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Some coincidence point theorems in ordered metric spaces via w-distances

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ABSTRACT. The purpose of this paper is to prove some existence theorems of coincidence points for generalized weak contractions in the setting of partially ordered sets with a metric via w-distances and give some example to illustrate our main results.

1. INTRODUCTION

In 1996, Kada et al. [7] introduced the generalized metric, which called the *w-distance*, and gave some examples of the w-distance. Also, they improved Caristi's fixed point theorem, Ekeland's variational principle and the nonconvex minimization theorem according to the results of Takahashi [11].

On the other hand, in 1997, Alber and Guerre-Delabriere [1] introduced the concept of weak contractions in Hilbert spaces. Later, in 2001, Rhoades [9] showed that the results of Alber and Guerre-Delabriere are also valid in complete metric spaces. In 2008, Dutta and Choudhury [3] extended the notion of weak contractions by using the concept of two altering distance functions. In 2012, Imdad and Rouzkard [5] proved some fixed point theorems in complete metric spaces equipped with a partial order via the w-distance. Recently, in 2014, Roshan et. al. [10], using the concept of weak contractions, proved some existence theorems of coincidence points for some generalized contractions in the framework of ordered b-metric spaces.

In this paper, we prove some existence theorems of coincidence points for generalized weak contractions in the setting of partially ordered sets with a metric via the w-distance. Also, we give some example to illustrate our main result.

2. PRELIMINARIES

First, we give some definitions, some examples and lemmas for our main results.

Definition 2.1. Let X be an nonempty set and $f, g : X \to X$ be two mappings. A point $x \in X$ is called a *coincidence point* of f and g if $fx = gx$, where $C(f, g)$ denote the sets of coincidence points of f and g.

Definition 2.2. [6] Let (X, d) be a metric space and $f, g : X \to X$ be two mappings. The pair (f, g) is said to be *compatible* if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some $z \in X$.

Definition 2.3. Let $(X, \leq)$ be a partially ordered set. The elements $x, y \in X$ are said to be *comparable* with respect to $\leq$ if either $x \leq y$ or $y \leq x$.

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Definition 2.4. [2] A triple $(X, d, \leq)$ is called an *ordered metric space* if (X, d) is a metric space with the partial order $\leq$.

Let $X_{\leq} \subset X \times X$ be defined by $X_{\leq} = \{(x, y) \in X \times X : x \leq y \text{ or } y \leq x\}$.

Definition 2.5. [4] (1) An ordered metric space $(X, d, \leq)$ is said to have the *sequential g-monotone property* if it satisfies the following properties:

(a) if $\{x_m\}$ is a non-decreasing sequence and $\lim_{m\to\infty} x_m = x$, then $gx_m \leq gx$ for all $m \geq 1$;
(b) if $\{y_m\}$ is a non-increasing sequence and $\lim_{m\to\infty} y_m = y$, then $gy_m \geq gy$ for all $m \geq 1$.

(2) If g is the identity mapping, then $(X, d, \leq)$ is said to have the *sequential monotone property*.

In 1984, Khan et al. [8] introduced the concept of an altering distance function as follows:

Definition 2.6. [8] A function $\psi : [0, \infty) \to [0, \infty)$ is called an *altering distance function* if the following properties are satisfied:

(a) ψ is continuous and monotone nondecreasing;
(b) $\psi(t) = 0$ if and only if $t = 0$.

Now, Ψ denotes the family of all altering distance functions and we give some examples of the altering distance function as follow:

Example 2.1. For each $i \in \{1, 2\}$, let $\varphi_i : [0, \infty) \to [0, \infty)$ be a function defined by

(φ_1) $\varphi_1(t) = t^k$ for all $t \in [0, \infty)$, for any $k > 0$;
(φ_2) $\varphi_2(t) = a^t - 1$ for all $t \in [0, \infty)$,for any $a > 0$ with $a \neq 1$.
Then φ_i is an altering distance function for each $i \in \{1, 2\}$.

Now, we recall the concept of w-distances and some useful lemmas for the main results.

Definition 2.7. [7] Let (X, d) be a metric space. A function $p : X \times X \to [0, \infty)$ is called a *w-distance* on X if the following are satisfied:

(a) $p(x, z) \leq p(x, y) + p(y, z)$ for all $x, y, z \in X$;
(b) for any $x \in X, p(x, \cdot) : X \to [0, \infty)$ is lower semi-continuous (i.e., if $x \in X$ and $y_n \to y \in X$, then $p(x, y) \leq \liminf_{n\to\infty} p(x, y_n)$);
(c) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $d(x, y) \leq \varepsilon$.

Let X be a metric space with the metric d. We recall some example in [12] to show that the w-distance is a generalization of the metric d.

Example 2.2. Let (X, d) be a metric space. A function $p : X \times X \to [0, \infty)$ defined by $p(x, y) = c$ for all $x, y \in X$ is a w-distance on X, where c is a positive real number. But p is not a metric since $p(x, x) = c \neq 0$ for any $x \in X$.

Lemma 2.1. [7, 12] *Let (X, d) be a metric space with the w-distance p. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X, whereas $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $[0, \infty)$ converging to zero. Then the following conditions hold: for all $x, y, z \in X$,*

(1) *If $p(x_n, y) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for all $n \geq 1$, then $y = z$. In particular, if $p(x, y) = 0$ and $p(x, z) = 0$, then $y = z$;*
(2) *If $p(x_n, y_n) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for all $n \geq 1$, then $\{y_n\}$ converges to z;*
(3) *If $p(x_n, y_m) \leq \alpha_n$ for all $n, m \geq 1$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence;*
(4) *If $p(y, x_n) \leq \alpha_n$ for all $n \geq 1$, then $\{x_n\}$ is a Cauchy sequence.*

Lemma 2.2. [7] *Let (X, d) be a metric space with the w-distance p. Let $\{x_n\}$ be sequences in X such that, for each $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $m > n > N_\varepsilon$ implies $p(x_n, x_m) < \varepsilon$ or $\lim_{m,n\to\infty} p(x_n, x_m) = 0$. Then $\{x_n\}$ is a Cauchy sequence.*

Next, we give the concept of compatible mappings in metric space with the w-distance.

Definition 2.8. Let (X, d) be a metric space with the w-distance p. The mappings $f, g : X \to X$ are said to be *compatible* if

$$\lim_{n\to\infty} fgx_n = \lim_{n\to\infty} gfx_n$$

with $\lim_{n\to\infty} p(fgx_n, gfx_n) = \lim_{n\to\infty} p(gfx_n, fgx_n)$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some $z \in X$.

Remark 2.1. If $p = d$, then Definition 2.8 become to Definition 2.2.

3. Main results

In this section, we establish some existence theorems of coincidence points for generalized weak contractions in partially ordered metric spaces via the w-distances. Also, we give some example to illustrate our main results.

Theorem 3.1. *Let $(X, d, \le)$ be a complete ordered metric space equipped with the w-distance p and $f, g : X \to X$ be two mappings such that f has the mixed g-monotone property on X, $f(X) \subseteq g(X)$ and g is continuous and compatible with f. Assume that there exist $\psi, \varphi \in \Psi$ such that*

$$\psi(p(fx, fy)) \le \psi\left(\mathcal{M}_p^g(x, y)\right) - \varphi\left(\mathcal{M}_p^g(x, y)\right) \tag{3.1}$$

for all $x, y \in X$, where

$$\mathcal{M}_p^g(x, y) = \max\{p(gx, gy), \min\{p(gx, fx), p(gy, fy), p(fx, gx), p(fy, gy)\}\}$$

for any $(gx, gy) \in X_\le$, and one of the following holds:

(a) *f is continuous;*
(b) *X has the sequential g-monotone property.*

Suppose that there exist $x_0 \in X$ such that $(g(x_0), f(x_0)) \in X_\le$. Then f and g have at least one coincidence point. Furthermore, If the sequence $\{gx_n\}$ converges to a point $x_\star \in X$, then

$$\lim_{n\to\infty} p(gfx_n, fx_\star) = 0 = \lim_{n\to\infty} p(fgx_n, gx_\star).$$

Proof. If we have $g(x_0) = f(x_0)$ for some $x_0 \in X$, then there is nothing to prove. Suppose that $x_0 \in X$ such that $g(x_0) \neq f(x_0)$ and $(g(x_0), f(x_0)) \in X_\le$. Since $f(X) \subseteq g(X)$, it follows that there exits $x_1 \in X$ such that $f(x_0) = g(x_1)$ and so $(g(x_0), g(x_1)) \in X_\le$. By the mixed g-monotone property of f, we have $(f(x_0), f(x_1)) \in X_\le$. Again, since $f(X) \subseteq g(X)$, there exits $x_2 \in X$ such that $f(x_1) = g(x_2)$ and hence $(g(x_1), g(x_2)) \in X_\le$. Continuing this way, we have a sequence $\{gx_n\}$ such that $(gx_n, gx_m) \in X_\le$ for any $m, n \in \mathbb{N}$.

Now, we show that

$$\lim_{n\to\infty} p(gx_n, gx_{n+1}) = 0. \tag{3.2}$$

For any $n \in \mathbb{N}$, by (3.1), we have

$$\begin{aligned} \psi(p(gx_{n+1}, gx_{n+2})) &= \psi(p(fx_n, fx_{n+1})) \\ &\le \psi\left(\mathcal{M}_p^g(x_n, x_{n+1})\right) - \varphi\left(\mathcal{M}_p^g(x_n, x_{n+1})\right). \end{aligned} \tag{3.3}$$

Note that

$$\begin{aligned}\mathcal{M}_p^g(x_n, x_{n+1}) &= \max\{p(gx_n, gx_{n+1}), \min\{p(gx_n, fx_n), p(gx_{n+1}, fx_{n+1}), \\ &\qquad p(fx_n, gx_n), p(fx_{n+1}, gx_{n+1})\}\} \\ &= \max\{p(gx_n, gx_{n+1}), \min\{p(gx_n, gx_{n+1}), p(gx_{n+1}, gx_{n+2}), \\ &\qquad p(gx_{n+1}, gx_n), p(gx_{n+2}, gx_{n+1})\}\}.\end{aligned}$$

Case I. If

$$\min\{p(gx_n, gx_{n+1}), p(gx_{n+1}, gx_{n+2}), p(gx_{n+1}, gx_n), p(gx_{n+2}, gx_{n+1})\} = p(gx_n, gx_{n+1}),$$

then we have $\mathcal{M}_p^g(x_n, x_{n+1}) = p(gx_n, gx_{n+1})$.

Case II. If

$$\begin{aligned}&\min\{p(gx_n, gx_{n+1}), p(gx_{n+1}, gx_{n+2}), p(gx_{n+1}, gx_n), p(gx_{n+2}, gx_{n+1})\} \\ &\neq p(gx_n, gx_{n+1}),\end{aligned}$$

then we have

$$\begin{aligned}&\min\{p(gx_n, gx_{n+1}), p(gx_{n+1}, gx_{n+2}), p(gx_{n+1}, gx_n), p(gx_{n+2}, gx_{n+1})\} \\ &< p(gx_n, gx_{n+1})\end{aligned}$$

and hence $\mathcal{M}_p^g(x_n, x_{n+1}) = p(gx_n, gx_{n+1})$. Therefore, by Cases I, II and (3.3), we have

$$\begin{aligned}\psi(p(gx_{n+1}, gx_{n+2})) &\leq \psi(p(gx_n, gx_{n+1})) - \varphi(p(gx_n, gx_{n+1})) \\ &\leq \psi(p(gx_n, gx_{n+1})).\end{aligned}$$

By the property of ψ, the sequence $\{p(gx_n, gx_{n+1})\}$ is non-increasing and converges to some $r \geq 0$. Taking $n \to \infty$ in the above inequality, we have

$$\psi(r) \leq \psi(r) - \varphi(r) \leq \psi(r),$$

which implies that $r = 0$ and hence (3.2) hold. Using the same method, we can see that

$$\lim_{n\to\infty} p(gx_{n+1}, gx_n) = 0. \tag{3.4}$$

Next, we claim that, for any $m, n \in \mathbb{N}$,

$$\lim_{n\to\infty} p(gx_n, gx_m) = 0. \tag{3.5}$$

Suppose that (3.5) does not hold. Then there exists $\delta > 0$ for which we can find subsequences $\{gx_{m_k}\}$ and $\{gx_{n_k}\}$ of $\{gx_n\}$ with $n_k > m_k \geq k$ such that

$$p(gx_{m_k}, gx_{n_k}) \geq \delta \tag{3.6}$$

and n_k is the smallest number such that (3.6) holds, but

$$p(gx_{m_k}, gx_{n_k-1}) < \delta. \tag{3.7}$$

This, in view of (3.6) and (3.7), gives that

$$\begin{aligned}\delta &\leq d(gx_{m_k}, gx_{n_k}) \\ &\leq d(gx_{m_k}, gx_{n_k-1}) + p(gx_{n_k-1}, gx_{n_k}) \\ &< \delta + d(gx_{n_k-1}, gx_{n_k}).\end{aligned}$$

Then, by using (3.2), we have $\lim_{k\to\infty} p(gx_{m_k}, gx_{n_k}) = \delta$.

Next, we prove that

$$\lim_{k\to\infty} \sup p(gx_{m_k+1}, gx_{n_k+1}) < \delta. \tag{3.8}$$

If $\lim_{k\to\infty} \sup p(gx_{m_k+1}, gx_{n_k+1}) \geq \delta$, then there exists a subsequence $\{k_r\}$ of $\{k : k \geq 1\}$ such that

$$\lim_{r\to\infty} p(gx_{m_{k_r}+1}, gx_{n_{k_r}+1}) = \varepsilon \geq \delta.$$

Since $(gx_n, gx_m) \in X_{\leq}$, by (3.1), we have

$$\begin{aligned}\psi(p(gx_{m_{k_r}+1}, gx_{n_{k_r}+1})) &= \psi(p(fx_{m_{k_r}}, fx_{n_{k_r}})) \\ &\leq \psi(\mathcal{M}_p^g(x_{m_{k_r}}, x_{n_{k_r}})) - \varphi(\mathcal{M}_p^g(x_{m_{k_r}}, gx_{n_{k_r}})) \\ &\leq \psi(\mathcal{M}_p^g(x_{m_{k_r}}, x_{n_{k_r}})).\end{aligned} \tag{3.9}$$

Note that

$$\begin{aligned}&\mathcal{M}_p^g(x_{m_{k_r}}, x_{n_{k_r}}) \\ =\ &\max\{p(gx_{m_{k_r}}, gx_{n_{k_r}}), \min\{p(gx_{m_{k_r}}, fx_{m_{k_r}}), p(gx_{n_{k_r}}, fx_{n_{k_r}}), \\ &\qquad p(fx_{m_{k_r}}, gx_{m_{k_r}}), p(fx_{n_{k_r}}, gx_{n_{k_r}})\}\} \\ =\ &\max\{p(gx_{m_{k_r}}, gx_{n_{k_r}}), \min\{p(gx_{m_{k_r}}, gx_{m_{k_r}+1}), p(gx_{n_{k_r}}, gx_{n_{k_r}+1}), \\ &\qquad p(gx_{m_{k_r}+1}, gx_{m_{k_r}}), p(gx_{n_{k_r}+1}, gx_{n_{k_r}})\}\}.\end{aligned}$$

Then we have

$$\lim_{r\to\infty} \mathcal{M}_p^g(x_{m_{k_r}}, x_{n_{k_r}}) = \max\{\delta, \min\{0,0,0,0\}\} = \delta. \tag{3.10}$$

Letting $r \to \infty$ in (3.9) and using (3.10), we have

$$\psi(\varepsilon) \leq \psi(\delta) - \varphi(\delta) < \psi(\delta).$$

So, we have $\delta = 0$, which is a contradiction and hence (3.8) hold.

Thus, from (3.6), (3.2) and (3.4), it follows that

$$\begin{aligned}\delta &\leq \lim_{k\to\infty} d(gx_{m_k}, gx_{n_k}) \\ &\leq \lim_{k\to\infty} d(gx_{m_k}, gx_{m_k+1}) + \lim_{k\to\infty} p(gx_{m_k+1}, gx_{n_k+1}) \\ &\quad + \lim_{k\to\infty} p(gx_{n_k+1}, gx_{n_k}) \\ &\leq \lim_{k\to\infty} \sup p(gx_{m_k+1}, gx_{n_k+1}) \\ &< \delta,\end{aligned}$$

which is a contradiction and thus we obtain the claim (3.5). By Lemma 2.2, the sequence $\{gx_n\}$ is a Cauchy sequence. Since X is a complete ordered metric space, the sequence $\{gx_n\}$ converges to a point $x_\star \in X$ and

$$\lim_{n\to\infty} fx_{n-1} = \lim_{n\to\infty} gx_n = x_\star. \tag{3.11}$$

If f is continuous, since g is continuous and the pair (f, g) compatible, then we have $gx_\star = \lim_{n\to\infty} gfx_n = \lim_{n\to\infty} fgx_n = fx_\star$, that is, $x_\star$ is a coincidence of f and g.

Suppose that the assumption (b) holds. Since $(gx_{n-1}, gx_n) \in X_{\leq}$, it follows from (3.1) that

$$\psi(p(fgx_{n-1}, fgx_n) \leq \psi(\mathcal{M}_p^g(gx_{n-1}, gx_n)) - \varphi(\mathcal{M}_p^g(gx_{n-1}, gx_n))$$

and

$$\begin{aligned}\mathcal{M}_p^g(gx_{n-1}, gx_n) &= \max\{p(ggx_{n-1}, ggx_n, \min\{p(ggx_{n-1}, fgx_{n-1}), p(ggx_n, fgx_n), \\ &\qquad p(fgx_{n-1}, ggx_{n-1}), p(fgx_n, ggx_n)\}\}.\end{aligned}$$

By (3.11), the pair (f, g) compatible and the mapping g is continuous, we have

$$\begin{aligned}&\lim_{n\to\infty} \mathcal{M}_p^g(gx_{n-1}, gx_n) \\ =\ &\lim_{n\to\infty} \max\{p(ggx_{n-1}, ggx_n), \min\{p(ggx_{n-1}, fgx_{n-1}), p(ggx_n, fgx_n), \\ &\qquad p(fgx_{n-1}, ggx_{n-1}), p(fgx_n, ggx_n)\}\} \\ =\ &\lim_{n\to\infty} \max\{p(ggx_{n-1}, ggx_n), \min\{p(ggx_{n-1}, gfx_{n-1}), p(ggx_n, gfx_n), \\ &\qquad p(gfx_{n-1}, ggx_{n-1}), p(gfx_n, ggx_n)\}\} \\ =\ &\max\{p(gx_\star, gx_\star), \min\{p(gx_\star, gx_\star), p(gx_\star, gx_\star)\}\} \\ =\ &p(gx_\star, gx_\star).\end{aligned}$$

Hence we have

$$\begin{aligned}\psi(p(gx_\star, gx_\star)) &= \lim_{n\to\infty} \psi(p(gfx_{n-1}, gfx_n) \\ &= \lim_{n\to\infty} \psi(p(fgx_{n-1}, fgx_n) \\ &\leq \lim_{n\to\infty} \big(\psi(\mathcal{M}_p^g(gx_{n-1}, gx_n)) - \varphi(\mathcal{M}_p^g(gx_{n-1}, gx_n))\big) \\ &= \psi(p(gx_\star, gx_\star)) - \varphi(p(gx_\star, gx_\star)) \\ &\leq \psi(p(gx_\star, gx_\star)).\end{aligned}$$

So, we have $p(gx_\star, gx_\star) = 0$. Furthermore, we have

$$\lim_{n\to\infty} p(fgx_n, gx_\star) = \lim_{n\to\infty} p(gfx_n, gx_\star) = p(gx_\star, gx_\star)$$

Let

$$p(fgx_n, gx_\star) \leq \alpha_n \tag{3.12}$$

for a sequence $\{\alpha_n\}$ converging to zero. On the other hand, since $\{gx_n\}$ converges to $x_\star$, by the assumption (b), we have $(ggx_n, gx_\star) \in X_\leq$ for any $n \in \mathbb{N}$. Then we have

$$\psi(p(fgx_n, fx_\star) \leq \psi(\mathcal{M}_p^g(gx_n, x_\star)) - \varphi(\mathcal{M}_p^g(gx_n, x_\star)) \tag{3.13}$$

and

$$\begin{aligned}\mathcal{M}_p^g(gx_n, x_\star) &= \max\{p(ggx_n, gx_\star, \min\{p(ggx_n), fgx_n), p(gx_\star, fx_\star), \\ &\qquad p(fgx_n, ggx_n), p(fx_\star, gx_\star)\}\}.\end{aligned}$$

Since

$$\begin{aligned}&\lim_{n\to\infty} \mathcal{M}_p^g(gx_n, x_\star) \\ &= \lim_{n\to\infty} \max\{p(ggx_n, gx_\star), \min\{p(ggx_n, fgx_n), p(gx_\star, fx_\star), \\ &\qquad p(fgx_n, ggx_n), p(fx_\star, gx_\star)\}\} \\ &= \lim_{n\to\infty} \max\{p(ggx_n, gx_\star), \min\{p(ggx_n, gfx_n), p(gx_\star, fx_\star), \\ &\qquad p(gfx_n, ggx_n), p(fx_\star, gx_\star)\}\} \\ &= \lim_{n\to\infty} \max\{p(gx_\star, gx_\star), \min\{p(gx_\star, gx_\star), p(gx_\star, fx_\star), \\ &\qquad p(gx_\star, gx_\star), p(fx_\star, gx_\star)\}\} \\ &= \max\{0, \min\{0, p(gx_\star, fx_\star), 0, p(fx_\star, gx_\star)\}\} \\ &= 0,\end{aligned}$$

by taking $n \to \infty$ in (3.13), we have

$$\lim_{n\to\infty} \psi(p(fgx_n, fx_\star) \leq \psi(0) - \varphi(0) \leq \psi(0),$$

which implies that $\lim_{n\to\infty} p(fgx_n, fx_\star) = 0$ and thus let $p(fgx_n, fx_\star) \leq \beta_n$ for a sequence $\{\beta_n\}$ converging to zero. Therefore, by Lemma 2.1 (1), we have $fx_\star = gx_\star$. This completes the proof. □

If we have $\mathcal{M}_p^g(x, y) = p(gx, gy)$ in Theorem 3.1, then we obtain the following:

Corollary 3.1. *Let $(X, d, \leq)$ be a complete ordered metric space equipped with the w-distance p and $f, g : X \to X$ be two mappings such that f has the mixed g-monotone property on X, $f(X) \subseteq g(X)$ and g is continuous and compatible with f. Assume that there exist $\psi, \varphi \in \Psi$ such that*

$$\psi(p(fx, fy)) \leq \psi\left(p(gx, gy)\right) - \varphi\left(p(gx, gy)\right)) \tag{3.14}$$

for all $x, y \in X$ for which $(gx, gy) \in X_\leq$ and one of the following hold:

(a) *f is continuous;*

(b) *X has the sequential g-monotone property.*

Suppose that there exist $x_0 \in X$ *such that* $(g(x_0), f(x_0)) \in X_{\leq}$. *Then* f *and* g *have at least one coincidence point.*

Next, we give an example to illustrate Theorem 3.1.

Example 3.3. Let $X = [0, \infty)$ with the Euclidean metric and the usual order equipped with the w-distance p define by $p(x, y) = y$ for all $x, y \in X$. Let f and g be the self-mappings on X defined by

$$f(x) = \sinh^{-1} \frac{x}{2}, \quad g(x) = \sinh(2x)$$

for all $x \in X$. Then f and g are continuous and, furthermore, $f(X) \subseteq g(X)$. Let $x_\star \in X$ be such that $\lim_{n\to\infty} gx_n = x_\star = \lim_{n\to\infty} fx_n$. Then we have

$$\sinh(2x_n) = \lim_{n\to\infty} gx_n = x_\star = \lim_{n\to\infty} fx_n = \sinh^{-1} \frac{x_n}{2}.$$

Further, by the continuity of f and g, we have

$$\frac{\sinh^{-1} x_\star}{2} = \lim_{n\to\infty} x_n = 2 \sinh x_\star,$$

which gives $\sinh^{-1} x_\star = 4 \sinh x_\star$. From

$$\sinh^{-1} x_\star = 4 \sinh x_\star \iff x_\star = 0,$$

it follows that

$$\lim_{n\to\infty} fgx_n = f\big(\lim_{n\to\infty} gx_n\big) = fx_\star = gx_\star = g\big(\lim_{n\to\infty} fx_n\big) = \lim_{n\to\infty} gfx_n.$$

Now, we show that f and g satisfy (3.1) with the altering distance functions $\psi, \varphi : [0, \infty) \to [0, \infty)$ defined by $\psi(t) = \lambda t$ and $\varphi(t) = (\lambda - 1)t$ for all $t \in [0, \infty)$, where $\lambda \in (1, 2)$. Let $x, y \in X$ be such that $(x, y) \in X_{\leq}$. Then we have

$$\begin{aligned}
\psi(p(fx, fy)) &= \psi\Big(\sinh^{-1} \frac{y}{2}\Big) = \lambda \sinh^{-1} \frac{y}{2} \leq \frac{\lambda}{2}(y) \\
&\leq \frac{\lambda}{4}(2y) \leq \frac{\lambda}{4} \sinh(2y) \leq p(gx, gy) \leq \mathcal{M}_p^g(x, y) \\
&= \psi(\mathcal{M}_p^g(x, y)) - \varphi(\mathcal{M}_p^g(x, y)).
\end{aligned}$$

Therefore, all the conditions of Theorem 3.1 are satisfied. Moreover, $0 \in X$ is a coincidence point of f and g.

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