 3.3 we obtained following result which is a special case of that theorem.

Corollary 3.2. *Let a complete metric space (X, d) and Ω be a measure space. Let $T : \Omega \times X \to X$ be a random F_p-contraction with self mapping and $g : X \to X$ is an isometry, where p is the w_s-distance. Then there exists a unique random p-fixed point $\xi(\omega)$ in X such that $p(g\xi(\omega), T\xi(\omega)) = g\xi(\omega)$.*

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