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Dedicated to Professor Yeol Je Cho on the occasion of his retirement

Amenability and Fan–Glicksberg theorem for set-valued mappings

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ABSTRACT. In this paper, we begin by discussion of some well known results on the existence of left invariant means in the spaces: $LUC(S)$, $AP(S)$ and $WAP(S)$ with Hahn-Banach extension theorem. We then give a new and precise proof of the well known Fan–Glicksberg fixed point theorem. This is then followed by a discussion on some related open problems.

1. INTRODUCTION

Throughout this paper, we assume that E is a real separated locally convex space. All topologies in this paper are assumed to be Hausdorff.

Let $P : E \to \mathbb{R}$. We say that P is *sublinear* if $P(x+y) \leq P(x)+P(y)$ and $P(\lambda x) = \lambda P(x)$ for all $x, y \in E, \lambda \geq 0$.

Let S be a *semitopological semigroup*, i.e., S is a semigroup with Hausdorff topology such that for every $a \in S$, the mappings $s \mapsto sa$ and $s \mapsto as$ from S into S are continuous.

Let $\ell^\infty(S)$ denote the space of all bounded real-valued functions on S with the supremum norm: $\|\cdot\|_\infty$. For each $a \in S$ and $f \in \ell^\infty(S)$, let $l_a f$ and $r_a f$ denote the *left and right translate* of f by a respectively, i.e., $(l_a f)(s) := f(as)$ and $(r_a f)(s) := f(sa), \quad \forall s \in S$. Let Y be a closed subspace of $\ell^\infty(S)$ containing constants and *invariant under translations* (i.e., $l_a(Y) \subseteq Y$ and $r_a(Y) \subseteq Y$, $\forall a \in S$). Then a linear functional $m \in Y^*$ is called a *mean* if $\|m\| = m(1) = 1$. We say that a mean m is a *left invariant mean* on Y, denoted by LIM, if

$$\langle m, l_a f\rangle = \langle m, f\rangle, \qquad \forall a \in S, \ \forall f \in Y.$$

Let $CB(S)$ denote the space of all bounded continuous real-valued functions on S with the supremum norm: $\|\cdot\|_\infty$. Let $LUC(S)$ be the space of all $f \in CB(S)$ such that the mappings $a \to l_a f$ from S into $CB(S)$ are continuous. If G is a topological group, then $LUC(G)$ is precisely the space of bounded right uniformly continuous functions [15]. Set $\mathcal{LO}(f) := \{l_s f \mid s \in S\}$ and $\mathcal{RO}(f) := \{r_s f \mid s \in S\}$, where $f \in CB(S)$.

Let $AP(S)$ and $WAP(S)$ be denoted by *space of almost periodic functions* and the *space of weakly almost periodic functions* on S, respectively. More precisely, the spaces $AP(S)$ and

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$WAP(S)$ are defined by the followings:

$$AP(S) := \text{ the space of all } f \in CB(S) \text{ such that } \mathcal{LO}(f) \text{ (or equivalently, } \mathcal{RO}(f) \text{ [3])}$$
$$\text{is relatively compact in the norm topology of } CB(S);$$
$$WAP(S) := \text{ the space of all } f \in CB(S) \text{ such that } \mathcal{LO}(f) \text{ (or equivalently, } \mathcal{RO}(f) \text{ [3])}$$
$$\text{is relatively compact in the weak topology of } CB(S).$$

In general, we have the following inclusions.

$$AP(S) \subseteq LUC(S) \subseteq CB(S) \quad \text{and} \quad AP(S) \subseteq WAP(S) \subseteq CB(S).$$

Note that $LUC(S)$, $WAP(S)$ and $AP(S)$ are closed subalgebras of $CB(S)$ invariant under left and right translations.

We say that S is *left amenable* if $LUC(S)$ has a left invariant mean (LIM).

Let S be a semitopological semigroup. An *action* of S on E is a mapping from $S \times E$ to E, denoted by $(s, x) \to s \cdot x$.

Let $e \in E$. We say that e is an *invariant element* if $e = s \cdot e$ for every $s \in S$. Let F be a nonempty subset of E. We say that F is an *invariant set* if $s \cdot x \in F$ for every $s \in S$ and every $x \in F$. Let $F \subseteq E$ be an invariant set and $f : F \to \mathbb{R}$. We say that f is an *invariant function* on F if for every $s \in S$ and every $x \in F$,

$$f(s \cdot x) = f(x).$$

Let S be a semitopological semigroup. Then *a continuous (resp. weakly continuous) right linear action* of S on E is an action of S on E satisfying the following.

(i) $(ab) \cdot x = b \cdot (a \cdot x)$ for all $a, b \in S$ and $x \in E$.
(ii) For each $s \in S$, the map $x \mapsto s \cdot x$ is a continuous linear mapping from E into E.
(iii) For each $x \in E$, the map $s \mapsto s \cdot x$ is continuous from S into E (resp. weak topology).

The rest of this paper are organized as follows. In Section 2, we introduce some classical characterizations on the existence of left invariant means on the spaces $LUC(S)$, $AP(S)$ and $WAP(S)$. In Section 3, we present a new proof for the well known Fan–Glicksberg fixed point theorem: Theorem 3.4. Some open interesting problems are listed in Section 4.

2. AMENABILITY OF SEMIGROUP AND HAHN-BANACH EXTENSION PROPERTY

The following result (Theorem 2.1) shows that the amenability of a semitopological semigroup S is equivalent to Hahn-Banach extension properties. Theorem 2.1 (a)⇒(b) is due to Silverman [25] for the case when S has the discrete topology (see also [11, Page 4] and [26, Page 576]).

Theorem 2.1. (See [20, Theorem 1].) *Let S be a semitopological semigroup. The following conditions on S are equivalent:*

(a) *S is left amenable (i.e., $LUC(S)$ has a left invariant mean).*
(b) *For any continuous right linear action of S on E, if p is a continuous sublinear function on E such that $p(s \cdot x) \leq p(x)$ for all $s \in S, x \in E$, and if ϕ is an invariant linear functional on an invariant subspace F of E such that $\phi \leq p$ on F, then there exists a continuous invariant linear extension $\widetilde{\phi}$ of ϕ to E such that $\widetilde{\phi} \leq p$.*
(c) *For any continuous right linear action of S on E, if U is an invariant open convex subset of E containing an invariant element, and M is an invariant subspace of E which does not meet U, then there exists a closed invariant hyperplane H of E such that H contains M and H does not meet U.*

(d) *For any continuous right linear action of S on E with a base of the neighbourhoods of the origin consisting of invariant open convex sets, then any two distinct points in*

$$E_f := \{x \in E \mid s \cdot x = x \quad \text{for all } s \in S\}$$

can be separated by a continuous invariant linear functional on E.

Corollary 2.1. (See [20, Corollary].) *Let S be a semitopological semigroup. If S is abelian, a solvable group, or a compact semigroup with finite intersection property for right ideals, then S has properties (b), (c) and (d) of Theorem 2.1.*

In the following, we will present some characterization of S on the existence of a left invariant mean on $AP(S)$ and $WAP(S)$ in terms of Hahn-Banach extension properties.

Let S be a semitopological semigroup. We say that the action of S on E is *almost periodic* (resp. *weakly almost periodic*) if for each $x \in E$, the *orbit*: $\{s \cdot x \mid s \in S\}$ is relatively compact in the topology of E (resp. weak topology).

Theorem 2.2 generalized [13, Theorems 1 and 3] by Fan and Silverman's result: Theorem 15.A (see [25, Theorem 15.A]).

Theorem 2.2. (See [22, Theorem 1].) *Let S be a semitopological semigroup. The following conditions on S are equivalent:*

(a) *$AP(S)$ has a left invariant mean (LIM).*

(b) *For any almost periodic continuous right linear action of S on E, if P is a continuous sublinear function on E such that $P(s \cdot x) \le P(x)$ for all $s \in S$, $x \in E$, and if L is an invariant linear functional on an invariant subspace F of E such that $L \le P$ on F, then there exists a continuous invariant linear extension $\widetilde{L}$ of L to E such that $\widetilde{L} \le P$.*

(c) *For any almost periodic continuous right linear action of S on E, if F is an invariant subspace of E and K is a convex subset of E such that $K - x_0$ is invariant for some $x_0 \in F \cap \operatorname{int} K$, then for each invariant linear functional L on F such that $L(x) \le \alpha$ for all $x \in F \cap K$ and some fixed real number α, then there exists a continuous invariant linear extension $\widetilde{L}$ of L to E such that $\widetilde{L}(x) \le \alpha$ for all $x \in K$.*

With a proof similar to that of the above Theorem 2.2, we can have the following result for $WAP(S)$.

Theorem 2.3. (See [22, Theorem 2].) *Let S be a semitopological semigroup. The following conditions on S are equivalent:*

(a) *$WAP(S)$ has a left invariant mean (LIM).*

(b) *For any weakly almost periodic weakly continuous right linear action of S on E, if P is a continuous sublinear function on E such that $P(s \cdot x) \le P(x)$ for all $s \in S$, $x \in E$, and if L is an invariant linear functional on an invariant subspace F of E such that $L \le P$ on F, then there exists a continuous invariant linear extension $\widetilde{L}$ of L to E such that $\widetilde{L} \le P$.*

(c) *For any weakly almost periodic weakly continuous right linear action of S on E, if F is an invariant subspace of E and K is a convex subset of E such that $K - x_0$ is invariant for some $x_0 \in F \cap \operatorname{int} K$, then for each invariant linear functional L on F such that $L(x) \le \alpha$ for all $x \in F \cap K$ and some fixed real number α, then there exists a continuous invariant linear extension $\widetilde{L}$ of L to E such that $\widetilde{L}(x) \le \alpha$ for all $x \in K$.*

Remark 2.1. See also Fan [13].

3. Fan–Glicksberg theorem

Let $T\colon E \rightrightarrows E$ be a *set-valued operator* (also known as multifunction) from E to E, i.e., for every $x \in E$, $Tx \subseteq E$, and let $\operatorname{gra} T := \{(x, y) \in E \times E \mid y \in Tx\}$ be the *graph* of T.

The set of the *fixed points* for T is $\operatorname{Fix} T := \{x \in E \mid x \in Tx\}$. Some interesting generalized variational inequalities for set-valued mappings can be found in [1, 2, 4].

Given a set $C \subseteq E$, the *closure* of C is $\overline{C}$ and the *interior* of C is $\operatorname{int} C$. Set $\mathbb{N} := \{1, 2, 3, \ldots\}$.

In this section, we will present a new and precise proof for the following well known Fan–Glicksberg fixed point theorem (see [12, 14]). Our proof is inspired by [14, Theorem] and [24, Theorem 5.28, page 143].

Theorem 3.4 (Fan-Glicksberg)**.** (See [12, Theorem 1] and [14].) *Let $C \subseteq E$ be a nonempty compact convex set. Let $T : C \rightrightarrows C$ be such that $\operatorname{gra} T$ is closed and that Tx is a nonempty convex set for all $x \in C$. Then $\operatorname{Fix} T \neq \varnothing$.*

Proof. Suppose to the contrary that $\operatorname{Fix} T = \varnothing$. Set

$$\Delta := \big\{(x, x) \in E \times E \mid x \in C\big\}.$$

Thus $\operatorname{gra} T \cap \Delta = \varnothing$. Then [24, Theorem 1.10, page 15] implies that there exists an open convex set V with $0 \in V$ and $V = -V$ such that

$$(\operatorname{gra} T + V \times V) \cap (\Delta + V \times V) = \varnothing.$$

Hence

$$(Tx + V) \cap (x + V) = \varnothing, \qquad \forall x \in C. \tag{3.1}$$

Since C is compact, there exist $x_1, x_2, \cdots, x_m \in C$ with $m \in \mathbb{N}$ such that

$$C \subseteq \bigcup_{i=1}^{m} \left(x_i + \frac{1}{2}V\right). \tag{3.2}$$

Set

$$K := \text{the convex hull of } \{x_1, x_2, \cdots, x_m\}.$$

Then K is a compact convex set by [24, Theorem 3.20(a), page 72] and $K \subseteq C$. Thus, we define $A : K \rightrightarrows K$ by

$$Ax := \left(Tx + \frac{1}{2}\overline{V}\right) \cap K, \qquad \forall x \in K.$$

Then $\operatorname{gra} A \subseteq K \times K$ is a closed set by the compactness of C and closeness of $\operatorname{gra} T$ (see also the corresponding lines in the proof of [14, Theorem]). We have

$$Ax \quad \text{is a nonempty convex set}, \qquad \forall x \in K.$$

Indeed, let $x \in K$ and then take $y \in Tx$. By (3.2), there exists $1 \leq i_0 \leq m$ such that $y \in x_{i_0} + \frac{1}{2}V$. Thus $x_{i_0} \in y - \frac{1}{2}V = y + \frac{1}{2}V$ and then $x_{i_0} \in \big(Tx + \frac{1}{2}V\big) \cap K \subseteq Ax$. Therefore, $Ax \neq \varnothing$. By the assumption that Tx is a nonempty convex set, Ax is a nonempty convex set.

Thus by Kakutani's fixed point theorem (see [18]), there exists $z \in K$ such that

$$z \in Az \subseteq Tz + \frac{1}{2}\overline{V} \quad \text{and hence} \quad \left(Tz + \frac{1}{2}\overline{V}\right) \cap (z + V) \neq \varnothing.$$

Since $\frac{1}{2}\overline{V} \subseteq \frac{1}{2}V + \frac{1}{2}V = V$ (see [24, Theorem 1.13(a), page 11]),

$$(Tz + V) \cap (z + V) \neq \varnothing,$$

which contradicts (3.1). Hence $\operatorname{Fix} T \neq \varnothing$. □

4. Some remarks and open problems

Remark 4.2. In the case of a locally compact group, there is an analogue of separation property and Hahn-Banach extension properties for the set of positive definite functions [5, 6, 7, 8, 9].

Problem 4.1. *Can we extend Theorem 2.2 and Theorem 2.3 for set-valued mappings?*

Problem 4.2. *Can we use the ideas in Theorem 2.2 and Theorem 2.3 to extend Day's fixed point theorem in* [10] *for set-valued mappings* [23, 16]*?*

Problem 4.3. *Let $C \subseteq E$ be a nonempty compact convex set, and let S be a nonempty set. For each $s \in S$, let $T_s : C \rightrightarrows C$ be such that $\operatorname{gra} T_s$ is closed and that $T_s x$ is a nonempty convex set for all $x \in C$. Under what condition is the intersection of the fixed point sets of T_s nonempty?*

Problem 4.4. *Can we extend the fixed point properties in* [19, 21] *for set-valued mappings* [17]*?*

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