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Dedicated to Professor Yeol Je Cho on the occasion of his retirement

The Dunkl generalization of Stancu type q-Szász-Mirakjan-Kantorovich operators and some approximation results

M. MURSALEEN and MOHD. AHASAN

ABSTRACT. In this paper, a Dunkl type generalization of Stancu type q-Szász-Mirakjan-Kantorovich positive linear operators of the exponential function is introduced. With the help of well-known Korovkin's theorem, some approximation properties and also the rate of convergence for these operators in terms of the classical and second-order modulus of continuity, Peetre's K-functional and Lipschitz functions are investigated.

1. INTRODUCTION

S. Bernstein [5] introduced a sequence of operators which are known as the Bernstein operators:

$$B_n(f;y) = \sum_{k=0}^{n} \binom{n}{k} y^k (1-y)^{n-k} f\left(\frac{k}{n}\right), \tag{1.1}$$

for $n \in \mathbb{N}$, $y \in [0,1]$ and for a real valued function f defined and bounded on $[0,1]$.
In 1950, for $f \in C[0,\infty)$ Szász [18] introduced the operators

$$S_n(f;y) = e^{-ny} \sum_{k=0}^{\infty} \frac{(ny)^k}{k!} f\left(\frac{k}{n}\right), \tag{1.2}$$

where $n \in \mathbb{N}$ and $y \geq 0$ whenever the series in (1.2) converges absolutely.
In the sequel, we recall some notations and basic definitions of q-calculus which have an important role in the present paper. For related work, we refer to [1]–[4], [6], [9], [11]–[15], [17].

Definition 1.1. For the given value of $|q| < 1$, the q-integer $[m]_q$ is defined by

$$[m]_q = \begin{cases} \frac{1-q^m}{1-q}, & (m \in \mathbb{C}) \\ \sum\limits_{l=0}^{m-1} q^l = 1 + q + q^2 + \cdots + q^{m-1} & (m \in \mathbb{N}). \end{cases} \tag{1.3}$$

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Corresponding author: M. Mursaleen; mursaleenm@gmail.com

Definition 1.2. For the given value of $|q|<1$, the q-factorial $[m]_q!$ is defined as

$$(1.4)\qquad [m]_q! = \begin{cases} 1, & (m=0) \\ \prod_{l=1}^{m} [l]_q & (m \in \mathbb{N}). \end{cases}$$

A generalization of exponential function [16] is defined as follows:

$$e_\mu(y) = \sum_{m=0}^{\infty} \frac{y^m}{\gamma_\mu(m)},$$

where

$$\gamma_\mu(2l) = \frac{2^{2l} l! \Gamma\left(l+\mu+\frac{1}{2}\right)}{\Gamma\left(\mu+\frac{1}{2}\right)},$$

and

$$\gamma_\mu(2l+1) = \frac{2^{2l+1} l! \Gamma\left(l+\mu+\frac{3}{2}\right)}{\Gamma\left(\mu+\frac{1}{2}\right)}.$$

For γ_μ, the recursion formula is defined as

$$\gamma_\mu(l+1) = (l+1+2\mu\theta_{l+1})\gamma_\mu(l), \qquad \left(l \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; \mu > -\frac{1}{2}\right)$$

where

$$\theta_l = \begin{cases} 0, & \text{if } l \in 2\mathbb{N} \\ 1, & \text{if } l \in 2\mathbb{N}+1. \end{cases}$$

A Dunkl analogue of the Szász operators via $e_\mu(x)$ was defined by Sucu [19] and is given by

$$(1.5)\qquad S_n^*(f;y) := \frac{1}{e_\mu(ny)} \sum_{l=0}^{\infty} \frac{(ny)^l}{\gamma_\mu(l)} f\left(\frac{l+2\mu\theta_l}{n}\right),$$

where $y \geq 0, \mu \geq 0,\ n \in \mathbb{N}$ *and* $f \in C[0,\infty)$.

Cheikh et al. [6] defined the q-Dunkl analogues of the q-exponential functions and studied the q-Dunkl classical q-Hermite type polynomials, together with their recurrence relations given by

$$(1.6)\qquad e_{\mu,q}(y) = \sum_{n=0}^{\infty} \frac{y^n}{\gamma_{\mu,q}(n)}, \quad for\ y \in [0,\infty),\ q \in (0,1),\ \mu > -\frac{1}{2}$$

$$(1.7)\qquad E_{\mu,q}(y) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}} y^n}{\gamma_{\mu,q}(n)}, \quad y \in [0,\infty)$$

$$(1.8)\qquad \gamma_{\mu,q}(n+1) = \left(\frac{1-q^{2\mu\theta_{n+1}+(n+1)}}{1-q}\right)\gamma_{\mu,q}(n), \quad n \in \mathbb{N},$$

where

$$\theta_n = \begin{cases} 0, & \text{if } n \in 2\mathbb{N}, \\ 1, & \text{if } n \in 2\mathbb{N}+1. \end{cases}$$

An explicit formula for $\gamma_{\mu,q}(n)$ is

$$\gamma_{\mu,q}(n) = \frac{(q^{2\mu+1},q^2)_{[\frac{n+1}{2}]}(q^2,q^2)_{[\frac{n}{2}]}}{(1-q)^n}\gamma_{\mu,q}(n), \quad n \in \mathbb{N}.$$

Some values of $\gamma_{\mu,q}(n)$ are defined as

$$\begin{aligned}
\gamma_{\mu,q}(0) &= 1,\\
\gamma_{\mu,q}(1) &= \frac{1-q^{2\mu+1}}{1-q},\\
\gamma_{\mu,q}(2) &= \left(\frac{1-q^{2\mu+1}}{1-q}\right)\left(\frac{(1-q)(1+q)}{1-q}\right),\\
\gamma_{\mu,q}(3) &= \left(\frac{1-q^{2\mu+1}}{1-q}\right)\left(\frac{(1-q)(1+q)}{1-q}\right)\left(\frac{1-q^{2\mu+3}}{1-q}\right),\dots\\
\gamma_{\mu,q}(n) &= \left(\frac{1-q^{2\mu\theta_n+n}}{1-q}\right)\gamma_{\mu,q}(n-1), \quad n\in\mathbb{N}.
\end{aligned}$$

İçöz and Çekim [8] introduced q-Szász operators [9] and Kantorovich integral generalization of q-Szász operators via Dunkl generalization.
Here, we construct Stancu type q-Szász-Mirakjan-Kantorovich operators of the exponential functions by Dunkl generalization. For these operators, we get some approximation results by well-known korovkin's theorem. Further, in terms of the classical, weighted and second order modulus of continuity we obtain the rate of convergence of the operators.

2. CONSTRUCTION OF OPERATORS

Srivastava et al. [17] defined the Dunkl generalization of q-Szász-Mirakjan-Kantorovich type operators for $n\in\mathbb{N}$, $y\in[0,\infty)$, $\mu>\frac{1}{2}$ and $q\in(0,1)$. We have

$$\tilde{K_{n,q}}(f;y)=\frac{[n]_q}{E_{\mu,q}([n]_qy)}\sum_{k=0}^{\infty}\frac{([n]_qy)^k}{\gamma_{\mu,q}(k)}q^{\frac{k(k-1)}{2}}\int_{\frac{[k+2\mu\theta_k]_q}{q^{k-2}[n]_q}}^{\frac{[k+1+2\mu\theta_k]_q-1}{q^{k-1}[n]_q}+\frac{1}{[n]_q}}f(t)d_qt, \tag{2.9}$$

where, f is a continuously nondecreasing function on the interval $[0,\infty)$.
Now we recall the following lemma which is proved in [17].

Lemma 2.1. *Let the q-operator* $\tilde{K_{n,q}}(f;y)$ *be given by* (2.9). *Then*
(1) $\tilde{K_{n,q}}(f;y)=1,\quad$ *when* $f(t)=1$
(2) $\tilde{K_{n,q}}(f;y)=\frac{1}{[2]_q[n]_q}+\frac{2q}{[2]_q}y,\quad$ *when* $f(t)=t$
(3) *When* $f(t)=t^2$, *then we have*

$$\begin{aligned}
\frac{1}{[3]_q[n]_q^2}+\frac{3q\left(1+q^{2\mu}[1-2\mu]_q\right)y}{[3]_q[n]_q}+\frac{3q}{[3]_q}y^2\le\tilde{K_{n,q}}(t^2;y) &\le \frac{1}{[3]_q[n]_q^2}+\frac{3}{[3]_q}y^2\\
&+ \frac{3\left(1+[1+2\mu]_q\right)y}{[3]_q[n]_q}.
\end{aligned}$$

Now we introduce Stancu type q-Szász-Mirakjan-Kantorovich operators via Dunkl generalization, for any $n\in\mathbb{N}$, $y\in[0,\infty)$, $0<q<1$ and $\mid\mu\mid\le\frac{1}{2}$, we have
(2.10)

$$K_{n,q}^{\alpha,\beta}(f;y)=\frac{[n]_q}{E_{\mu,q}([n]_qy)}\sum_{k=0}^{\infty}\frac{([n]_qy)^k}{\gamma_{\mu,q}(k)}q^{\frac{k(k-1)}{2}}\int_{\frac{[k+2\mu\theta_k]_q}{q^{k-2}[n]_q}}^{\frac{[k+1+2\mu\theta_k]_q-1}{q^{k-1}[n]_q}+\frac{1}{[n]_q}}f\left(\frac{[n]_qt+\alpha}{[n]_q+\beta}\right)d_qt.$$

If $\alpha=\beta=0$ in (2.10), then $K_{n,q}^{\alpha,\beta}(f;y)$ reduce to the operators (2.9).

Lemma 2.2. *For each* $y\ge0$, *we have*
(1) $K_{n,q}^{\alpha,\beta}(1;y)=1,\qquad$ *for* $f(t)=1$

(2) $K_{n,q}^{\alpha,\beta}(t;y)=\frac{\alpha}{([n]_q+\beta)}+\frac{1}{[2]_q([n]_q+\beta)}+\frac{2q[n]_q}{[2]_q([n]_q+\beta)}y,$ *for* $f(t)=t$

(3) *For* $f(t)=t^2$, *we have*

$$\frac{\left(1+3q[n]_qy+3q[n]_qq^{2\mu}[1-2\mu]_qy+3q[n]_q^2y^2\right)}{[3]_q([n]_q+\beta)^2}+\frac{2\alpha}{[2]_q([n]_q+\beta)^2}(1+2q[n]_qy)+\frac{\alpha^2}{([n]_q+\beta)^2}$$

$$\leq K_{n,q}^{\alpha,\beta}\left(t^2;y\right)\leq\frac{1}{[3]_q([n]_q+\beta)^2}\left(1+3[n]_qy+3[n]_q[1+2\mu]_qy+3[n]_q^2y^2\right)$$

$$+\frac{2\alpha}{[2]_q([n]_q+\beta)^2}\left(1+2q[n]_qy\right)+\frac{\alpha^2}{([n]_q+\beta)^2}.$$

Lemma 2.3. *We have*

(1) $K_{n,q}^{\alpha,\beta}(t-1;y)=\frac{\alpha}{([n]_q+\beta)}-1+\frac{1}{[2]_q([n]_q+\beta)}\left(1+2q[n]_qy\right),$

(2) $K_{n,q}^{\alpha,\beta}(t-y;y)=\frac{\alpha}{([n]_q+\beta)}+\frac{1}{[2]_q([n]_q+\beta)}+\left(\frac{2q[n]_q}{[2]_q([n]_q+\beta)}-1\right)y,$

(3)
$$\left(\frac{\left(3q[n]_q+3[n]_qq^{2\mu+1}[1-2\mu]_q\right)}{[3]_q([n]_q+\beta)^2}+\frac{2}{([n]_q+\beta)}\left(\frac{2\alpha q[n]_q}{[2]_q([n]_q+\beta)}-\alpha-\frac{1}{[2]_q}\right)\right)y$$
$$+\frac{1}{([n]_q+\beta)^2}\left(\alpha^2+\frac{1}{[3]_q}+\frac{2\alpha}{[2]_q}\right)+\left(1+\frac{3q[n]_q^2}{[3]_q([n]_q+\beta)^2}-\frac{4q[n]_q}{[2]_q([n]_q+\beta)}\right)y^2$$
$$\leq K_{n,q}^{\alpha,\beta}\left((t-y)^2;y\right)\leq\left(\frac{(3[n]_q+3[n]_q[1+2\mu]_q)}{[3]_q([n]_q+\beta)^2}+\frac{2}{([n]_q+\beta)}\left(\frac{2\alpha q[n]_q}{[2]_q([n]_q+\beta)}-\alpha-\frac{1}{[2]_q}\right)\right)y$$
$$+\frac{1}{([n]_q+\beta)^2}\left(\alpha^2+\frac{1}{[3]_q}+\frac{2\alpha}{[2]_q}\right)+\left(1+\frac{3[n]_q^2}{[3]_q([n]_q+\beta)^2}-\frac{4q[n]_q}{[2]_q([n]_q+\beta)}\right)y^2.$$

3. Korovkin type approximation properties

Here, the main aim is to derive some well-known Korovkin type and weighted Korovkin type approximation results for the operators $K_{n,q}^{\alpha,\beta}(f;y)$ defined in (2.10).

Theorem 3.1. ([10]) *Let* $L_n:C[a,b]\to C[a,b]$ *is a sequence of positive linear operators. Then for all* $f\in C[a,b]$, $\lim_{n\to\infty}L_nf=f$ *uniformly on* $[a,b]$ *if* $\lim_{n\to\infty}L_nt^r=t^r$ $(r=0,1,2)$ *uniformly on* $[a,b]$.

Let $C_B(\mathbb{R}^+)$ is a normed linear space equipped with the norm:

$$\| f\|_{C_B}=\sup_{y\geq 0}|f(y)|,$$

where, $C_B(\mathbb{R}^+)$ is the set of all bounded and continuous functions on $\mathbb{R}^+=[0,\infty)$. Also

$$H:=\{f(y):y\in[0,\infty),\frac{f(y)}{1+y^2}\text{ is convergent as }y\to\infty\}.$$

For the operators $K_{n,q}^{\alpha,\beta}(f;y)$, we obtain the convergence results if we take $q=q_n$, where $q_n\in(0,1)$ satisfying

$$\lim_n q_n\to 1,\quad \lim_n q_n^n\to a. \tag{3.11}$$

Theorem 3.2. *Let* $q=q_n$, *for* $0<q_n<1$ *satisfy* (3.11). *Then, on each compact subset of* $[0,\infty)$ *and any function* $f\in C[0,\infty)\cap H$,

$$\lim_{n\to\infty}K_{n,q}^{\alpha,\beta}(f;y)=f(y),$$

converges uniformly.

Proof. We prove this theorem regarding the convergence of a sequence of positive linear operators with the help of well known Korovkin's theorem, if we prove the following conditions, then it is enough

$$K_{n,q}^{\alpha,\beta}(t^j;y)=y^j,\ j=0,1,2,\ \{\text{as }n\to\infty\}$$

converges uniformly on the interval $[0,a]$, where $a>0$.
From (3.11) and

$$\frac{1}{[n]_{q_n}} \to 0 \ (n\to\infty),$$

we have

$$\begin{aligned}\lim_{n\to\infty} K_{n,q}^{\alpha,\beta}(t;y) &= \lim_{n\to\infty}\frac{1}{([n]_q+\beta)}\left(\alpha+\frac{1}{[2]_q}+\frac{2q[n]_q}{[2]_q}y\right)\\ &= y\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty} K_{n,q}^{\alpha,\beta}\left(t^2;y\right) &= \lim_{n\to\infty}\left(\frac{\left(1+3[n]_q y+3[n]_q[1+2\mu]_q y+3[n]_q^2y^2\right)}{[3]_q([n]_q+\beta)^2}\right)\\ &+ \lim_{n\to\infty}\left(\frac{2\alpha}{[2]_q([n]_q+\beta)^2}\left(1+2q[n]_q y\right)+\frac{\alpha^2}{([n]_q+\beta)^2}\right)\\ &= y^2.\end{aligned}$$

□

Gadzhiev [7] proved weighted Korovkin-type theorems. If a real function $\rho(y)=1+y^2$ is continuous on $\mathbb{R}$ then it is known as a weight function if

$$\lim_{|y|\to\infty}\rho(y)=\infty \quad \text{and } \rho(y)\geq 1 \text{ for all } y\in\mathbb{R}.$$

We have some weighted spaces of the functions defined on $\mathbb{R}^+$ as follows

$$\begin{aligned}P_\rho(\mathbb{R}^+) &= \{f: \mid f(y)\mid\leq K_f\rho(y)\},\\ Q_\rho(\mathbb{R}^+) &= \left\{f:\ f\in P_\rho(\mathbb{R}^+)\cap C[0,\infty)\right\} \text{ and}\\ Q_\rho^M(\mathbb{R}^+) &= \left\{f:\ f\in Q_\rho(\mathbb{R}^+) \text{ and } \lim_{y\to\infty}\frac{f(y)}{\rho(y)}=M,\ M \text{ is a constant}\right\},\end{aligned}$$

where K_f is a constant depend only on f, $\rho(y)$ is defined as above and $Q_\rho(\mathbb{R}^+)$ be a normed space with $\parallel f\parallel_\rho=\sup\limits_{y\geq 0}\frac{|f(y)|}{\rho(y)}$.

Theorem 3.3. *Let* $q=q_n$ *satisfy* (3.11), *for* $0<q<1$. *Then for any function* $f\in Q_\rho^k(\mathbb{R}^+)$, *we have*

$$\lim_{n\to\infty}\parallel K_{n,q}^{\alpha,\beta}(f;y)-f(y)\parallel_\rho=0.$$

Proof. By using Lemma 2.2, the first condition of Lemma 2.2 is fulfilled for $\tau=0$. Now for $\tau=1,2$ it is easy to see from conditions (2) and (3) of Lemma 2.2, we have

$$\parallel K_{n,q}^{\alpha,\beta}(t^\tau;y)-y^\tau\parallel_\rho=0.$$

By using the weighted Korovkin's type theorem we get the desired result. □

4. THE RATES OF CONVERGENCE

In this section, we use Lipschitz type maximal functions and modulus of continuity for calculating the rate of convergence of the operators (2.10).
Let $\omega(f,\delta)$ denotes the modulus of continuity of f and $f\in C[0,\infty)$. For any interval of length does not exceed $\delta>0$, $\omega(f,\delta)$ gives the maximum oscillation of f and we have the following relation:

$$\omega(f,\delta)=\sup_{|w-v|\leq\delta}\mid f(w)-f(v)\mid,\quad v,w\in[0,\infty). \tag{4.12}$$

But we know that $\lim_{\delta\to 0+}\omega(f,\delta)=0$,
for any $f\in C[0,\infty)$, $\delta>0$, we have

$$| f(w)-f(v) |\leq \left(\frac{| w-v |}{\delta}+1\right)\omega(f,\delta). \tag{4.13}$$

Theorem 4.4. *Let the operators $K_{n,q}^{\alpha,\beta}(f;y)$ is defined by* (2.10)*. Then for a given $q\in(0,1)$, $y\geq 0$, and $f\in C^*[0,\infty)$, we have*

$$| K_{n,q}^{\alpha,\beta}(f;y)-f(y) |\leq \left\{1+\sqrt{\phi^*(y)}\right\}\omega\left(f;\frac{1}{\sqrt{([n]_q+\beta)}}\right), \tag{4.14}$$

where

$$\begin{aligned}\phi^*(y) &= \left(\frac{(3[n]_q+3[n]_q[1+2\mu]_q)}{[3]_q}+2([n]_q+\beta)\left(\frac{2\alpha q[n]_q}{[2]_q([n]_q+\beta)}-\alpha-\frac{1}{[2]_q}\right)\right)y\\ &+ \left(\alpha^2+\frac{1}{[3]_q}+\frac{2\alpha}{[2]_q}\right)+\left(([n]_q+\beta)^2+\frac{3[n]_q^2}{[3]_q}-\frac{4q[n]_q([n]_q+\beta)}{[2]_q}\right)y^2,\end{aligned}$$

$\omega(f,\delta)$ is defined as above and $C^*[0,\infty)$ denotes the space of all uniformly continuous function on $\mathbb{R}^+$.

Proof. Now using the above relations (4.12), (4.13) and Cauchy-Schwarz inequality, we get

$$\begin{aligned}&| K_{n,q}^{\alpha,\beta}(f;y)-f(y) |\\ \leq\ & \frac{[n]_q}{E_{\mu,q}([n]_qy)}\sum_{k=0}^{\infty}\frac{([n]_qy)^k}{\gamma_{\mu,q}(k)}q^{\frac{k(k-1)}{2}}\int_{\frac{[k+2\mu\theta_k]_q}{q^{k-2}[n]_q}}^{\frac{[k+1+2\mu\theta_k]_q-1}{q^{k-1}[n]_q}+\frac{1}{[n]_q}}|f(t)-f(y)|d_q(t)\\ \leq\ & \frac{[n]_q}{E_{\mu,q}([n]_qy)}\sum_{k=0}^{\infty}\frac{([n]_qy)^k}{\gamma_{\mu,q}(k)}q^{\frac{k(k-1)}{2}}\int_{\frac{[k+2\mu\theta_k]_q}{q^{k-2}[n]_q}}^{\frac{[k+1+2\mu\theta_k]_q-1}{q^{k-1}[n]_q}+\frac{1}{[n]_q}}\left(1+\frac{| t-y |}{\delta}\right)d_q(t)\omega(f;\delta)\\ =\ & \left\{1+\frac{1}{\delta}\left(\frac{[n]_q}{E_{\mu,q}([n]_qy)}\sum_{k=0}^{\infty}\frac{([n]_qy)^k}{\gamma_{\mu,q}(k)}q^{\frac{k(k-1)}{2}}\int_{\frac{[k+2\mu\theta_k]_q}{q^{k-2}[n]_q}}^{\frac{[k+1+2\mu\theta_k]_q-1}{q^{k-1}[n]_q}+\frac{1}{[n]_q}}| t-y | d_q(t)\right)\right\}\\ \times\ & \omega(f;\delta)\\ \leq\ & \left\{1+\frac{1}{\delta}\left(\frac{[n]_q}{E_{\mu,q}([n]_qy)}\sum_{k=0}^{\infty}\frac{([n]_qy)^k}{\gamma_{\mu,q}(k)}q^{\frac{k(k-1)}{2}}\int_{\frac{[k+2\mu\theta_k]_q}{q^{k-2}[n]_q}}^{\frac{[k+1+2\mu\theta_k]_q-1}{q^{k-1}[n]_q}+\frac{1}{[n]_q}}(t-y)^2d_q(t)\right)^{\frac{1}{2}}\right\}\\ \times\ & \omega(f;\delta)\\ =\ & \left\{1+\frac{1}{\delta}\left(K_{n,q}^{\alpha,\beta}((t-y)^2;y)\right)^{\frac{1}{2}}\right\}\omega(f;\delta),\quad where\quad t=\frac{[n]_qt+\alpha}{[n]_q+\beta}.\end{aligned}$$

We easily obtain the result (4.14) if $\delta=\delta_n=\sqrt{\frac{1}{([n]_q+\beta)}}$. □

Let $Lip_S(\vartheta)$ denotes the usual Lipschitz class for $f\in C[0,\infty)$, $S>0$, $0<\vartheta\leq 1$ and defined by

$$Lip_S(\vartheta)=\left\{f :| f(u_1)-f(u_2) |\leq S | u_1-u_2 |^{\vartheta},\ u_1,u_2\in[0,\infty)\right\}. \tag{4.15}$$

The following theorem gives the rate of convergence, in terms of the elements of the usual Lipschitz class $Lip_S(\vartheta)$ of the operators $K_{n,q}^{\alpha,\beta}(f;y)$ defined in (2.10).

Theorem 4.5. *Let the operators* $K_{n,q}^{\alpha,\beta}(f;y)$ *defined in* (2.10). *Then for each* $f \in Lip_S(\vartheta),\ S > 0$ *and* $0 < \vartheta \leq 1$ *satisfying* (4.15), *we have*

$$| K_{n,q}^{\alpha,\beta}(f;y) - f(y) | \leq S\left(\lambda_n(y)\right)^{\frac{\vartheta}{2}}$$

where, $\lambda_n(y) = K_{n,q}^{\alpha,\beta}((t-y)^2;y)$ *and* $t = \frac{[n]_q t+\alpha}{[n]_q+\beta}$.

Proof. For proving the above theorem we need (4.15) and using the Hölder's inequality, we have

$$\begin{aligned} | K_{n,q}^{\alpha,\beta}(f;y) - f(y) | &\leq | K_{n,q}^{\alpha,\beta}(f(t)-f(y);y) | \\ &\leq K_{n,q}^{\alpha,\beta}\left(| f(t)-f(y) |;y\right) \\ &\leq SK_{n,q}^{\alpha,\beta}\left(| t-y |^{\vartheta};y\right). \end{aligned}$$

Further,

$$\begin{aligned} &| K_{n,q}^{\alpha,\beta}(f;y) - f(y) | \\ &\leq S\frac{[n]_q}{E_{\mu,q}([n]_q y)}\sum_{k=0}^{\infty}\frac{([n]_q y)^k}{\gamma_{\mu,q}(k)}q^{\frac{k(k-1)}{2}}\int_{\frac{[k+2\mu\theta_k]_q}{q^{k-2}[n]_q}}^{\frac{[k+1+2\mu\theta_k]_q-1}{q^{k-1}[n]_q}+\frac{1}{[n]_q}} | t-y |^{\vartheta}\, d_q(t) \\ &\leq S\frac{[n]_q}{E_{\mu,q}([n]_q y)}\sum_{k=0}^{\infty}\left(\frac{([n]_q y)^k q^{\frac{k(k-1)}{2}}}{\gamma_{\mu,q}(k)}\right)^{\frac{2-\vartheta}{2}} \\ &\times \left(\frac{([n]_q y)^k q^{\frac{k(k-1)}{2}}}{\gamma_{\mu,q}(k)}\right)^{\frac{\vartheta}{2}}\int_{\frac{[k+2\mu\theta_k]_q}{q^{k-2}[n]_q}}^{\frac{[k+1+2\mu\theta_k]_q-1}{q^{k-1}[n]_q}+\frac{1}{[n]_q}} | t-y |^{\vartheta}\, d_q(t) \\ &\leq S\left(\frac{[n]_q}{(E_{\mu,q}([n]_q y))}\sum_{k=0}^{\infty}\frac{([n]_q y)^k q^{\frac{k(k-1)}{2}}}{\gamma_{\mu,q}(k)}\int_{\frac{[k+2\mu\theta_k]_q}{q^{k-2}[n]_q}}^{\frac{[k+1+2\mu\theta_k]_q-1}{q^{k-1}[n]_q}+\frac{1}{[n]_q}} d_q(t)\right)^{\frac{2-\vartheta}{2}} \\ &\times \left(\frac{[n]_q}{(E_{\mu,q}([n]_q y))}\sum_{k=0}^{\infty}\frac{([n]_q y)^k q^{\frac{k(k-1)}{2}}}{\gamma_{\mu,q}(k)}\int_{\frac{[k+2\mu\theta_k]_q}{q^{k-2}[n]_q}}^{\frac{[k+1+2\mu\theta_k]_q-1}{q^{k-1}[n]_q}+\frac{1}{[n]_q}} | t-y |^{2}\, d_q(t)\right)^{\frac{\vartheta}{2}} \\ &= S\left(K_{n,q}^{\alpha,\beta}((t-y)^2;y)\right)^{\frac{\vartheta}{2}}. \end{aligned}$$

□

5. Conclusion

In this paper, we introduce the dunkl generalization of Stancu type q-Szász-Mirakjan-Kantorovich positive linear operator of the exponential functions. Also, with the help of well-known Korovkin's theorem we studied some approximation properties and also determined the rate of convergence for the same operators.

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DEPARTMENT OF MATHEMATICS
ALIGARH MUSLIM UNIVERSITY
202002 ALIGARH, INDIA
Email address: mursaleenm@gmail.com
Email address: ahasan.amu@gmail.com