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Dedicated to Professor Yeol Je Cho on the occasion of his retirement

A Stackelberg quasi-equilibrium problem via quasi-variational inequalities

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ABSTRACT. In this paper, we consider a class of Stackelberg quasi-equilibrium problem with two players in finite dimensional spaces. Existence and location of the Stackelberg quasi-equilibrium is discussed by employing the quasi-variational inequality techniques and the fixed point arguments. The results presented in this paper generalize some corresponding ones due to Nagy [Nagy, S., *Stackelberg equilibia via variational inequalities and projections*, J. Global Optim., **57** (2013), 821–828].

1. INTRODUCTION

Stackelberg equilibrium model is a game with two players, in which the leader one moves first and then the follower one moves sequentially [13]. Recently, Nagy [9] studied the existence and location for a class of Stackelberg equilibrium problem with two players by employing variational inequality methods and fixed point arguments. We note that the strategy sets of the two players are assumed to be fixed in the study of Nagy [9]. However, in some real situations, the strategy sets of the players may depend on their choices, such as the generalized Nash equilibria considered in [10]. Thus, it is important and interesting to extend the Stackelberg equilibrium problem considered by Nagy [9] to the case that the strategy sets of the two players depend on their choices. The motivation of the present work is to make an attempt in this direction.

Assume that $f, g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are the payoff/loss functions for the two players. Let K_1, K_2 be two set-valued mappings from $\mathbb{R}^n$ to $\mathbb{R}^n$ with nonempty values. In this paper, we consider the following Stackelberg quasi-equilibrium problem (SQEP):

$$\begin{aligned} &\min_{x \in K_1(x)} f(x, y) \\ &\text{s.t.} \quad y \in \arg\min_{y \in K_2(y)} g(x, y). \end{aligned}$$

We note that, if $K_1(x) = K_1$ and $K_2(y) = K_2$ for all x, y in $\mathbb{R}^n$, where K_1, K_2 are two subsets of $\mathbb{R}^n$, then (SQEP) reduces to the Stackelberg equilibrium model considered by Nagy [9].

As discussed by Nagy [9], the first step is to define the best response set of the follower in (SQEP) as

$$R_{SQE}(x) = \{y^* \in K_2(y^*) : g(x, y) - g(x, y^*) \geq 0, \ \forall y \in K_2(y^*)\}, \ x \in \mathbb{R}^n,$$

i.e.,

$$R_{SQE}(x) = \arg\min_{y \in K_2(y)} g(x, y), \ x \in \mathbb{R}^n.$$

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Assume that $R_{SQE}(x) \neq \emptyset$ and $r(x)$ is a selector of $R_{SQE}(x)$. Then, the Stackelberg quasi-equilibrium leader set in (SQEP) can be defined as

$$S_{SQE} = \{x^* \in K_1(x^*) : f(x, r(x)) - f(x^*, r(x^*)) \geq 0, \ \forall x \in K_1(x^*)\},$$

i.e.,

$$S_{SQE} = \underset{x \in K_1(x)}{\arg\min} f(x, r(x)).$$

In order to locate the points of the Stackelberg quasi-equilibrium response set, we define a slightly larger set than the best response set of the follower in (SQEP) by means of the quasi-variational inequality [2, 3, 6, 8, 10, 12, 14]. More precisely, based on the C^1 class functions $f(x, y)$ and $g(x, y)$, we can define the Stackelberg quasi-variational response set of the follower in (SQEP) as

$$R_{SQV}(x) = \left\{ y^* \in K_2(y^*) : \left\langle \left. \frac{\partial g(x,y)}{\partial y} \right|_{y=y^*}, y - y^* \right\rangle \geq 0, \ \forall y \in K_2(y^*) \right\}, \quad x \in \mathbb{R}^n,$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^n$. Assume that $R_{SQV}(x) \neq \emptyset$ and $r(x)$ is a selector of $R_{SQV}(x)$. Then, the Stackelberg quasi-variational leader set in (SQEP) can be defined as

$$S_{SQV} = \left\{ x^* \in K_1(x^*) : \left\langle \left. \frac{\partial f(x, r(x))}{\partial x} \right|_{x=x^*}, x - x^* \right\rangle \geq 0, \ \forall x \in K_1(x^*) \right\}.$$

It is well known that the (quasi) variational inequality techniques and the fixed point arguments are popular for solving equilibrium problems (see, for example, [4, 5, 7, 11, 15, 16] and the references therein). The main purpose of this paper is to show some new existence theorems concerned with solutions of (SQEP) by employing the quasi-variational inequality method and the fixed point arguments. This paper is organized as follows. Section 2 presents some basic definitions and results which are needed for the further investigations. In Section 3, we give the main results of this paper in the cases of the compactness and non-compactness for the strategy sets of the two players.

2. Preliminaries

In this section, we give some basic properties concerned with the Stackelberg quasi-variational response set. Throughout this section, the notions are the same as in previous section.

Proposition 2.1. *Let g be a function of class C^1 and $K_2(x)$ be nonempty and convex for all $x \in \mathbb{R}^n$. Then, we have the following conclusions:*

(a) *$R_{SQE}(x) \subseteq R_{SQV}(x)$ for every $x \in \mathbb{R}^n$;*
(b) *If $g(x, \cdot)$ is convex for some $x \in \mathbb{R}^n$, then $R_{SQE}(x) = R_{SQV}(x)$.*

Proof. (a) For any $x \in \mathbb{R}^n$ and $y^* \in R_{SQE}(x)$, it follows from the definition that $y^* \in K_2(y^*)$ and

$$g(x, y) - g(x, y^*) \geq 0, \quad \forall y \in K_2(y^*).$$

Since g is a function of class C^1, by the definition, one has

$$\left\langle \left. \frac{\partial g(x,y)}{\partial y} \right|_{y=y^*}, h \right\rangle = \lim_{t \to 0^+} \frac{g(x, y^* + th) - g(x, y^*)}{t}, \quad \forall h \in \mathbb{R}^n.$$

Since $K_2(x)$ is nonempty and convex, we know that $y^* + t(y - y^*) \in K_2(y^*)$ for all $y \in K_2(y^*)$ and every $t \in [0, 1]$. Thus, taking $h = y - y^* \in \mathbb{R}^n$ in the above expression, we have

$$\left\langle \frac{\partial g(x,y)}{\partial y}\bigg|_{y=y^*}, y-y^* \right\rangle \geq 0, \quad \forall y \in K_2(y^*), \tag{2.1}$$

which shows that $y^* \in R_{SQV}(x)$ and so $R_{SQE}(x) \subseteq R_{SQV}(x)$ for every $x \in \mathbb{R}^n$.

(b) We only need to show that $R_{SQE}(x) \supseteq R_{SQV}(x)$. For any given $y^* \in R_{SQV}(x)$, we know that $y^* \in K_2(y^*)$ and (2.1) holds. Since $g(x,\cdot)$ is convex and of class C^1, it follows that

$$g(x,y) - g(x,y^*) \geq \left\langle \frac{\partial g(x,y)}{\partial y}\bigg|_{y=y^*}, y-y^* \right\rangle, \quad \forall y \in \mathbb{R}^n. \tag{2.2}$$

Thus, by (2.1) and (2.2), we have

$$g(x,y) - g(x,y^*) \geq 0, \quad \forall y \in K_2(y^*)$$

and so $y^* \in R_{SQE}(x)$. This completes the proof. □

Remark 2.1. Proposition 2.1 is a generalization of Proposition 2.1 of Nagy [9].

Assume that $K \subset \mathbb{R}^n$ is a nonempty, closed and convex subset. Let $P_K : \mathbb{R}^n \to K$ be the metric projection defined as follows:

$$P_K(x) = \left\{ y \in K : \|x-y\| = \inf_{z\in K} \|z-x\| \right\}, \quad \forall x \in \mathbb{R}^n.$$

It is well known that

$$z = P_K(x) \Leftrightarrow \langle z-x, y-z \rangle \geq 0, \quad \forall y \in K. \tag{2.3}$$

Moreover, P_K is a nonexpansive mapping, i.e.,

$$\|P_K(x) - P_K(y)\| \leq \|x-y\|, \quad \forall x,y \in \mathbb{R}^n.$$

Now we turn to the investigation of an element of follower's Stackelberg quasi-variational response set $R_{SQV}(x)$. The element acts as a solution to the parametric quasi-variational inequality problem: find $y^* \in K_2(y^*)$ such that

$$\left\langle \frac{\partial g(x,y)}{\partial y}\bigg|_{y=y^*}, y-y^* \right\rangle \geq 0, \quad \forall y \in K_2(y^*).$$

For any given $x \in \mathbb{R}^n$, let

$$A^x_\rho(y) = P_{K_2(y)}\left[y - \rho \frac{\partial g(x,y)}{\partial y} \right], \quad \rho > 0.$$

Then we have the following result.

Proposition 2.2. *Suppose that g is a function of class C^1 and $K_2(x)$ is nonempty, closed and convex for all $x \in \mathbb{R}^n$. Let $X^* \in \mathbb{R}^n$. Then the following assertions are equivalent:*

(i) $y^* \in R_{SQV}(x^*)$;

(ii) $y^* = A^{x^*}_\rho(y^*)$ *for any* $\rho > 0$;

(iii) $y^* = A^{x^*}_\rho(y^*)$ *for some* $\rho > 0$.

Proof. Clearly, $y^* \in R_{SQV}(x^*)$ if and only if

$$\left\langle \frac{\partial g(x^*,y)}{\partial y}\bigg|_{y=y^*}, y-y^* \right\rangle \geq 0, \quad \forall y \in K_2(y^*).$$

which is equivalent to

$$\left\langle y^* - \left[y^* - \rho \left.\frac{\partial g(x^*, y)}{\partial y}\right|_{y=y^*} \right], y - y^* \right\rangle \geq 0, \quad \forall y \in K_2(y^*)$$

for all/some $\rho > 0$. Thus, it follows from (2.3) that

$$R_{SQV}(x^*) = \left\{ y^* \in K_2(y^*) : y^* = P_{K_2(y^*)} \left(y^* - \rho \left.\frac{\partial g(x^*, y)}{\partial y}\right|_{y=y^*} \right) \right\}.$$

By the definition of $A_\rho^{x^*}$, we know the claim is proved. This completes the proof. □

Remark 2.2. Proposition 2.2 is a generalization of Proposition 2.2 of Nagy [9].

At the end of this section, we give a result concerned with the Stackelberg quasi-variational leader set. More precisely, by the definitions, we have the following result.

Proposition 2.3. *Let f be a function of class C^1 and $K_1(x)$ be nonempty, closed and convex for all $x \in \mathbb{R}^n$. Assume that $x \mapsto R_{SQV}(x)$ is a single-valued function of class C^1. Then $S_{SQE} \subset S_{SQV}$.*

Remark 2.3. Proposition 2.3 is a generalization of Proposition 2.3 of Nagy [9].

3. Stackelberg quasi-variational response sets

By Proposition 2.2, to find elements in $R_{SQV}(x)$ is equivalent to find fixed points of A_ρ^x for $\rho > 0$. To this end, we discuss two cases: compact and non-compact strategy sets.

Theorem 3.1. *Let f and g be two functions of class C^1, and $K_1, K_2 \subset \mathbb{R}^n$ be two nonempty, convex and compact subsets. Assume that $K_i : \mathbb{R}^n \to 2^{K_i}$ $(i = 1, 2)$ are two set-valued mappings such that $K_i(x)$ $(i = 1, 2)$ are nonempty, closed and convex, for each $x \in \mathbb{R}^n$. Then the following statements are true:*

(a) *$R_{SQV}(x)$ is nonempty for each given $x \in K_1$;*
(b) *If $R_{SQV}(x)$ is a singleton for each $x \in K_1$, and the mapping $x \mapsto R_{SQV}(x)$ is of class C^1, then S_{SQV} is nonempty.*

Proof. (a) For fixed $x \in K_1$ and $\rho > 0$, since $P_{K_2(y)}$ is nonexpansive for any $y \in \mathbb{R}^n$ and g is a function of class C^1, we know that $A_\rho^x : K_2 \to K_2$ is a continuous mapping. Thus, by employing the Brouwer fixed point theorem, A_ρ^x has a fixed point $y^* \in K_2$. Now Proposition 2.2 shows that $y^* \in R_{SQV}(x)$.

(b) Let $\eta > 0$. Since $R_{SQV}(x)$ is a singleton for each $x \in K_1$, we can define a mapping $B_\eta : K_1 \to K_1$ as follows:

$$B_\eta(x) = P_{K_1(x)} \left[x - \eta \frac{\partial f(x, r(x))}{\partial x} \right], \quad r(x) = R_{SQV}(x).$$

Since $P_{K_1(x)}$ is nonexpansive for any $x \in \mathbb{R}^n$, f and $r(x)$ are two functions of class C^1, we know that $B_\eta : K_1 \to K_1$ is a continuous mapping. Thus, the Brouwer fixed point theorem shows that B_η has a fixed point $x^* \in K_1$. Similar to the proof of Proposition 2.2, we know that $x^* \in B_\eta(x^*)$ is and only if $x^* \in S_{SQV}$. This completes the proof. □

Remark 3.4. We would like to mention that Theorem 3.1 presented in this paper reduces to Theorem 3.1 of Nagy [9] when $K_1(x) = K_1$ and $K_2(y) = K_2$ for all x, y in $\mathbb{R}^n$.

Next we consider the non-compact case. As pointed out by Nagy [9], in order to ensure the existence of the the elements concerned with the Stackelberg quasi-equilibrium/quasi-variational response set in the non-compact case, some additional assumptions are needed beside regular conditions of functions.

Theorem 3.2. *Let g be a function of class C^1 and $K_2 : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be a set-valued mapping with nonempty, closed and convex valued such that*

$$\left\|P_{K_2(y_1)}z - P_{K_2(y_2)}z\right\| \leq k_2\|y_1 - y_2\|, \quad \forall y_1, y_2, z \in \mathbb{R}^n, \tag{3.4}$$

where $k_2 > 0$ is a constant. For any given $x \in \mathbb{R}^n$, assume that there exist two positive constants κ_g and L_g such that

$$\left\langle \left.\frac{\partial g(x,y)}{\partial y}\right|_{y=y_1} - \left.\frac{\partial g(x,y)}{\partial y}\right|_{y=y_2}, y_1 - y_2 \right\rangle \geq \kappa_g\|y_1 - y_2\|^2, \quad \forall y_1, y_2 \in \mathbb{R}^n \tag{3.5}$$

and

$$\left\| \left.\frac{\partial g(x,y)}{\partial y}\right|_{y=y_1} - \left.\frac{\partial g(x,y)}{\partial y}\right|_{y=y_2} \right\| \leq L_g\|y_1 - y_2\|, \quad \forall y_1, y_2 \in \mathbb{R}^n. \tag{3.6}$$

If

$$k_2 + \sqrt{1 - 2\rho\kappa_g + \rho^2 L_g^2} < 1, \tag{3.7}$$

then there exists a unique point $y^* \in \mathbb{R}^n$ such that $R_{SQV}(x) = \{y^*\}$. Moreover, the sequence $\{y^k\}$ generated by

$$\begin{cases} y^{k+1} = A_\rho^x(y^k), \\ y^0 \in \mathbb{R}^n, \end{cases} \tag{3.8}$$

converges to y^*.

Proof. For any $y_1, y_2 \in \mathbb{R}^n$, it follows from (3.4) that

$$\begin{aligned} &\|A_\rho^x(y_1) - A_\rho^x(y_2)\| \\ = &\left\| P_{K_2(y_1)}\left[y_1 - \rho\left.\frac{\partial g(x,y)}{\partial y}\right|_{y=y_1}\right] - P_{K_2(y_2)}\left[y_2 - \rho\left.\frac{\partial g(x,y)}{\partial y}\right|_{y=y_2}\right]\right\| \\ \leq &\; k_2\|y_1 - y_2\| + \left\|\left[y_1 - \rho\left.\frac{\partial g(x,y)}{\partial y}\right|_{y=y_1}\right] - \left[y_2 - \rho\left.\frac{\partial g(x,y)}{\partial y}\right|_{y=y_2}\right]\right\|. \end{aligned} \tag{3.9}$$

By (3.5) and (3.6), one has

$$\begin{aligned} &\left\|\left[y_1 - \rho\left.\frac{\partial g(x,y)}{\partial y}\right|_{y=y_1}\right] - \left[y_2 - \rho\left.\frac{\partial g(x,y)}{\partial y}\right|_{y=y_2}\right]\right\|^2 \\ = &\; \|y_1 - y_2\|^2 - 2\rho\left\langle \left.\frac{\partial g(x,y)}{\partial y}\right|_{y=y_1} - \left.\frac{\partial g(x,y)}{\partial y}\right|_{y=y_2}, y_1 - y_2 \right\rangle \\ &+ \rho^2\left\| \left.\frac{\partial g(x,y)}{\partial y}\right|_{y=y_1} - \left.\frac{\partial g(x,y)}{\partial y}\right|_{y=y_2} \right\|^2 \\ \leq &\; \|y_1 - y_2\|^2 - 2\rho\kappa_g\|y_1 - y_2\|^2 + \rho^2 L_g^2\|y_1 - y_2\|^2 \\ = &\; (1 - 2\rho\kappa_g + \rho^2 L_g^2)\|y_1 - y_2\|^2. \end{aligned} \tag{3.10}$$

Thus, from (3.9) and (3.10), we have

$$\|A_\rho^x(y_1) - A_\rho^x(y_2)\| \leq h\|y_1 - y_2\|, \quad \forall y_1, y_2 \in \mathbb{R}^n \tag{3.11}$$

with

$$0 < h = k_2 + \sqrt{1 - 2\rho\kappa_g + \rho^2 L_g^2}.$$

It follows from (3.7) that $0 < h < 1$ and so by the Banach fixed point theorem, A^x_ρ has a unique fixed point $y^* \in \mathbb{R}^n$. Moreover, it is easy to know that the sequence $\{y^k\}$ generated by (3.8) converges to y^*. This completes the proof. □

Remark 3.5. It is clear that Theorem 3.2 generalizes discrete case of Theorem 3.2 of Nagy [9].

Theorem 3.3. *Assume that all the conditions of Theorem 3.2 hold. If*

$$\left\| \frac{\partial g(x_1, y)}{\partial y} - \frac{\partial g(x_2, y)}{\partial y} \right\| \le L'_g \|x_1 - x_2\|, \quad \forall x_1, x_2, y \in \mathbb{R}^n, \tag{3.12}$$

where $L'_g > 0$ is a constant, then

$$\|R_{SQV}(x_1) - R_{SQV}(x_2)\| \le C\|x_1 - x_2\|, \quad \forall x_1, x_2 \in \mathbb{R}^n,$$

where C is a positive constant.

Proof. From Theorem 3.2, we know that $R_{SQV}(x) = \{y^*\}$. For any $x_1, x_2 \in \mathbb{R}^n$, let $y_i^* = R_{SQV}(x_i)$ with $i = 1, 2$. Then, it follows from (3.11) and (3.12) that

$$\begin{aligned} \|R_{SQV}(x_1) - R_{SQV}(x_2)\| &= \|y_1^* - y_2^*\| \\ &= \|A^{x_1}_\rho(y_1^*) - A^{x_2}_\rho(y_2^*)\| \\ &\le \|A^{x_1}_\rho(y_1^*) - A^{x_1}_\rho(y_2^*)\| + \|A^{x_1}_\rho(y_2^*) - A^{x_2}_\rho(y_2^*)\| \\ &\le h\|y_1^* - y_2^*\| + \rho \left\| \left.\frac{\partial g(x_1, y)}{\partial y}\right|_{y=y_2^*} - \left.\frac{\partial g(x_2, y)}{\partial y}\right|_{y=y_2^*} \right\| \\ &\le h\|y_1^* - y_2^*\| + \rho L'_g \|x_1 - x_2\|. \end{aligned}$$

In view of fact that $0 < h < 1$, we have

$$\|y_1^* - y_2^*\| \le \frac{\rho L'_g}{1 - h} \|x_1 - x_2\|$$

and so

$$\|R_{SQV}(x_1) - R_{SQV}(x_2)\| \le C\|x_1 - x_2\|$$

with a positive constant $C = \frac{\rho L'_g}{1-h}$. This completes the proof. □

Next we will consider the existence result for the Stackelberg quasi-variational leader set. To this end, we need the following assumptions. Suppose that there exist constants $\kappa_f > 0$ and $L_f > 0$ such that, for all $x_1, x_2 \in \mathbb{R}^n$,

$$\left\langle \left.\frac{\partial f(x, r(x_2))}{\partial x}\right|_{x=x_1} - \left.\frac{\partial f(x, r(x))}{\partial x}\right|_{x=x_2}, x_1 - x_2 \right\rangle \ge \kappa_f \|x_1 - x_2\|^2 \tag{3.13}$$

and

$$\left\| \left.\frac{\partial f(x, r(x_2))}{\partial x}\right|_{x=x_1} - \left.\frac{\partial f(x, r(x))}{\partial x}\right|_{x=x_2} \right\| \le L_f \|x_1 - x_2\|. \tag{3.14}$$

Moreover, we assume that

$$\left\| \left.\frac{\partial f(x, r(x))}{\partial x}\right|_{x=x_1} - \left.\frac{\partial f(x, r(x_2))}{\partial x}\right|_{x=x_1} \right\| \le L'_f \|r(x_1) - r(x_2)\|, \quad \forall x_1, x_2 \in \mathbb{R}^n. \tag{3.15}$$

When the mapping $x \mapsto R_{SQV}(x)$ is single-valued, we denote $r(x) = R_{SQV}(x)$. For any $\eta > 0$, we can define an operator $B_\eta : \mathbb{R}^n \to \mathbb{R}^n$ as follows:

$$B_\eta(x) = P_{K_1(x)} \left[x - \eta \frac{\partial f(x, r(x))}{\partial x} \right], \quad x \in \mathbb{R}^n. \tag{3.16}$$

Theorem 3.4. *Assume that all the conditions of Theorems 3.2 and 3.3 hold. Moreover, suppose that conditions* (3.13)-(3.15) *are satisfied and*

$$\left\|P_{K_1(x_1)}z - P_{K_1(x_2)}z\right\| \le k_1\|x_1 - x_2\|, \quad \forall x_1, x_2, z \in \mathbb{R}^n, \tag{3.17}$$

where $k_1 > 0$ is a constant. If there exist constants $\rho > 0$ and $\eta > 0$ such that

$$h = k_2 + \sqrt{1 - 2\rho\kappa_g + \rho^2 L_g^2} < 1, \quad k_1 + \sqrt{1 - 2\eta\kappa_f + \eta^2 L_f^2} + \frac{\eta\rho L_g' L_f'}{1-h} < 1, \tag{3.18}$$

then there exists a unique $x^* \in \mathbb{R}^n$ such that $S_{SQV} = \{x^*\}$ and $y^* = r(x^*)$ with $R_{SQV}(x^*) = \{y^*\}$.

Proof. By (3.16) and (3.17), for any $x_1, x_2 \in \mathbb{R}^n$, we have

$$\begin{aligned}
&\|B_\eta(x_1) - B_\eta(x_2)\| \\
= &\left\| P_{K_1(x_1)}\left[x_1 - \eta \left.\frac{\partial f(x, r(x))}{\partial x}\right|_{x=x_1}\right] - P_{K_1(x_2)}\left[x_2 - \eta \left.\frac{\partial f(x, r(x))}{\partial x}\right|_{x=x_2}\right]\right\| \\
\le &\; k_1\|x_1 - x_2\| + \left\|(x_1 - x_2) - \eta\left[\left.\frac{\partial f(x, r(x))}{\partial x}\right|_{x=x_1} - \left.\frac{\partial f(x, r(x))}{\partial x}\right|_{x=x_2}\right]\right\| \\
\le &\; k_1\|x_1 - x_2\| + \left\|(x_1 - x_2) - \eta\left[\left.\frac{\partial f(x, r(x_2))}{\partial x}\right|_{x=x_1} - \left.\frac{\partial f(x, r(x))}{\partial x}\right|_{x=x_2}\right]\right\| \\
&+ \eta\left\|\left.\frac{\partial f(x, r(x))}{\partial x}\right|_{x=x_1} - \left.\frac{\partial f(x, r(x_2))}{\partial x}\right|_{x=x_1}\right\|.
\end{aligned} \tag{3.19}$$

It follows from (3.13) and (3.14) that

$$\left\|(x_1 - x_2) - \eta\left[\left.\frac{\partial f(x, r(x_2))}{\partial x}\right|_{x=x_1} - \left.\frac{\partial f(x, r(x))}{\partial x}\right|_{x=x_2}\right]\right\|^2 \le (1 - 2\eta\kappa_f + \eta^2 L_f^2)\|x_1 - x_2\|^2$$

and so

$$\left\|(x_1 - x_2) - \eta\left[\left.\frac{\partial f(x, r(x_2))}{\partial x}\right|_{x=x_1} - \left.\frac{\partial f(x, r(x))}{\partial x}\right|_{x=x_2}\right]\right\| \le \sqrt{1 - 2\eta\kappa_f + \eta^2 L_f^2}\|x_1 - x_2\|.$$

By (3.15) and Theorem 3.3, one has

$$\left\|\left.\frac{\partial f(x, r(x))}{\partial x}\right|_{x=x_1} - \left.\frac{\partial f(x, r(x_2))}{\partial x}\right|_{x=x_1}\right\| \le L_f'\|r(x_1) - r(x_2)\| \le \frac{\rho L_g' L_f'}{1-h}\|x_1 - x_2\|.$$

These inequalities mentioned above together with (3.19) imply that

$$\|B_\eta(x_1) - B_\eta(x_2)\| \le h'\|x_1 - x_2\|, \quad \forall x_1, x_2 \in \mathbb{R}^n,$$

where

$$h' = k_1 + \sqrt{1 - 2\eta\kappa_f + \eta^2 L_f^2} + \frac{\eta\rho L_g' L_f'}{1-h}.$$

Now (3.18) shows that B_η is a contractive mapping. Thus, it follows from the Banach fixed point theorem that there exists a unique $x^* \in \mathbb{R}^n$ such that $x^* = B_\eta(x^*)$ and so $S_{SQV} = \{x^*\}$. Moreover, by Theorem 3.1, there exists a unique $y^* \in \mathbb{R}^n$ such that $y^* = r(x^*)$ with $R_{SQV}(x^*) = \{y^*\}$. This completes the proof. □

Remark 3.6. We would like to mention that conditions (3.4) and (3.17) are similar to condition (b) in Theorem 1 of Aussel et al. [1].

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