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Dedicated to Professor Yeol Je Cho on the occasion of his retirement

A new iterative method for the split feasibility problem

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ABSTRACT. The split feasibility problem (SFP) has many applications, which can be a model for many inverse problems where constraints are imposed on the solutions in the domain of a linear operator as well as in the operator's range. In this paper, we introduce a new projection method to solve the SFP and prove its convergence under standard assumptions. Our results improve previously known corresponding methods and results of this area. The preliminary numerical experiments illustrates the advantage of our proposed methods.

1. INTRODUCTION

We consider the split feasibility problem (shortly, SFP) which was first Censor and Elfving [5] and is formulated as follows:

Find a point x^* with the property

$$x^* \in C \text{ and } Ax^* \in Q, \tag{1.1}$$

where A an $M \times N$ real matrix, $C \subseteq \mathbb{R}^N$ and $Q \subseteq \mathbb{R}^M$ are nonempty closed convex sets. The SFP has a variety of specific applications in real world, such as medical care, image reconstruction and signal processing (see [4] for details).

A great deal of projection methods were proposed to approximate the solutions of the SFP [7, 8, 9, 10, 11, 14, 17, 18]. Byrne [2, 3] first presented the so-called *CQ-method* in which he used fixed stepsize. In general, a method with fixed stepsize may be slow. To improve the choice of the stepsize, Qu and Xiu [14] determined the stepsizes self-adaptively by adopting Armijo–like searches and López et al. [13] presented a way to choose the stepsize directly. Zhao and Yang used two-step projection methods to solve the SFP, such as the well-known extragradient method [20] and Tseng's method [21].

Inspired by the work of He [12], we introduce a new iterative method to solve the SFP. The convergence of the proposed method are proved under standard assumptions. Preliminary numerical experiments are presented to illustrate the advantage of our method by comparing it with extragradient method and Tseng's method.

2. PRELIMINARIES

In this section, we review some definitions and lemmas which are used in the main results.

The projection is an important tool for our work in this paper. Let C be a closed convex subset of a real Hilbert space H. Recall that the *nearest point* or *metric projection* from H

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onto C, which is denoted P_C, is defined as follows: for each $x \in H$, $P_C x$ is the unique point in C such that

$$\|x - P_C x\| = \min\{\|x - z\| : z \in C\}.$$

The following two lemmas are useful characterizations of projections:

Lemma 2.1. *For any $x \in H$ and $z \in C$, then $z = P_C x$ if and only if*

$$\langle x - z, y - z\rangle \leq 0, \quad \forall y \in C.$$

Lemma 2.2. [1] *For any $x, y \in H$ and $z \in C$, the following hold:*

(1) $\|P_C(x) - P_C(y)\|^2 \leq \langle P_C(x) - P_C(y), x - y\rangle$;
(2) $\|P_C(x) - z\|^2 \leq \|x - z\|^2 - \|P_C(x) - x\|^2$;
(3) $\langle (I - P_C)x - (I - P_C)y, x - y\rangle \geq \|(I - P_C)x - (I - P_C)y\|^2$.

Lemma 2.3. [1] *Let K be a nonempty closed convex subset of a Hilbert space H. Let $\{x^k\}$ be a bounded sequence which satisfies the following properties:*

(1) *every weak limit point of $\{x^k\}$ lies in K;*
(2) $\lim_{n\to\infty} \|x^k - x\|$ *exists for every $x \in X$.*

Then $\{x^k\}$ converges weakly to a point in K.

In this paper, we are concerned with the case whenever the involved subsets are composed of level sets. Namely, we consider the case whenever C and Q in (1.1) are defined by

$$C = \{x \in \mathbb{R}^N : c(x) \leq 0\}, \quad Q = \{y \in \mathbb{R}^M : q(y) \leq 0\},$$

where $c : \mathbb{R}^N \to \mathbb{R}$ is a convex function, and $q : \mathbb{R}^M \to \mathbb{R}$ is a convex function.

For the functions c and q, we make the following assumptions:

(1) c and q are subdifferentiable on C and Q. (Note that the convex function is subdifferentiable everywhere in $\mathbb{R}^N$.) For any $x \in \mathbb{R}^N$, at least one subgradient $\xi \in \partial c(x)$ can be calculated, where $\partial c(x)$ is defined as follows:

$$\partial c(x) = \{z \in \mathbb{R}^N : c(u) \geq c(x) + \langle u - x, z\rangle, \quad \text{for all } u \in \mathbb{R}^N\}.$$

For any $y \in \mathbb{R}^M$, at least one subgradient $\eta \in \partial q(y)$ can be calculated, where

$$\partial q(x) = \{w \in \mathbb{R}^M : q(v) \geq q(y) + \langle v - y, w\rangle, \quad \text{for all } v \in \mathbb{R}^M\}.$$

(2) c and q are bounded on bounded sets. (Note that this condition is automatically satisfied if $\mathbb{R}^N$ and $\mathbb{R}^M$ are finite dimensional.)

From Banach-Steinhaus Theorem (see Theorem 2.5 in [15]), it is easy to get the following result.

Remark 2.1. Assumption (4) guarantees that if $\{x^k\}$ is a bounded sequence in $\mathbb{R}^N$ (resp. $\mathbb{R}^M$) and $\{y^k\}$ is a sequence in $\mathbb{R}^N$ (resp. $\mathbb{R}^M$) such that $y^k \in \partial c(x^k)$ (resp. $y^k \in \partial q(x^k)$) for each k, then $\{y^k\}$ is bounded.

3. Main results

In this section, we present a new projection method and establish its convergence under the standard assumptions.

Throughout this paper, we assume that the solution set of the SFP (1.1), denoted by

$$\Gamma = \{x \,|\, x \in C \quad \text{and} \quad Ax \in Q\},$$

is nonempty.

Let

$$F(x^k) = (x^k - P_{C_k}(x^k)) + A^*(I - P_{Q_k})A(x^k)).$$

Next we define a new algorithm.

Algorithm 3.1 For any $\sigma > 0$, $\rho \in (0,1)$ and $\theta \in (0,1)$, take arbitrarily $x_0 \in R^N$ and let

$$y^k = P_X(x^k - \alpha_k F(x^k)), \tag{3.2}$$

where $\alpha_k = \sigma\rho^{m_k}$ and m_k is the smallest nonnegative integer such that

$$\alpha_k \|F(x^k) - F(y^k)\| \le \theta \|x^k - y^k\|. \tag{3.3}$$

Let the corrections of the method be, respectively, defined by:

$$x_I^{k+1} = x^k - \gamma\varrho_k d(x^k, y^k) \tag{3.4}$$

and

$$x_{II}^{k+1} = P_X(x^k - \gamma\varrho_k\alpha_k F(y^k)), \tag{3.5}$$

where $\gamma \in (0,2)$,

$$d(x^k, y^k) := (x^k - y^k) - \alpha_k(F(x^k) - F(y^k)) \tag{3.6}$$

and

$$\varrho_k := \frac{\langle x^k - y^k, d(x^k, y^k)\rangle + \alpha_k(\|(I - P_{C_k})y^k\|^2 + \|(I - P_{Q_k})Ay^k\|^2)}{\|d(x^k, y^k)\|^2}. \tag{3.7}$$

Remark 3.2. There are some illustrations and observations to Algorithm 3.1.

(1) For convenience, we call the projection algorithms which use update forms (3.4) and (3.5) Algorithm 3.1 (I) and Algorithm 3.1 (II), respectively.
(2) Algorithm 3.1 is a two-step method, which contains two methods since the second step are two different. There is no projection onto X in x_I^{k+1}, while x_{II}^{k+1} involves a projection onto X.
(3) The set X in Algorithm 3.1 can be chosen variously. It can be chosen to be a simple bounded subset of Hilbert spaces that contains at least one solution of the SFP, it can also be directly chosen as $X = \mathbb{R}^N$. In fact, it can be more generally chosen to be a dynamically changing set X_k, provided $\cap_{k=1}^{\infty} X_k$ contains a solution of the SFP. This does not affect the convergence result (see, e.g. [16]).

Following the line of the proof of Lemma 3.4 in [20], we get the following lemma, which shows that the inner loop in the stepsize calculation in (3.3) is always finite.

Lemma 3.4. *The line rule* (3.3) *is well defined. Besides,* $\underline{\alpha} \le \alpha_k \le \sigma$, *where* $\underline{\alpha} = \min\{\sigma, \frac{\mu\rho}{L}\}$.

Remark 3.3. By the definitions of y^k and $d(x^k, y^k)$, we get

$$y^k = P_X(y^k - (\alpha_k F(y^k) - d(x^k, y^k))). \tag{3.8}$$

From Lemma 2.1, it follows that

$$\langle x - y^k, \alpha_k F(y^k) - d(x^k, y^k)\rangle \ge 0, \quad \forall x \in X. \tag{3.9}$$

Lemma 3.5. *Let* $\{x^k\}$ *and* $\{y^k\}$ *be the iterations generated by Algorithm 3.1 and* $d(x^k, y^k)$ *be given by* (3.6). *Then we have*

$$\langle x^k - x^*, d(x^k, y^k)\rangle \ge \varrho_k \|d(x^k, y^k)\|^2, \quad \forall x^* \in \Gamma. \tag{3.10}$$

Proof. Take arbitrarily $x^* \in \Gamma$, that is, $x^* \in C, Ax^* \in Q$. By setting $x = x^*$ in (3.9), we get

$$\langle x^* - y^k, \alpha_k F(y^k) - d(x^k, y^k)\rangle \geq 0. \tag{3.11}$$

which implies that

$$\langle y^k - x^*, d(x^k, y^k)\rangle \geq \langle y^k - x^*, \alpha_k F(y^k)\rangle. \tag{3.12}$$

By the definition of F, Lemma 2.2 (2) and the choice of x^*, we have

$$\begin{aligned}\langle y^k - x^*, F(y^k)\rangle &= \langle y^k - x^*, (I - P_{C_k})y^k + A^*(I - P_{Q_k})Ay^k\rangle \\ &= \langle y^k - x^*, (I - P_{C_k})y^k\rangle + \langle y^k - x^*, A^*(I - P_{Q_k})Ay^k\rangle \\ &= \langle y^k - x^*, (I - P_{C_k})y^k\rangle + \langle Ay^k - Ax^*, (I - P_{Q_k})Ay^k\rangle \\ &\geq \|(I - P_{C_k})y^k\|^2 + \|(I - P_{Q_k})Ay^k\|^2.\end{aligned} \tag{3.13}$$

So, we get

$$\begin{aligned}\langle x^k - x^*, d(x^k, y^k)\rangle &= \langle x^k - y^k, d(x^k, y^k)\rangle + \langle y^k - x^*, d(x^k, y^k)\rangle \\ &\geq \langle x^k - y^k, d(x^k, y^k)\rangle + \alpha_k(\|(I - P_{C_k})y^k\|^2 + \|(I - P_{Q_k})Ay^k\|^2) \\ &= \varrho_k\|d(x^k, y^k)\|^2.\end{aligned} \tag{3.14}$$

So, we get the (3.10).

Lemma 3.6. *Let $\{x^k\}$ and $\{y^k\}$ be the iterations generated by Algorithm 3.1. Let $d(x^k, y^k)$ and ϱ_k be given by* (3.6) *and* (3.7)*, respectively. Then we have*

$$\langle x^k - y^k, d(x^k, y^k)\rangle \geq (1 - \theta)\|x^k - y^k\|^2 \tag{3.15}$$

and

$$\varrho_k \geq \frac{1 - \theta}{1 + \theta^2}. \tag{3.16}$$

Proof. We obtain

$$\begin{aligned}\langle x^k - y^k, d(x^k, y^k)\rangle &= \|x^k - y^k\|^2 - \alpha_k\langle x^k - y^k, F(x^k) - F(y^k)\rangle \\ &\geq \|x^k - y^k\|^2 - \alpha_k\|x^k - y^k\|\|F(x^k) - F(y^k)\| \\ &\geq \|x^k - y^k\|^2 - \theta\|x^k - y^k\|^2 \\ &= (1 - \theta)\|x^k - y^k\|^2\end{aligned} \tag{3.17}$$

which completes the proof of (3.15).

From (2.2), we obtain

$$\begin{aligned}&- 2\alpha_k\langle x^k - y^k, F(x^k) - F(y^k)\rangle \\ =& -2\alpha_k\langle x^k - y^k, (I - P_{C_k})(x^k) - (I - P_{C_k})(y^k)\rangle \\ &- 2\alpha_k\langle x^k - y^k, A^*(I - P_{Q_k})A(x^k) - A^*(I - P_{Q_k})A(y^k)\rangle \\ =& -2\alpha_k\langle x^k - y^k, (I - P_{C_k})(x^k) - (I - P_{C_k})(y^k)\rangle \\ &- 2\alpha_k\langle Ax^k - Ay^k, (I - P_{Q_k})A(x^k) - (I - P_{Q_k})A(y^k)\rangle \\ \leq& -2\alpha_k\|(I - P_{C_k})(x^k) - (I - P_{C_k})(y^k)\|^2 \\ &- 2\alpha_k\|A\| \cdot \|(I - P_{Q_k})A(x^k) - (I - P_{Q_k})A(y^k)\|^2 \leq 0.\end{aligned} \tag{3.18}$$

So, we obtain

$$\begin{aligned}\|d(x^k,y^k)\|^2 &= \|x^k-y^k\|^2+\alpha_k^2\|F(x^k)-F(y^k)\|^2-2\alpha_k\langle x^k-y^k,F(x^k)-F(y^k)\rangle \\ &\le (1+\theta^2)\|x^k-y^k\|^2.\end{aligned} \tag{3.19}$$

Thus we obtain (3.16).

Theorem 3.1. *Assume that Γ is nonempty. Then the iteration $\{x^k\}$ generated by Algorithm 3.1 (I) converges to a solution of the SFP* (1.1).

Proof. From Lemma (3.5), we obtain

$$\begin{aligned}\|x_I^{k+1}-x^*\|^2 &= \|x^k-\gamma\varrho_k d(x^k,y^k)-x^*\|^2 \\ &= \|x^k-x^*\|^2+\gamma^2\varrho_k^2\|d(x^k,y^k)\|^2-2\gamma\varrho_k\langle x^k-x^*,d(x^k,y^k)\rangle \\ &\le \|x^k-x^*\|^2+\gamma^2\varrho_k^2\|d(x^k,y^k)\|^2-2\gamma\varrho_k^2\|d(x^k,y^k)\|^2 \\ &= \|x^k-x^*\|^2-(2-\gamma)\gamma\varrho_k^2\|d(x^k,y^k)\|^2.\end{aligned} \tag{3.20}$$

Since $\gamma\in(0,2)$, (3.20) implies that the sequence $\{\|x^k-x^*\|^2\}$ is monotonically decreasing and thus convergent, moreover, $\{x^k\}$ is bounded. This implies

$$\lim_{k\to\infty}\varrho_k^2\|d(x^k,y^k)\|^2=0. \tag{3.21}$$

From definition of ρ_k and Lemma 3.6, we have

$$\begin{aligned}&\varrho_k^2\|d(x^k,y^k)\|^2 \\ &= \varrho_k(\langle x^k-y^k,d(x^k,y^k)\rangle+\alpha_k(\|(I-P_{C_k})y^k\|^2+\|(I-P_{Q_k})A(y^k)\|^2)) \\ &\ge \varrho_k((1-\theta)\|x^k-y^k\|^2+\alpha_k(\|(I-P_{C_k})y^k\|^2+\|(I-P_{Q_k})A(y^k)\|^2)) \\ &\ge \frac{(1-\theta)^2}{1+\theta^2}\left(\|x^k-y^k\|^2+\underline{\alpha}(\|y^k-P_{C_k}y^k\|^2+\|A(y^k)-P_{Q_k}A(y^k)\|^2))\right),\end{aligned} \tag{3.22}$$

which with (3.21) implies

$$\lim_{k\to\infty}\|x^k-y^k\|=0, \tag{3.23}$$

and

$$\lim_{k\to\infty}\|A(y^k)-P_{Q_k}A(y^k)\|=0. \tag{3.24}$$

Let $\bar{x}$ be a cluster point of $\{x^k\}$ with a subsequence $\{x^{k_l}\}$ converging to $\bar{x}$. From (3.23), it follows that $\{y^{k_l}\}$ also converges to $\bar{x}$. We will show that $\bar{x}$ is a solution of the SFP (1.1).

From the definition of C_{k_l}, we have

$$c(x^{k_l})+\langle\xi^{k_l},y^{k_l}-x^{k_l}\rangle\le 0,$$

where $\xi^{k_l}\in\partial c(x^{k_l})$. By the assumption that ξ^{k_l} is bounded and (3.23), we have

$$c(x^{k_l})\le-\langle\xi^{k_l},y^{k_l}-x^{k_l}\rangle\le\|\xi^{k_l}\|\|y^{k_l}-x^{k_l}\|\to 0$$

as $l\to\infty$, which implies $c(\bar{x})\le 0$, i.e., $\bar{x}\in C$.

From the definition of Q_{k_l}, we have P_Q

$$q(Ay^{k_l})+\langle\eta^{k_l},P_{Q_{k_l}}(Ay^{k_l})-Ay^{k_l}\rangle\le 0,$$

where $\eta^{k_l}\in\partial q(Ay^{k_l})$. From the assumption that ξ^{k_l} is bounded and (3.24), it follows that

$$q(Ay^{k_l})\le-\langle\eta^{k_l},P_{Q_{k_l}}(Ay^{k_l})-Ay^{k_l}\rangle\le\|\eta^{k_l}\|\|P_{Q_{k_l}}(Ay^{k_l})-Ay^{k_l}\|\to 0$$

as $l\to\infty$, which implies $q(\bar{x})\le 0$, i.e., $A\bar{x}\in Q$. Thus $\bar{x}$ is a solution of the SFP. Therefore, $\bar{x}\in\Gamma$.

Now we can apply Lemma 2.3 to $K := \Gamma$ to get that the full sequence $\{x^k\}$ converges weakly to a point in Γ. This completes the proof.

Theorem 3.2. *Assume that* Γ *is nonempty. Then the iteration* $\{x^k\}$ *generated by Algorithm 3.1 (II) converges to a solution of the SFP* (1.1).

Proof. From Lemma (2.2) (2), we obtain

$$\begin{aligned}\|x_{II}^{k+1} - x^*\|^2 &\le \|x^k - \gamma\varrho_k\alpha_k F(y^k) - x^*\|^2 - \|x^k - \gamma\varrho_k\alpha_k F(y^k) - x_{II}^{k+1}\|^2 \\ &= \|x^k - x^*\|^2 - \|x^k - x_{II}^{k+1}\|^2 - 2\gamma\varrho_k\langle x_{II}^{k+1} - x^*, \alpha_k F(y^k)\rangle.\end{aligned} \tag{3.25}$$

By setting $x = x^{k+1}$ in (3.9), we get

$$\begin{aligned}-2\gamma\varrho_k\langle x_{II}^{k+1} - y^k, \alpha_k F(y^k)\rangle &\le -2\gamma\varrho_k\langle x_{II}^{k+1} - y^k, d(x^k, y^k)\rangle \\ &= -2\gamma\varrho_k\langle x^k - y^k, d(x^k, y^k)\rangle - 2\gamma\varrho_k\langle x_{II}^{k+1} - x^k, d(x^k, y^k)\rangle.\end{aligned} \tag{3.26}$$

It holds

$$\begin{aligned}&- 2\gamma\varrho_k\langle x_{II}^{k+1} - x^k, d(x^k, y^k)\rangle \\ &= -\|x^k - x_{II}^{k+1} - \gamma\varrho_k d(x^k, y^k)\|^2 + \|x^k - x_{II}^{k+1}\|^2 + \gamma^2\varrho_k^2\|d(x^k, y^k)\|^2.\end{aligned} \tag{3.27}$$

Substituting (3.27) in the right hand side of (3.26) and using $x^k - \gamma\varrho_k d(x^k, y^k) = x_I^{k+1}$, we obtain

$$\begin{aligned}-2\gamma\varrho_k\langle x_{II}^{k+1} - y^k, \alpha_k F(y^k)\rangle &\le -2\gamma\varrho_k\langle x^k - y^k, d(x^k, y^k)\rangle \\ &\quad - \|x_I^{k+1} - x_{II}^{k+1}\|^2 + \|x^k - x_{II}^{k+1}\|^2 + \gamma^2\varrho_k^2\|d(x^k, y^k)\|^2.\end{aligned} \tag{3.28}$$

From (3.13), we have

$$-2\gamma\varrho_k\alpha_k\langle y^k - x^*, F(y^k)\rangle \le -2\gamma\varrho_k\alpha_k(\|(I - P_{C_k})y^k\|^2 + \|(I - P_{Q_k})Ay^k\|^2). \tag{3.29}$$

So, adding (3.28) and (3.29) and using the definition of ϱ_k, we obtain

$$\begin{aligned}&- 2\gamma\varrho_k\langle x_{II}^{k+1} - x^*, \alpha_k F(y^k)\rangle \\ &\le -2\gamma\varrho_k(\langle x^k - y^k, d(x^k, y^k)\rangle + \alpha_k(\|(I - P_{C_k})y^k\|^2 + \|(I - P_{Q_k})Ay^k\|^2)) \\ &\quad - \|x_I^{k+1} - x_{II}^{k+1}\|^2 + \|x^k - x_{II}^{k+1}\|^2 + \gamma^2\varrho_k^2\|d(x^k, y^k)\|^2 \\ &\le -2\gamma\varrho_k^2\|d(x^k, y^k)\|^2 + \gamma^2\varrho_k^2\|d(x^k, y^k)\|^2 - \|x_I^{k+1} - x_{II}^{k+1}\|^2 + \|x^k - x_{II}^{k+1}\|^2 \\ &\le -(2-\gamma)\gamma\varrho_k^2\|d(x^k, y^k)\|^2 - \|x_I^{k+1} - x_{II}^{k+1}\|^2 + \|x^k - x_{II}^{k+1}\|^2.\end{aligned} \tag{3.30}$$

Adding (3.25) and (3.30), we obtain

$$\|x_{II}^{k+1} - x^*\|^2 \le \|x^k - x^*\|^2 - (2-\gamma)\gamma\varrho_k^2\|d(x^k, y^k)\|^2 - \|x_I^{k+1} - x_{II}^{k+1}\|^2. \tag{3.31}$$

Similar with Theorem 3.1, we obtain that the whole sequence $\{x^k\}$ weakly converges to a solution of the SFP (1.1), which completes proof.

4. Numerical experiments

In this section, we present a numerical example to compare Algorithm 3.1 with the extragradient method in [20] and Tseng's methods in [21].

For convenience, we denote the vector with all elements 0 by e_0 and the vector with all elements 1 by e_1 in what follows. In the numerical results listed in the following table, 'Iter.' and 'Sec.' denote the number of iterations and the cpu time in seconds, respectively. And "InIt." denotes the number of total iterations of finding suitable α_k in (3.3).

Example 4.1. [19] Consider the SFP, where $A = (a_{ij})_{M\times N} \in R^{M\times N}$ and $a_{ij} \in (0,1)$ generated randomly and

$$C = \{x \in \mathbb{R}^N | c(x) \leq 0\} \quad \text{where} \quad c(x) = -x_1 + x_2^2 + \cdots + x_N^2,$$

and

$$Q = \{y \in \mathbb{R}^M | q(y) \leq 0\} \quad \text{where} \quad q(x) = y_1 + 20y_2^2 + \cdots + 20y_M^2 - 1.$$

Note that C is the set above the function $x_1 = x_2^2 + \cdots + x_N^2$ and Q is the set below the function $y_1 = -20y_2^2 - \cdots - 20y_M^2 + 1$. The initial point $x^0 \in (0, 100e_1)$ is randomly chosen. The parameters $\mu = 0.95$, $\rho = 0.4$ and $\sigma = 5$ and $\gamma = 1.5$.

TABLE 1. Computational results for example 4.1 with different dimensions.

(N, M)		Extragradient method	Tseng's method	Alg.3.1(I)	Alg.3.1(II)
	Iter.	552	516	211	107
(50,50)	Inlt.	4004	3564	1792	887
	Sec.	2.2344	1.7969	0.9063	0.5313
	Iter.	1944	1840	434	398
$(100, 150)$	Inlt.	19400	17640	4530	4119
	Sec.	22.4375	16.3594	4.3594	3.2656
	Iter.	3024	2731	546	361
$(250, 200)$	Inlt.	33305	30157	6171	4326
	Sec.	57.0781	49.0313	10.1563	8.2656

The numerical results listed in Table 1 illustrate that Algorithm 3.1 behaves better than the extragradient method and Tseng's method from the number of iterations and the cpu time in seconds. We also conclude that Algorithm 3.1(II) outperforms Algorithm 3.1(I) since Algorithm 3.1(II) has better contraction inequality (3.31) than the contraction inequality (3.20) of Algorithm 3.1(I).

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