CARPATHIAN J. MATH.
Volume **34** (2018), No. 3,
Pages 305 - 312

Online version at https://www.carpathian.cunbm.utcluj.ro/
Print Edition: ISSN 1584 - 2851; Online Edition: ISSN 1843 - 4401
DOI: https://doi.org/10.37193/CJM.2018.03.04

Dedicated to Professor Yeol Je Cho on the occasion of his retirement

Common fixed point theorems for Geraghty's type contraction mapping with two generalized metrics endowed with a directed graph in JS-metric spaces

PHAKDI CHAROENSAWAN

ABSTRACT. The purpose of this work is to present some existence results for common fixed point theorems for Geraghty contraction mappings with two generalized metrics endowed with a directed graph in JS-metric spaces. Some examples supported our main results are also presented.

1. INTRODUCTION

Fixed point theory and optimization are strongly intertwined. Many applications of fixed point theory appear in optimization problem-solving. To be more precise, employing the tools from fixed point theory can help us in the discovery of solutions of structural optimizations and inverse problem, As a consequence, fixed points and related objects as well as the sufficient conditions to generate them are very popular among researchers. Many spaces endowed with functions with nice properties can help illuminating results in this field.

Geraghty [7] introduced an interesting class Θ of functions $\theta : [0, \infty) \to [0, 1)$ satisfying that:

$$\theta(t_n) \to 1 \quad \Longrightarrow \quad t_n \to 0,$$

and shown some results which is a generalization of the Banach's contraction principle in 1973.

When equipping a metric space with two metrics, Juan Martínez-Moreno, Wutiphol Sintunavarat and Yeol Je Cho [11] showed some new common fixed point theorems for Geraghty's type contraction mappings using the monotone property and g-uniform continuity defined as follows.

Definition 1.1 ([11]). Let (X, d) and (Y, d') be two metric spaces. Let $f : X \to Y$, and $g : X \to X$ be two mappings. A mapping f is said to be *g-uniformly continuous* on X if, for any real number $\epsilon > 0$, there exists $\delta > 0$ such that $d'(fx, fy) < \epsilon$ whenever $x, y \in X$ and $d(gx, gy) < \delta$. If g is the identity mapping, it is obvious that f is uniformly continuous on X.

In 2015, Jleli and Samet [9] presented a new generalization of metric spaces that recovers a large class of topological spaces including standard metric spaces, b-metric spaces, dislocated metric spaces, and modular spaces as follows;

Let X be a nonempty set, and let $D : X \times X \to [0, \infty]$ be a given mapping. For every $x \in X$, let us define the set

$$C(D, X, x) = \{\{x_n\} \subset X : \lim_{n\to\infty} D(x_n, x) = 0\}$$

Received: 28.08.2017. In revised form: 21.02.2018. Accepted: 15.07.2018
2010 *Mathematics Subject Classification.* 47H10; 47H05.
Key words and phrases. *common fixed point, coincidence point, Geraghty, JS-metric spaces, generalized metric.*

Definition 1.2. [9] A generalized metric on a set X is a mapping $D : X \times X \to [0, +\infty)$ satisfying the following conditions, for any $x, y, z \in X$:

(D_1) if $D(x, y) = 0$ then $x = y$,
(D_2) $D(x, y) = D(y, x)$,
(D_3) there exists $K > 0$ such that if $x, y \in X$, $\{x_n\} \in C(D, X, x)$, then

$$D(x, y) \leq K \limsup_{n\to\infty} D(x_n, y).$$

Then (X, D) is known as a JS-metric space.

Definition 1.3. [9] Let (X, D) be a JS-metric space. Let $\{x_n\}$ be a sequencence in X and $x \in X$. We say that

(i) $\{x_n\}$ D-converges to x if $\{x_n\} \in C(D, X, x)$.
(ii) $\{x_n\}$ is a D-Cauchy sequence if $\lim_{m,n\to\infty} D(x_n, x_{n+m}) = 0$.
(iii) X is D-complete if every D-Cauchy sequence in X is D-convergent to some element in X.

Proposition 1.1. [9] *Let (X, D) be a JS-metric space. Let $\{x_n\}$ be a sequencence in X and $x, y \in X$. If $\{x_n\} \in C(D, X, x)$ and $\{x_n\} \in C(D, X, y)$, then $x = y$.*

Definition 1.4. [9]Let (X, D) be a JS-metric space. A mapping $f : X \to X$ is called continuous at a point $x_0 \in X$, if $\{x_n\} \in C(D, X, x_0)$ then $\{fx_n\} \in C(D, X, fx_0)$.

A mappping f is continuous if it is continuous at each x in X.

Let (X, d) be a metric space, and Δ be a diagonal of $X \times X$. Let G be a directed graph such that the set $V(G)$ of its vertices coincides with X and $\Delta \subseteq E(G)$, where $E(G)$ is the set of the edges of the graph. Assume also that G has no parallel edges and, thus, one can identify G with the pair $(V(G), E(G))$.

Throughout the paper we shall say that G with the above mentioned properties satisfies the *standard conditions*.

The fixed point theorem using the context of metric spaces endowed with a graph was initiated by Jachymski [8],which generalizes the Banach contraction principle to mappings on a metric spaces with a graph. Also, the definitions of G-continuous were given.

Definition 1.5 ([8]). Let (X, D) be JS-metric spaces. A mapping $f : (X, D) \to (X, D)$ is called G-continuous if for any $x \in X$ such that there exists a sequence $\{x_n\} \in C(D, X, x)$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$, then $\{fx_n\} \in C(D, X, fx)$.

The purpose of this work is to present some existence results for common fixed point theorems for Geraghty contraction mappings with two generalized metrics endowed with a directed graph in JS-metric spaces.

2. MAIN RESULTS

First, we introduce some concepts which will be useful for proving our main results.

Definition 2.6. Let G be a directed graph, and let $f, g : X \to X$ be two mappings. We say that f is *g-edge preserving w.r.t G* if

$$(gx, gy) \in E(G) \Rightarrow (fx, fy) \in E(G).$$

Definition 2.7. Let (X, D) be a JS-metric space, $g : X \to X$ be a self-map on X and suppose that G is a directed graph. We say that the triple (X, D, G) has the property A_g, if for any sequence $\{x_n\}$ in $C(D, X, x)$, and $(x_n, x_{n+1}) \in E(G)$, for $n \in \mathbb{N}$, we have $(gx_n, gx) \in E(G)$.

Definition 2.8. Let (X, D) be a JS-metric space, and let $E(G)$ be the set of the edges of the graph. We say that $E(G)$ satisfies the transitivity property if and only if, for all $x, y, a \in X$

$$(x, a), (a, y) \in E(G) \Rightarrow (x, y) \in E(G).$$

Definition 2.9. Let (X, D) be a JS-metric space, and let $f, g : X \to X$ be a pair of mappings. f and g are commuting if for every $x \in X$,

$$gf(x) = fg(x).$$

We introduce a class of the Geraghty type contractions in the following definition.

Definition 2.10. Let (X, D) be a JS-metric space endowed with a directed graph G, and let $f, g : X \to X$ be given mappings. The pair (f, g) is called a θ-contraction w.r.t D if :

(1) f is g-edge preserving w.r.t G;
(2) there exists functions $\theta \in \Theta$ such that for all $x, y \in X$ such that $(gx, gy) \in E(G)$,

$$D(fx, fy) \leq \theta(D(gx, gy))D(gx, gy). \tag{2.1}$$

Let X with a directed graph G satisfying the standard conditions, and let two mappings $f, g : X \to X$ be given. We define important subsets of X as follows

$$C(f, g) :=\{u \in X : fu = gu\},$$

i.e., the set of all coincidence points of mappings f and g,

$$Cm(f, g) :=\{u \in X : fu = gu = u\},$$

i.e., the set of all common fixed points of mappings f and g, and,

$$X(f, g) :=\{u \in X : (gu, fu) \in E(G) \quad \text{and} \quad \sup\{D(gu, fy) : y \in X\} < \infty\}.$$

We give a concept of g-Cauchy mapping defined below.

Definition 2.11. Let (X, D) and (Y, D') be two JS-metric spaces, and let $f : X \to Y$ and $g : X \to X$ be two mappings. The mapping f is said to be g-*Cauchy* on X if, for any sequence $\{x_n\}$ in X such that $\{gx_n\}$ is D-Cauchy sequence in (X, D), then $\{fx_n\}$ is D′-Cauchy sequence in (Y, D').

Let D', D be two generalize metrics on X. By $D < D'$ (resp., $D \leq D'$), we mean $D(x, y) < D'(x, y)$ (resp., $D(x, y) \leq D'(x, y)$) for all $x, y \in X$.

Now we are ready to present and prove the main results.

Theorem 2.1. *Let (X, D') be a complete JS-metric space endowed with a directed graph G, and let D be another generalize metric on X. Suppose that $f, g : X \to X$ and (f, g) is a θ-contraction w.r.t D. Suppose that*

(1) *$g : (X, D') \to (X, D')$ is continuous;*
(2) *$f(X) \subseteq g(X)$;*
(3) *$E(G)$ satisfies the transitivity property;*
(4) *if $D \not\geq D'$, assume that $f : (X, D) \to (X, D')$ is* g-Cauchy *on X;*
(5) *$f : (X, D') \to (X, D')$ is G-continuous;*
(6) *f and g are commuting.*

Then, under these conditions,

$$\text{if} \;\; X(f, g) \neq \emptyset, \text{then} \;\; C(f, g) \neq \emptyset.$$

Proof. Suppose $X(f, g) \neq \emptyset$. We have $x_0 \in X$ such that $(gx_0, fx_0) \in E(G)$ and $\sup\{D(gx_0, fy) : y \in X\} < \infty$. By the assumption that $f(X) \subseteq g(X)$ and $f(x_0) \in X$, it is easy to construct sequences $\{x_n\}$ in X for which

$$gx_n = fx_{n-1}$$

for all $n \in \mathbb{N}$. If $gx_{n_0} = gx_{n_0-1}$ for some $n_0 \in \mathbb{N}$, then x_{n_0-1} is a coincidence point of the mappings g and f. Therefore, we assume that, for each $n \in \mathbb{N}$, $gx_n \neq gx_{n-1}$ holds.

Since $(gx_0, fx_0) = (gx_0, gx_1) \in E(G)$ and f is g-edge preserving w.r.t G, we have $(fx_0, fx_1) = (gx_1, gx_2) \in E(G)$. Continue inductively, we obtain that

$$(gx_n, gx_{n+1}) \in E(G) \quad \text{for each} \quad n \in \mathbb{N}. \tag{2.2}$$

We assert that $\{gx_n\}$ is a D-Cauchy sequence. Suppose, on the contrary, that $\{gx_n\}$ is not a Cauchy sequence. Thus, there exists $\epsilon > 0$ such that, for all $k \in \mathbb{N}$, there exists $n_k, m_k \in \mathbb{N}$ such that $n_k, m_k \geq k$ with satisfying the condition below

$$D(gx_{n_k}, gx_{n_k+m_k}) \geq \epsilon.$$

By (2.2) and the transitivity property of $E(G)$, we get $(gx_{n_k}, gx_{n_k+m_k}) \in E(G)$ for all k. Consider

$$\begin{aligned} D(gx_{n_k+1}, gx_{n_k+m_k+1}) &= D(fx_{n_k}, fx_{n_k+m_k}) \\ &\leq \theta(D(gx_{n_k}, gx_{n_k+m_k}))D(gx_{n_k}, gx_{n_k+m_k}). \end{aligned}$$

Continuing this process, we get that

$$D(gx_{n_k+1}, gx_{n_k+m_k+1}) \leq \prod_{i=0}^{n_k} \theta(D(gx_{n_k-i}, gx_{n_k+m_k-i}))D(gx_0, gx_{m_k}).$$

We choose $0 \leq i_k \leq n_k$ such that

$$\theta(D(gx_{n_k-i_k}, gx_{n_k+m_k-i_k})) = \max\{\theta(D(gx_{n_k-i}, gx_{n_k+m_k-i})) : 0 \leq i \leq n_k\}$$

Define $\delta := \limsup_{k\to\infty} \theta(D(gx_{n_k-i_k}, gx_{n_k+m_k-i_k}))$.

If $\delta < 1$, then $\lim_{k\to\infty} D(gx_{n_k+1}, gx_{n_k+m_k+1}) = 0$, which contradicts to the assumption.

If $\delta = 1$, by passing through a subsequence, then we may assume that

$$\lim_{k\to\infty} \theta(D(gx_{n_k-i_k}, gx_{n_k+m_k-i_k})) = 1.$$

Since $\theta \in \Theta$, we have

$$\lim_{k\to\infty} D(gx_{n_k-i_k}, gx_{n_k+m_k-i_k}) = 0.$$

So, there exists $k_0 \in N$ such that

$$D(gx_{n_{k_0}-i_{k_0}}, gx_{n_{k_0}+m_{k_0}-i_{k_0}}) < \epsilon/2.$$

Then, we have

$$\begin{aligned} \epsilon &\leq D(gx_{n_{k_0}}, gx_{n_{k_0}+m_{k_0}}) \\ &\leq \prod_{j=1}^{i_{k_0}} \theta(D(gx_{n_{k_0}-j}, gx_{n_{k_0}+m_{k_0}-j}))D(gx_{n_{k_0}-i_{k_0}}, gx_{n_{k_0}+m_{k_0}-i_{k_0}}) \\ &< \epsilon/2, \end{aligned}$$

which is a contradiction. So, we conclude that $\{gx_n\}$ is a D-Cauchy sequence in (X, D).

Next, we claim that $\{gx_n\}$ is a D'-Cauchy sequence in (X, D').

If $D \geq D'$, it is trivial.

Thus, suppose $D \ngeq D'$. Since $\{gx_n\}$ is a D-Cauchy sequence in (X, D) and f is *g-Cauchy* on X, we have $\{fx_n\}$ is a D'-Cauchy sequence in (X, D'). Then

$$\lim_{n,m\to\infty} D'(gx_{n+1}, gx_{n+m+1}) = \lim_{n,m\to\infty} D'(fx_n, fx_{n+m}) = 0$$

whenever $n, m \geq N_0$. So $\{gx_n\}$ is a D'-Cauchy sequence.

Since (X, D') is a complete metric space, there exists $u \in X$ such that

$$\lim_{n\to\infty} D'(gx_n, u) = \lim_{n\to\infty} D'(fx_n, u) = 0.$$

This means that

$$\{gx_n\}, \{fx_n\} \in C(D', X, u) \tag{2.3}$$

Now, since the continuity of g, we have

$$\{gfx_n\} \in C(D', X, gu) \tag{2.4}$$

Suppose that $f : (X, D') \to (X, D')$ is G-continuous, from (2.2), (2.3) and f and g are commuting, we have

$$\{fgx_n\} = \{gfx_n\} \in C(D', X, fu). \tag{2.5}$$

By equality (2.4), (2.5) and Proposition 1.1, it follows that $gu = fu$. So u is a coincidence point of f and g.

□

If $D = D'$, we have the following theorem.

Theorem 2.2. *Let (X, D) be a complete JS-metric space endowed with a directed graph G. Suppose that $f, g : X \to X$ and (f, g) is a θ-contraction w.r.t D. Moreover, suppose that:*

(1) *g is continuous;*
(2) *$f(X) \subseteq g(X)$;*
(3) *$E(G)$ satisfies the transitivity property;*
(4) *f and g are commuting;*
(5) *assume that (a) f is G-continuous or (b) (X, D, G) has the property A_g.*

Then, under these conditions,

$$\text{if } \ X(f, g) \neq \emptyset \ \text{, then } \ C(f, g) \neq \emptyset.$$

Proof. In order to avoid the repetition, following from the same proof in Theorem 2.1, we can only consider (b) of the condition (5). Since $\{gx_n\}$ is a D-Cauchy sequence in a complete JS-metric space (X, D), there exists $u \in X$ such that

$$\lim_{n\to\infty} D(gx_n, u) = \lim_{n\to\infty} D(fx_n, u) = 0. \tag{2.6}$$

That is

$$\{gx_n\}, \{fx_n\} \in C(D, X, u). \tag{2.7}$$

Since g is continuous, we have

$$\{ggx_n\}, \{gfx_n\} \in C(D, X, gu). \tag{2.8}$$

Now, we show that u is a coincidence point of f and g. Since (X, D, G) has the property A_g and (2.7), we have $(ggx_n, gu) \in E(G)$ for each $n \in \mathbb{N}$.From (2.1), we have

$$D(fgx_n, fu)) \leq \theta(D(ggx_n, gu))D(ggx_n, gu) < D(ggx_n, gu).$$

By (2.8) and taking $n \to \infty$ in above inequality and since f and g are commuting, we have

$$\{fgx_n\} = \{gfx_n\} \in C(D, X, fu). \tag{2.9}$$

Form (2.8), (2.9) and the uniqueness of convergence, it follows that $gu = fu$. So u is a coincidence point of f and g.

□

Theorem 2.3. *In addition to the hypotheses of Theorem 2.1 or Theorem 2.2, assume that*

(K) *for any* $x, y \in C(f, g)$ *such that* $gx \neq gy$*, we have* $(gx, gy) \in E(G)$*.*

If $X(f, g) \neq \emptyset$*, then* $Cm(f, g) \neq \emptyset$*.*

Proof. Theorem 2.1 implies that there exists a coincidence point $x \in X$, that is, $gx = fx$. Suppose that there exists another coincidence point $y \in X$ such that $gy = fy$.

Assume that $gx \neq gy$. By assumption (K), $(gx, gy) \in E(G)$, we have

$$\begin{aligned} D(fx, fy) &\leq \theta(D(gx, gy))D(gx, gy) \\ &< D(gx, gy) = D(fx, fy), \end{aligned}$$

which is a contradiction. Therefore, $gx = gy$.

Now, let $p = gx$. Since f and g are commuting, we have $gp = gfx = fgx = fp$. This implies that p is another coincidence point of the mappings f and g. By the property we have just proved, it follows that $fp = gp = gx = p$ and so p is a common fixed point of g and f. This completes the proof. □

Example 2.1. Let $X = [0, \infty]$, and let the generalized metrics $D, D' : X \times X \to [0, \infty)$ be defined by

$$D(x, y) = \begin{cases} x + y & \text{if } x, y \in [0, \infty) \\ \\ \infty & \text{if } x = \infty \text{ or } y = \infty \end{cases}$$

and

$$D'(x, y) = \begin{cases} L(x + y) & \text{if } x, y \in [0, \infty) \\ \\ \infty & \text{if } x = \infty \text{ or } y = \infty \end{cases}$$

where L is a real number such that $L \in (1, \infty)$.

Now, we consider $E(G)$ given by

$$E(G) = \{(x, y) : x = y \text{ or } [x, y \in [0, 1] \text{ with } x \leq y]\}.$$

Consider the mappings $f : X \to X$ and $g : X \to X$ defined by

$$f(x) = \begin{cases} x^2 e^{-x^2} & \text{if } x \in [0, \infty) \\ \\ 0 & \text{if } x = \infty \end{cases}$$

and

$$g(x) = \begin{cases} x & \text{if } x \in [0, \infty) \\ \\ \infty & \text{if } x = \infty \end{cases}$$

Next, we show that the conditions (1)–(2) in Definition 2.10 hold as follows:

(1) Let $(gx, gy) \in E(G)$.

If $gx = gy$, it is easy to see that $(fx, fy) \in E(G)$.

If $gx, gy \in [0, 1]$ and $gx \leq gy$, we have $x, y \in [0, 1]$ and $x \leq y$. It is easy to show that $f(1) \approx 0.37$ is the absolute maximum of f on $[0, \infty)$ and f is increasing function on $[0, 1]$. For all $x, y \in [0, 1]$, we have $fx = x^2 e^{-x^2} \leq y^2 e^{-y^2} = fy$ and $f(x), f(y) \in [0, 1]$, thus $(fx, fy) \in E(G)$. So f is g-edge preserving w.r.t G;

(2) Let $\theta \in \Theta$ be defined by $\theta(t) = 1/2$. Let x, y be arbitrary points in X and $(gx, gy) \in E(G)$. Let $h(x) = 2xe^{-x^2}$, for all $x \in [0, \infty)$. We can show that $h(1/\sqrt{2}) \approx 0.86$ is the absolute maximum of h on $[0, \infty)$. Thus $2xe^{-x^2} \leq 1$ which means that

$$x^2 e^{-x^2} \leq x/2.$$

Then for all $x, y \in X$ such that $(gx, gy) \in E(G)$, we have

$$D(fx, fy) = fx + fy = x^2 e^{-x^2} + y^2 e^{-y^2} \leq \frac{1}{2}(x + y) = \theta(D(gx, gy))D(gx, gy).$$

Therefore, (f, g) is a θ-contraction.

Next, we show that the conditions (1)–(6) in Theorem 2.1 hold as follows:

(1) We can easily check that $g : (X, D') \to (X, D')$ is continuous.

(2) By the definition of f and g, we can see that $f(X) \subseteq g(X)$.

(3) It is easy to see that $E(G)$ satisfies the transitivity property.

(4) Since $D \leq D'$, then let $\{gx_n\}$ be a D-Cauchy sequence in (X, D) and $\epsilon > 0$. There exists $k \in N$ such that for all $m, n \geq k$, we have

$$D(gx_n, gx_{n+m}) = x_n + x_{n+m} < \frac{2\epsilon}{L},$$

where $L \in (1, \infty)$. For all $m, n \geq k$, we have

$$D'(fx_n, fx_{n+m}) = L(x_n^2 e^{-x_n{}^2} + x_{n+m}^2 e^{-(x_{n+m})^2}) < \frac{L}{2}(x_n + x_{n+m}) < \epsilon.$$

So, $f : (X, D) \to (X, D')$ is *g-Cauchy* on X.

(5) Since $x^2 e^{-x^2} \leq x/2$. It is easy to see that $f : (X, D') \to (X, D')$ is G-continuous.

(6) It is easy to see that f and g are commuting.

and we can see that $(0, 0) = (g(0), g(0)) \in E(G)$ and $\sup\{D(g(0), f(y)) = D(0, f(y)) : y \in X\} < \infty$, so $X(f, g) \neq \emptyset$.

Consequently, all the conditions of Theorem 2.1 hold. Therefore, g and f have a coincidence point and, further, the point 0 is a common fixed point of the mappings g and f.

Acknowledgements. This research was supported by Chiang Mai University.

References

[1] Abbas, M., Alfuraidan, M. R. and Nazir, T., *Common fixed points of multivalued F-contractions on metric spaces with a directed graph*, Carpathian J. Math., **32** (2016), No. 1, 1–12

[2] Berinde, V. and Păcurar, M., *Coupled and triple fixed point theorems for mixed monotone almost contractive mappings in partially ordered metric spaces*, J. Nonlinear Convex Anal., **18** (2017), No. 4, 651–659

[3] Choban, M. M. and Berinde, V., *A general concept of multiple fixed point for mappings defined on spaces with a distance*, Carpathian J. Math., **33** (2017), No. 3, 275–286

[4] Choban, M. M. and Berinde, V., *Two open problems in the fixed point theory of contractive type mappings on quasimetric spaces*, Carpathian J. Math., **33** (2017), No. 2, 169–180

[5] Choban, M. M. and Berinde, V., *Multiple fixed point theorems for contractive and Meir-Keeler type mappings defined on partially ordered spaces with a distance*, Appl. Gen. Topol., **18** (2017), No. 2, 317–330

[6] Fukhar-ud-din, H. and Berinde, V., *Fixed point iterations for Prešić-Kannan nonexpansive mappings in product convex metric spaces*, Acta Univ. Sapientiae Math., **10** (2018), No. 1, 56–69

[7] Geraghty, M., *On contractive mappings*, Proc. Amer. Math. Soc., **40** (1973), 604–608

[8] Jachymski, J., *The contraction principle for mappings on a metric space with a graph*, Proc. Amer. Math. Soc., **136** (2008), No. 4, 1359–1373

[9] Jleli, M., Samet, B., *A generalized metric space and related fixed point theorems*, Fixed Point Theory Appl., Vol. **61** (2015)

[10] Kumam, P. and Mongkolkeha, C., *Common best proximity points for proximity commuting mapping with Geraghty's functions*, Carpathian J. Math., **31** (2015), No. 3, 359–364

[11] Martínez-Moreno, J., Sintunavarat, W. and Cho, Y. J., *Common fixed point theorems for Geraghty's type contraction mappings using the monotone property with two metrics*, Fixed Point Theory Appl., 2015, 2015:174 doi:10.1186/s13663-015-0426-y

[12] Shukri, S. A., Berinde, V. and Khan, A. R., *Fixed points of discontinuous mappings in uniformly convex metric spaces*, Fixed Point Theory, **19** (2018), No. 1, 397–406

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
CENTER OF EXCELLENCE IN MATHEMATICS AND APPLIED MATHEMATICS
CHIANG MAI UNIVERSITY, CHIANG MAI 50200, THAILAND
Email address: phakdi@hotmail.com