CARPATHIAN J. MATH.
Volume **34** (2018), No. 2,
Pages 229 - 237

Online version at `https://www.carpathian.cunbm.utcluj.ro/`
Print Edition: ISSN 1584 - 2851; Online Edition: ISSN 1843 - 4401
DOI: https://doi.org/10.37193/CJM.2018.02.11

Proximal point algorithms involving fixed point iteration for nonexpansive mappings in CAT(κ) spaces

NUTTAPOL PAKKARANANG[1], POOM KUMAM[1,2], PRASIT CHOLAMJIAK[3],
RAWEEROTE SUPARATULATORN[4] and PARIN CHAIPUNYA[1]

ABSTRACT. In this paper, we propose a new modified proximal point algorithm involving fixed point iteration for nonexpansive mappings in CAT(1) spaces. Under some mild conditions, we prove that the sequence generated by our iterative algorithm Δ-converges to a common solution between certain convex optimization and fixed point problems.

1. INTRODUCTION

One of the most major analytical problems is the fixed points problem for nonlinear mappings. For a real number κ, a CAT(κ) space is defined by a geodesic space whose geodesic triangle is sufficiently thinner than the corresponding comparison triangle in a model space with curvature κ. The concept of such a space has been investigated by lots of researchers. In 2003, Kirk [20, 21] first proved the existence of fixed points for nonexpansive mappings in a CAT(κ) space for $\kappa \leq 0$. Later on Espánola and Fernández-León showed this result for $\kappa > 0$. In case of at least one fixed point exists, it is natural to ask whether such a fixed point can be approximated by simple iteration processes. There are many approximation methods of fixed points for nonexpansive mappings T. In 2012, He et al. [10] using one of the most successive approximation method is Mann algorithm [23] which is defined in CAT(κ) spaces by

$$\begin{cases} x_1 \in X, \\ x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n) T x_n \text{ for all } n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}$ is a real sequence in [0,1]. They proved Δ-convergence theorem for nonexpansive mapping in CAT(κ) spaces for $\kappa > 0$. Later on, there are many researchers have shown convergence results in CAT(κ) with curvature bounded above (see, for examples [14, 15, 16, 26]).

The proximal point algorithm has applications in various fields such as saddle point problem, variational inequality problems, convex minimization problems and equilibrium problems [28, 32, 4]. In 1970, this algorithm was first introduced by Martinet [24]. Later in 1976, Rockafellar [29] studied the proximal point algorithm for solving maximal monotone generalized equation, which can be reduced to solve for a minimizer of a certain convex function. In this particular case, let $g : X \to (-\infty, \infty]$ be a proper lower semi-continuos convex function in a Hilbert space H. The Proximal point algorithm is

Received: 29.09.2017. In revised form: 13.06.2018. Accepted: 20.06.2018
2010 *Mathematics Subject Classification.* 47H09, 47H10, 52A41.
Key words and phrases. *Proximal point algorithm, Mann iteration, Resolvent, Nonexpansive mappings, CAT(κ) spaces.*
Corresponding author: Poom Kumam; poom.kum@kmutt.ac.th

defined by

$$\begin{cases} x_1 \in H, \\ x_{n+1} = \arg\min_{y\in H} \left[g(y) + \frac{1}{2\lambda_n}\|y - x_n\|^2 \right], \end{cases}$$

for each $n \in \mathbb{N}$, where $\lambda_n > 0$ for all $n \in \mathbb{N}$.

In 2013, Bačák [3] introduced the proximal point algorithm in a CAT(κ) space for $\kappa \le 0$ so-called CAT(0) spaces (X, d) as follows:

$$\begin{cases} x_1 \in X, \\ x_{n+1} = \arg\min_{y\in X} \left[g(y) + \frac{1}{2\lambda_n} d^2(y, x_n) \right], \end{cases}$$

for each $n \in \mathbb{N}$, where $\lambda_n > 0$ for all $n \in \mathbb{N}$. He also proved that, if f has a minimizer and $\sum_{n=1}^{\infty} \lambda_n = \infty$, then the sequence $\{x_n\}$ Δ-converges to its minimizer (see also [1]). Later on, in 2014, Bačák [2] extend the proximal point algorithm to split version for minimizing a sum of convex functions in complete CAT(0) spaces. Some interesting results restricted to Hadamard manifolds can be found in [9, 22]. The general proximal point algorithm designed for variational inequalities and maximal monotone operators in CAT(0) spaces are also investigated in [13, 6], respectively.

Recently, Kimura and Kohsaka [19] introduced the proximal point algorithm in a CAT(1) space (X, d) as follows:

$$\begin{cases} x_1 \in X, \\ x_{n+1} = \arg\min_{y\in X} \left[g(y) + \frac{1}{\lambda_n} \tan d(y, x_n) \sin d(y, x_n) \right], \end{cases}$$

for each $n \in \mathbb{N}$, where $\lambda_n > 0$ for all $n \in \mathbb{N}$. They also proved that, if f has a minimizer and $\sum_{n=1}^{\infty} \lambda_n = \infty$, then the sequence $\{x_n\}$ Δ-converges to its minimizer.

In this paper, motivated and inspired by above results of the previous works, we propose the modified proximal point algorithm using the Mann algorithm for nonexpansive mappings in CAT(1) spaces for finding a common solution between a convex optimization and a fixed point problem. We also show the Δ-convergence of the proposed algorithm under some mild conditions.

2. Preliminaries

In this section, we give some fundamental concepts, definitions, and useful lemmas which will be used in the next section.

Let (X, d) be a metric space and $x, y \in X$ with $d(x, y) = l$. A geodesic path from x to y is an isometry $\gamma : [0, l] \to X$ such that $\gamma(0) = x, \gamma(l) = y$. The image of a geodesic path is called geodesic segment. The space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic, and X is a uniquely geodesic space if every two points of X are joined by only one geodesic segment. We write $(1-t)x \oplus ty$ for the unique point z in the geodesic segment joining x and y such that

$$d(x, z) = td(x, y) \qquad \text{and} \qquad d(y, z) = (1 - t)d(x, y)$$

for $t \in [0, 1]$. A set $E \subset X$ is said to be convex if E includes every geodesic segment joining any two of its points. The set E is said to be bounded if

$$\text{diam}(E) := \sup\{d(x, y) : x, y \in E\}.$$

Now we introduce the model spaces M_κ^n. Let $n \in N$. We denote by $\mathbb{E}^n$ the metric space $\mathbb{R}^n$ endowed with the usual Euclidean distance. We denote by $(\cdot|\cdot)$ the Euclidean scalar product in $\mathbb{R}^n$ such that

$$(x|y) = x_1y_1 + ... + x_ny_n$$

where $x = (x_1, ..., x_n), y = (y_1, ..., y_n)$.
Let $\mathbb{S}_n$ denote the *n-dimensional sphere* defined by

$$\mathbb{S}^n = \{x = (x_1, ..., x_{n+1}) \in \mathbb{R}^{n+1} : (x|x) = 1\},$$

with metric $d_{\mathbb{S}^n}(x, y) = arccos(x|y), x, y \in \mathbb{S}^n$. See [30] for more details on these spaces.

Let D be a positive number. A metric space (X, d) is called a *D-geodesic* space if any two points of X with the distance $< D$ are joined by a geodesic path. If this holds in a convex set E, then E is said to be *D-convex*. For a constant κ, we denote M_κ by the 2-dimensional, complete, simply connected spaces of curvature κ.

In the following, for each positve real number κ and define the diameter D_κ of M_κ by $D_\kappa = \frac{\pi}{\sqrt{\kappa}}$ for $k > 0$ and $D_\kappa = \infty$ for $\kappa = 0$. It is well known that any ball in X with radius $< D_\kappa/2$ is convex [5]. A geodesic triangle $\Delta(x, y, z)$ in the metric space (X, d) consists of three points x, y, z in X (the vertices of Δ) and three geodesic segments between each pair of vertices. For $\Delta(x, y, z)$ in a geodesic space X satisfying

$$d(x, y) + d(y, z) + d(z, x) < 2D_\kappa, \tag{2.1}$$

there exist points $\bar{x}, \bar{y}, \bar{z} \in M_\kappa^2$ such that $d(x, y) = d_\kappa(\bar{x}, \bar{y}),\ d(y, z) = d_\kappa(\bar{y}, \bar{z})$, and $d(z, x) = d_\kappa(\bar{z}, \bar{x})$ where d_κ is the metric of M_κ. The set Δ and $\bar{\Delta}$ defined by

$$\Delta = [x, y] \cup [y, z] \cup [z, x] \text{ and } \bar{\Delta} = [\bar{x}, \bar{y}] \cup [\bar{y}, \bar{z}] \cup [\bar{z}, \bar{x}]$$

are said to be a geodesic triangle having verticles x, y, z and a comparison triangle for $\Delta(x, y, z)$, respectively. A geodesic triangle of $\Delta(x, y, z)$ in X satisfying (2.1) is called to satisfy the *CAT(κ)* inequality if, for any $p, q \in \Delta(x, y, z)$ and for their comparison points $\bar{p}, \bar{q} \in \bar{\Delta}(\bar{x}, \bar{y}, \bar{z})$, then we have

$$d(p, q) \le d_\kappa(\bar{p}, \bar{q}).$$

Definition 2.1. A metric space (X, d) is called a CAT(κ) space if it is D_κ-geodesic and any geodesic triangle $\Delta(x, y, z)$ in X with $d(x, y) + d(y, z) + d(z, x) < 2D_\kappa$ satisfies the CAT(κ) inequality.

For more detailed discussion on geodesic spaces. The interested reader can consult, for instance, [5, 27, 7, 11, 31].

In the sequel, we denote by $F(T)$ the set of all fixed point of a self-mapping T of X. Then a self-mapping T of X is said to be *nonexpansive* if

$$d(Tx, Ty) \le d(x, y) \text{ for all } x, y \in X.$$

Since many results and basic concepts in CAT(κ) spaces can be deduced from those in CAT(1) spaces, we now sufficiently state useful lemmas on CAT(1) spaces.

It is well known that if (X, d) is a CAT(1) space, $x, y, z \in X$ satisfy (2.1) for $\kappa = 1$ and $\alpha \in [0, 1]$, then we have

$$\cos d(\alpha x \oplus (1 - \alpha)y, z) \ge \alpha \cos d(x, z) + (1 - \alpha) \cos d(y, z). \tag{2.2}$$

Let $R \in (0, 2]$. Recall that a geodesic space (X, d) is said to be $R - convex$ for R (see [25]) if any $\alpha \in [0, 1]$ and for any three points $x, y, z \in X$, we have

$$d^2(x,(1-\alpha)y \oplus \alpha z) \le (1-\alpha)d^2(x,y) + \alpha d^2(x,z) - \frac{R}{2}(1-\alpha)\alpha d^2(y,z). \tag{2.3}$$

Definition 2.2. A sequence $\{x_n\}$ in a CAT(1) space X is said to be Δ-*convergent* to a point $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\Delta - \lim_{n\to\infty} x_n = x$ of $\{x_n\}$ and denote $W_\Delta(x_n) := \cup\{A(\{u_n\})\}$, where the union is sum over all subsequences $\{u_n\}$ of $\{x_n\}$.

Let g be a function of X into $(-\infty,\infty]$. We denote by domain of g the set of all $x \in X$ such that $g(x) \in \mathbb{R}$. The function g is called *proper* if domain of g is nonempty. It is said to be *lower semi-continuous* if the set

$$K = \{x \in X : g(x) \le \beta\}$$

is closed in X for all $\beta \in \mathbb{R}$. We denote by $\arg\min_{y\in X}\ g(y)$ such that

$$g(u) \le \liminf_{n\to\infty} g(x_n)$$

whenever u is an element of X and $\{x_n\}$ is a sequence of X which is convergent to u.

The function g is said to be *convex* if

$$g(\alpha x \oplus (1-\alpha)y) \le \alpha g(x) + (1+\alpha)g(y)$$

for all $x,y \in X$ and $\alpha \in [0,1]$. See also [33, 12] for some interesting examples of convex functions in CAT(1) spaces.

Let $g : X \to (-\infty,\infty]$ be a proper lower semi-continuous convex function. For all $\lambda > 0$, define the resolvent of g in admissible CAT(1) spaces as

$$R_\lambda(x) = \arg\min_{y\in X}\left[g(y) + \frac{1}{\lambda}\tan d(y,x)\sin d(y,x)\right]$$

for all $x \in X$. The mapping R_λ is well define for all $\lambda > 0$ (see [17]). In particular, the set $F(R_\lambda)$ of fixed points of the resolvent associated with g coincides with the set $\arg\min_{y\in X}\ g(y)$ of minimizers of g.

For CAT(1) space (X,d) is admissible if $d(v,v') < \frac{\pi}{2}$ for all $v,v' \in X$. Moreover, the sequence $\{x_n\}$ in a CAT(1) space is spherical bounded if

$$\inf_{y\in X}\limsup_{n\to\infty} d(y,x_n) < \frac{\pi}{2}$$

Lemma 2.1. [19] *Let (X,d) be an admissible complete CAT(1) space and $g : X \to (-\infty,\infty]$ be a proper lower semi-continuous convex function. Then, for all $x \in X$, $u \in \arg\min_X g$ and $\lambda > 0$, we have two inequalities*

$$\lambda(g(R_\lambda x) - g(u)) \le \frac{\pi}{2}\left(\frac{1}{\cos d^2(R_\lambda x,x)} + 1\right)(\cos d(R_\lambda x,x)\cos d(u,R_\lambda x) - \cos d(u,x)) \tag{2.4}$$

and

$$\cos d(R_\lambda x,x)\cos d(u,R_\lambda x) \ge \cos d(u,x). \tag{2.5}$$

Lemma 2.2. [14] *Let (X,d) be a complete CAT(1) space and a sequence $\{x_n\}$ is spherical bounded in X. If $\{d(x_n,u)\}$ is convergent for all $u \in W_\Delta(\{x_n\})$, then a sequence is Δ-convergent.*

Lemma 2.3. [17] *Let (X,d) be an admissible complete CAT(1) space and $g : X \to (-\infty,\infty]$ be a proper lower semi-continuous convex function. Then g is Δ-lower semi-continuous.*

In 2014 Panyanak [26] proved demiclosedness principle for total asymptotically nonexpansive mapping in CAT(κ) spaces. We know that each nonexpansive mapping is in turn a total asymptotically nonexapnsive mapping, then we obtain result immediately as follows:

Corollary 2.1. *Let C be a nonempty closed and convex subset of a complete CAT(1) space (X, d). Let $T : C \to C$ be a nonexpansive mapping. If $\{x_n\}$ be a bounded sequence such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ and $\Delta - \lim_{n\to\infty} x_n = w$, then $w \in C$ and $w = Tw$.*

3. Main results

To simplify the proof of our main theorem, we first develop the following lemma which describes some boundedness properties and also the asymptotic regularity of the algorithm.

Lemma 3.4. *Let (X, d) be an admissible complete CAT(1) space and $g : X \to (-\infty, \infty]$ be a proper lower semi-continuos convex function. Let T be a nonexpansive mapping on X such that $F(T) \cap \arg\min_X g \neq \emptyset$. Assume that a sequence $\{\alpha_n\} \subseteq [a, b]$ such that $a, b \in (0, 1)$ for all $n \geq 1$ and for some a, b and $\{\lambda_n\}$ is a sequence such that $\lambda_n \geq \lambda > 0$ for all $n \geq 1$ and for some λ. Let for each $x_1 \in X$ and the sequence $\{x_n\}$ be generated in the following manner:*

$$\begin{cases} z_n := R_{\lambda_n}(x_n), \\ x_{n+1} := (1 - \alpha_n)x_n \oplus \alpha_n T z_n, \quad \forall n \geq 1. \end{cases} \tag{3.6}$$

Then we have the following statements, for any $p \in F(T) \cap \arg\min_X g$:
(s_1) $\lim_{n\to\infty} d(x_n, p)$ *exists.*
(s_2) $\lim_{n\to\infty} d(x_n, z_n) = 0$.
(s_3) $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

Proof. **Procedure 1**, we will show (s_1) that $\{x_n\}$ is spherical bounded. Note that $z_n = R_{\lambda_n} x_n$ for all $n \geq 1$. Let $p \in F(T) \cap \arg\min_X g$. By inequality (2.5) of Lemma 2.1, then we have

$$\begin{aligned} \min\{\cos d(p, z_n), \cos d(z_n, x_n)\} &\geq \cos d(p, z_n) \cos d(z_n, x_n) \\ &\geq \cos d(p, x_n), \end{aligned} \tag{3.7}$$

which implies

$$\max\{d(p, z_n), d(z_n, x_n)\} \leq d(p, x_n). \tag{3.8}$$

It follows from (2.2), T is nonexpansive and assumption admissible of X that

$$\begin{aligned} \cos d(p, x_{n+1}) &= \cos d(p, (1 - \alpha_n)x_n \oplus \alpha_n T z_n) \\ &\geq (1 - \alpha_n) \cos d(p, x_n) + \alpha_n \cos d(p, T z_n) \\ &\geq (1 - \alpha_n) \cos d(p, x_n) + \alpha_n \cos d(p, z_n) \\ &\geq \cos d(p, x_n). \end{aligned} \tag{3.9}$$

This implies that

$$d(p, x_{n+1}) \leq d(p, x_n) \leq d(p, x_1) < \frac{\pi}{2}. \tag{3.10}$$

Hence, by (3.8) and (3.10), we see that

$$\limsup_{n\to\infty} d(p, z_n) \leq \limsup_{n\to\infty} d(p, x_n) < \frac{\pi}{2}.$$

Thus sequences $\{x_n\}$ and $\{z_n\}$ are spherically bounded. Moreover, $\sup_n d(x_n, z_n) < \frac{\pi}{2}$ and $\lim_{n\to\infty} d(p, x_n) < \frac{\pi}{2}$ exists for all $p \in F(T) \cap \arg\min_X g$.

Procedure 2 Next, we claim that $\lim_{n\to\infty} d(x_n, z_n) = 0$.
From (3.7) and (3.9), we see that

$$\begin{aligned}
\cos d(p, x_{n+1}) &\geq (1-\alpha_n)\cos d(p, x_n) + \alpha_n \cos d(p, Tz_n) \\
&\geq (1-\alpha_n)\cos d(p, x_n) + \alpha_n \cos d(p, z_n) \\
&\geq (1-\alpha_n)\cos d(p, x_n) + \alpha_n \frac{\cos d(p, x_n)}{\cos d(z_n, x_n)} \\
&= \cos d(p, x_n) + \alpha_n \cos d(p, x_n)\left[\frac{1}{\cos d(z_n, x_n)} - 1\right].
\end{aligned}$$

Therefore

$$\begin{aligned}
\alpha_n\left(\frac{1}{\cos d(z_n, x_n)} - 1\right) &\leq \frac{\cos d(p, x_{n+1})}{\cos d(p, x_n)} - 1 \\
&\to 0.
\end{aligned}$$

Since $\alpha_n \geq a > 0$,

$$\lim_{n\to\infty} d(z_n, x_n) = 0, \tag{3.11}$$

that is,

$$\lim_{n\to\infty} d(R_{\lambda_n} x_n, x_n) = 0.$$

Procedure 3 Next, we show that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.
It follows from (2.5), we see that

$$\begin{aligned}
d^2(p, x_{n+1}) &= d^2(p, (1-\alpha_n)x_n \oplus \alpha_n Tz_n) \\
&\leq (1-\alpha_n)d^2(p, x_n) + \alpha_n d^2(p, Tz_n) - \frac{R}{2}\alpha_n(1-\alpha_n)d^2(x_n, Tz_n) \\
&\leq (1-\alpha_n)d^2(p, x_n) + \alpha_n d^2(p, z_n) - \frac{R}{2}abd^2(x_n, Tz_n) \\
&\leq (1-\alpha_n)d^2(p, x_n) + \alpha_n d^2(p, x_n) - \frac{R}{2}abd^2(x_n, Tz_n) \\
&= d^2(p, x_n) - \frac{R}{2}abd^2(x_n, Tz_n),
\end{aligned}$$

which equivalent to

$$d^2(x_n, Tz_n) \leq \frac{2}{Rab}[d^2(p, x_n) - d^2(p, x_{n+1})].$$

Which yields

$$\lim_{n\to\infty} d(x_n, Tz_n) = 0. \tag{3.12}$$

By triangle inequality, (3.11) and (3.12), we have

$$\begin{aligned}
d(z_n, Tz_n) &\leq d(z_n, x_n) + d(x_n, Tz_n) \\
&\to 0 \qquad (3.13)
\end{aligned}$$

as $n \to \infty$ and it

$$\begin{aligned}
d(x_n, Tx_n) &\leq d(x_n, Tz_n) + d(Tz_n, Tx_n) \\
&\leq d(x_n, Tz_n) + d(z_n, x_n) \\
&\to 0 \qquad (3.14)
\end{aligned}$$

as $n \to \infty$ and therefore we prove (s_3). This completes the proof. □

Now, we are ready to prove Δ-convergence theorem.

Theorem 3.1. *Let (X, d) be an admissible complete CAT(κ) space and $g : X \to (-\infty, \infty]$ be a proper lower semi-continuous convex function. Let T be a nonexpansive mapping on X such that $F(T) \cap \arg\min_X g \neq \emptyset$. Assume that a sequence $\{\alpha_n\} \subseteq [a, b]$ such that $a, b \in (0, 1)$ for all $n \geq 1$ and $\{\lambda_n\}$ is a sequence such that $\lambda_n > \lambda > 0$ for all $n \geq 1$ and for some λ. Let for each $x_1 \in X$ and the sequence $\{x_n\}$ be generated by* (3.6)*. Then the sequence $\{x_n\}$ Δ-converges to an element in $F(T) \cap \arg\min_X g$.*

Proof. Let $u \in F(T) \cap \arg\min_X g$. Then we have $g(u) \leq g(z_n)$. By inequality (2.4) of Lemma 2.1, we know that

$$\lambda_n(g(z_n) - g(u)) \leq \frac{\pi}{2}\left(\frac{1}{\cos^2 d(z_n, x_n)} + 1\right)(\cos d(z_n, x_n) \cos d(u, z_n) - \cos d(u, x_n)),$$

which gives

$$0 \leq g(z_n) - g(u) \leq \frac{\pi}{2\lambda_n}\left(\frac{1}{\cos^2 d(z_n, x_n)} + 1\right)(\cos d(z_n, x_n) \cos d(u, z_n) - \cos d(u, x_n)). \tag{3.15}$$

Since $\lambda_n > \lambda > 0$ for all $n \in \mathbb{N}$ and by Lemma 3.4, we have

$$d(z_n, x_n) \to 0, \quad \lim_{n\to\infty} d(u, x_n) \text{ and } \lim_{n\to\infty} d(u, z_n) \text{ exist.} \tag{3.16}$$

Combining (3.15) with (3.16), we get

$$\lim_{n\to\infty} g(z_n) = \inf g(X). \tag{3.17}$$

Next, we show that $W_\Delta(\{x_n\}) \subset F(T) \cap \arg\min_X g$. Let $z \in W_\Delta(\{x_n\})$, then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which is Δ-convergent to z. Since $\lim_{n\to\infty} d(z_n, x_n) = 0$, we can see by the definition of the Δ-convergence that the subsequence $\{z_{n_i}\}$ of $\{z_n\}$ is also Δ-convergent to z. Using Lemma 2.3 and (3.17), we obtain

$$g(z) \leq \liminf_{i\to\infty} g(z_{n_i}) = \lim_{n\to\infty} g(z_n) = \inf g(X).$$

Hence $z \in \arg\min_X g$ and thus $W_\Delta(\{x_n\}) \subset \arg\min_X g$. Moreover, since $d(x_n, Tx_n) \to 0$ and $\{x_{n_i}\}$ is Δ-convergent to z, we obtain from Corollary 2.1 that $z \in F(T)$. We thus conclude that $W_\Delta(\{x_n\}) \subset F(T) \cap \arg\min_X g$.

Now, by using (3.16) and $W_\Delta(\{x_n\}) \subset \arg\min_X g \cap F(T)$, we can see that $d(z, x_n)$ is convergent for all $z \in W_\Delta(\{x_n\})$. Apply Lemma 2.2, $\{x_n\}$ is Δ-convergence to an element in $F(T) \cap \arg\min_X g$. This completes the proof. □

Acknowledgments. Prasit Cholamjiak thanks the Thailand Research Fund and University of Phayao under grant RSA6180084. Furthermore, Parin Chaipunya is grateful to the joint support between KMUTT and the Thailand Research Fund (TRF) through the Royal Golden Jubilee Ph.D. (RGJ-PHD) Program (Grant no. PHD/0045/2555).

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[1]DEPARTMENT OF MATHEMATICS
KMUTTFIXED POINT RESEARCH LABORATORY
ROOM SCL 802 FIXED POINT LABORATORY
SCIENCE LABORATORY BUILDING, FACULTY OF SCIENCE
KING MONGKUT'S UNIVERSITY OF TECHNOLOGY THONBURI (KMUTT)
126 PRACHA-UTHIT ROAD, BANG MOD, THRUNG KHRU, BANGKOK 10140, THAILAND
Email address: nuttapol.pak@mail.kmutt.ac.th
Email address: parin.cha@mail.kmutt.ac.th

[2]DEPARTMENT OF MEDICAL RESEARCH
CHINA MEDICAL UNIVERSITY HOSPITAL
CHINA MEDICAL UNIVERSITY
TAICHUNG 40402, TAIWAN
Email address: poom.kum@kmutt.ac.th

[3]SCHOOL OF SCIENCE
UNIVERSITY OF PHAYAO
PHAYAO 56000, THAILAND
Email address: prasitch2008@yahoo.com

[4] DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
CHIANGMAI UNIVERSITY
CHIANGMAI 50200, THAILAND
Email address: raweerote.s@gmail.com