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Dedicated to Professor Yeol Je Cho on the occasion of his retirement

Subgradient algorithm for split hierarchical optimization problems

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ABSTRACT. In this paper we emphasize a split type problem of some integrating ideas of the split feasibility problem and the hierarchical optimization problem. Working on real Hilbert spaces, we propose a subgradient algorithm for approximating a solution of the introduced problem. We discuss its convergence results and present a numerical example.

1. INTRODUCTION

The classical theory of nonsmooth convex optimization deals with

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in C, \end{array}$$

where $f : \mathcal{H} \to \mathbb{R}$ is a convex lower semicontinuous function, and C is a nonempty closed and convex subset of a real Hilbert space $\mathcal{H}$. The nonsmooth convex phenomena occur frequently in many practical situations, for example, optimal control problems, economic modelings, computational chemistry and biology, and data analysis, see the book of Bagirov et al. [3] for some more applications and recent developments.

Among the earliest methods utilized for solving nonsmooth convex optimization problems were the *projected subgradient method* which is a method that generates an iterative sequence as

$$x_{k+1} := \text{proj}_C(x_k - \alpha_k g_k), \quad \forall k \geq 1,$$

where g_k is any subgradient of f at x_k, proj_C is the metric projection onto C, and $\{\alpha_k\}_{k\in\mathbb{N}}$ is a positive step size. One of the main features of this method is to require a single subgradient at each iteration, rather than the entire subdifferential. However, the method itself seems to have a weakness, of course, the computation of the metric projection onto complicated constraints might be hard to be implemented. This is because it may not have a closed form expression. This trouble can be overcome by transforming these complex constraints to a fixed point set of an appropriated operator $T : \mathcal{H} \to \mathcal{H}$ such that $\text{Fix}(T) = C$ and slightly modifying the projected subgradient method to be

$$x_{k+1} := T(x_k - \alpha_k g_k), \quad \forall k \geq 1. \tag{1.1}$$

This motivated us to consider the problem

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in \text{Fix}(T), \end{array}$$

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where $T : \mathcal{H} \to \mathcal{H}$ is a nonlinear operator, and we call the problem of this type by a *hierarchical optimization problem*. Note that this hierarchical problems can be applied to signal processing [11], power-control [12], and network bandwidth allocation [13] problems.

Let us come now to the split type problem of finding a feasible decision over a feasible region whose its image forms a feasible decision in another feasible region. Actually, in 1994, Censor and Elfving [6] introduced the notion of split feasibility problem, let $\mathcal{H}_1, \mathcal{H}_2$ be real Hilbert spaces, $C \subset \mathcal{H}_1, Q \subset \mathcal{H}_2$ be nonempty closed convex subsets, and $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator with its adjoint $A^* : \mathcal{H}_2 \to \mathcal{H}_1$. The celebrated *split feasibility problem* (**SFP**) is to find a point

$$x^* \in C \text{ such that } Ax^* \in Q.$$

For solving **SFP**, Byrne [4] offered the so-called CQ-procedure, which is defined by

$$x_{k+1} = \mathrm{proj}_C(x_k + \gamma A^*(\mathrm{proj}_Q - I)Ax_k), \quad \forall k \geq 1, \tag{1.2}$$

where an initial $x_1 \in \mathcal{H}_1$ is given, γ is a chosen positive constant, proj_C and proj_Q are the metric projections onto C and Q, respectively. There are several works related to the **SFP** in more general aspects, for example [9, 14, 15], but we will focus here the so-called *split fixed point problem* (**SFPP**): let $T : \mathcal{H}_1 \to \mathcal{H}_1, S : \mathcal{H}_2 \to \mathcal{H}_2$ be nonlinear operators having fixed points, Censor and Segal [8] presented the problem of finding a point

$$x^* \in \mathrm{Fix}(T) \text{ such that } Ax^* \in \mathrm{Fix}(S).$$

They also presented a method for solving this problem constructed by replacing $T := \mathrm{proj}_C$ and $S := \mathrm{proj}_Q$ in (1.2).

Inspired by the above convincing, in this work, we will consider a split type problem of some integrating ideas of the split feasibility problem and the hierarchical optimization problem. Working on real Hilbert settings, we propose the iterative method for approximation a solution of the introduced problem, and subsequently discuss its convergences.

2. Mathematical preliminaries and problem formulation

2.1. **Mathematical preliminaries.** Now, we summarize some notations, definitions and their properties, which will be used later. Let $\mathcal{H}$ be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$. The strong convergence and weak convergence of a sequence $\{x_k\}_{k\in\mathbb{N}}$ to $x \in \mathcal{H}$ are denoted by $x_k \to x$ and $x_k \rightharpoonup x$, respectively.

Let $T : \mathcal{H} \to \mathcal{H}$ be an operator. We denote the set of all fixed points of T by $\mathrm{Fix}(T)$, that is, $\mathrm{Fix}(T) := \{x \in \mathcal{H} : x = Tx\}$, and denote the identity operator on a Hilbert space by I. An operator T having a fixed point is said to be *quasi-nonexpansive* if $\|Tx - z\| \leq \|x - z\|$, for all $x \in \mathcal{H}, z \in \mathrm{Fix}(T)$. It should be note that the set of fixed points of a quasi-nonexpansive operator is a closed and convex set, see [5, Proposition 2.1.21].

An operator T with a nonempty fixed point is called *cutter* if $\langle x - Tx, z - Tx \rangle \leq 0$, for all $x \in \mathcal{H}$ and all $z \in \mathrm{Fix}(T)$. Note that if an operator T is cutter, then it is a quasi-nonexpansive operator. It is worth noticing that an operator T which has a nonempty fixed point set is cutter if and only if the inequality $\|Tx - z\|^2 + \|Tx - x\|^2 \leq \|x - z\|^2$ holds for all $x \in \mathcal{H}$ and for all $z \in \mathrm{Fix}(T)$.

We say that an operator T satisfies the *demiclosed principle* if whenever the sequence $\{x_k\}_{k\in\mathbb{N}} \subseteq \mathcal{H}$ converges weakly to an element $x \in \mathcal{H}$ and the sequence $\{Tx_k - x_k\}_{k\in\mathbb{N}}$ converges strongly to 0, then x is a fixed point of the operator T.

For any bounded linear operator A from a real Hilbert space $\mathcal{H}_1$ into a real Hilbert space $\mathcal{H}_2$, we denote its adjoint by A^*. We denote the range of A by $\mathrm{Ran}(A)$. For a subset $D \subset \mathcal{H}_2$, we denote the inverse image of D under A by $A^{-1}(D)$.

Let $f : \mathcal{H} \to \mathbb{R}$ and $\bar{x} \in \mathcal{H}$. An element $x^* \in \mathcal{H}$ satisfies the inequality

$$\langle x^*, x - \bar{x}\rangle + f(\bar{x}) \leq f(x), \quad \text{for all } x \in \mathcal{H},$$

is called a *subgradient* of f at $\bar{x}$, and the set of all such subgradient is called the *subdifferential* of f at $\bar{x}$; denoted by $\partial f(\bar{x})$. It is well known that if $f : \mathcal{H} \to \mathbb{R}$ is convex and lower semicontinuous, then $\partial f(\bar{x}) \neq \emptyset$, see [16, Theorem 2.4.4]. For a function $f : \mathcal{H} \to \mathbb{R}$ and a nonempty subset $C \subset \mathcal{H}$, we denote the solution set of optimization problem by

$$\arg\min_{x\in C} f(x).$$

Let C be a nonempty subset of $\mathcal{H}$. We say that a sequence $\{x_k\}_{k\in\mathbb{N}} \subset \mathcal{H}$ is *quasi-Fejér monotone* relative to C, if for all $c \in C$ there exists a sequence $\{\delta_k\}_{k\in\mathbb{N}} \subset [0, +\infty)$ such that $\sum_{k\in\mathbb{N}} \delta_k < +\infty$ and

$$\|x_{k+1} - c\|^2 \leq \|x_k - c\|^2 + \delta_k, \quad \forall k \geq 1.$$

The following proposition provides some essential properties of a quasi-Fejér monotone sequence, for further information the readers may consult [10].

Proposition 2.1. [10] *Let $\mathcal{H}$ be a real Hilbert space and $\{x_k\}_{k\in\mathbb{N}} \subset \mathcal{H}$ be a quasi-Fejér monotone sequence relative to a nonempty subset $C \subset \mathcal{H}$. Then,*

(i) $\lim_{k\to+\infty} \|x_k - c\|$ *exists for all $c \in C$.*
(ii) *If all weak cluster points of $\{x_k\}_{k\in\mathbb{N}}$ lies in C, then $\{x_k\}_{k\in\mathbb{N}}$ converges weakly to a point in C.*

In order to show the convergence results, we also need the following fact.

Proposition 2.2. [2] *Let $\{a_k\}_{k\in\mathbb{N}}$, $\{b_k\}_{k\in\mathbb{N}}$ and $\{c_k\}_{k\in\mathbb{N}}$ be real sequences. Assume that $\{a_k\}_{k\in\mathbb{N}}$ is bounded from below, $\{b_k\}_{k\in\mathbb{N}}$ is nonnegative. If there holds $a_{k+1} + b_k \leq a_k + c_k, \forall k \geq 1$ and $\sum_{k\in\mathbb{N}} c_k < +\infty$, then $\lim_{k\to+\infty} a_k$ exists and $\sum_{k\in\mathbb{N}} b_k < +\infty$.*

2.2. **Problem Formulation.** In this paper we are interested in the following problem.

Problem 2.1 (SHOP). *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces, $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator, $f : \mathcal{H}_1 \to \mathbb{R}$, $T : \mathcal{H}_1 \to \mathcal{H}_1$ be such that $\mathrm{Fix}(T) \neq \emptyset$, and $g : \mathcal{H}_2 \to \mathbb{R}$, $S : \mathcal{H}_2 \to \mathcal{H}_2$ be such that $A^{-1}(\mathrm{Fix}(S)) \neq \emptyset$. The split hierarchical optimization problem (in short,* **SHOP***) is to find*

$$x^* \in \arg\min_{x\in\mathrm{Fix}(T)} f(x),$$

and such that

$$Ax^* \in \arg\min_{y\in\mathrm{Ran}(A)\cap\mathrm{Fix}(S)} g(y).$$

Here, we will denote the solution set of **SHOP** by Γ, and the intersection $\mathrm{Fix}(T) \cap A^{-1}(\mathrm{Fix}(S))$ by Ω.

Next, we present a problem which is related to **SHOP**.

Example 2.1. Ansari *et al.* [1] also investigated **SHOP** in the case when the considered functions f and g are convex continuously differentiable, but in view of the following split hierarchical variational inequality problem: find

$$x^* \in \mathrm{Fix}(T) \text{ such that } \langle \nabla f(x^*), x - x^*\rangle \geq 0, \quad \text{for all } x \in \mathrm{Fix}(T),$$

and such that the point

$$Ax^* \in \mathrm{Fix}(S) \text{ such that } \langle \nabla g(Ax^*), y - Ax^*\rangle \geq 0, \quad \text{for all } y \in \mathrm{Ran}(A) \cap \mathrm{Fix}(S),$$

where ∇f and ∇g denote the gradient of f and g, respectively. Note that the split variational inequality problem was considered by Censor, Gibali and Reich [7] when the constraints are simple.

3. A Subgradient-Splitting algorithm and its convergence results

This section is dedicated to the formulation of a subgradient algorithm, which unifies the ideas of the conventional subgradient method (1.1) and the CQ-algorithm (1.2). Also, we give some corresponding convergence results.

Algorithm 3.1. ***(Subgradient-Splitting Method)***
Initialization: *Choose the positive sequences* $\{\alpha_k\}_{k\in\mathbb{N}}$ *and* $\{\gamma_k\}_{k\in\mathbb{N}}$ *and take arbitrary* $x_1 \in \mathcal{H}_1$.
Step 1: *For a given current iterate* $x_k \in \mathcal{H}_1$ $(\forall k \geq 1)$*, define* $z_k \in \mathcal{H}_2$ $(\forall k \geq 1)$ *by*

$$z_k := SAx_k - \alpha_k d_k, \qquad \text{where } d_k \in \partial g(SAx_k).$$

Step 2: *Evaluate* $y_k \in \mathcal{H}_1$ $(\forall k \geq 1)$ *as*

$$y_k := x_k + \gamma_k A^*(z_k - Ax_k).$$

Step 3: *Define* $x_{k+1} \in \mathcal{H}_1$ $(\forall k \geq 1)$ *by*

$$x_{k+1} := Ty_k - \alpha_k c_k, \qquad \text{where } c_k \in \partial f(Ty_k).$$

Update $k := k + 1$ *and go to* **Step 1**.

In order to prove the convergent results, the following assumption will be assumed.

Assumption 3.2. *The following statements hold:*

(I) $T : \mathcal{H}_1 \to \mathcal{H}_1$, $S : \mathcal{H}_2 \to \mathcal{H}_2$ *are cutter operators with nonempty fixed point sets, and both satisfy the demiclosed principle;*
(II) $f : \mathcal{H}_1 \to \mathbb{R}$, $g : \mathcal{H}_2 \to \mathbb{R}$ *are convex lower semicontinuous functions.*

Remark 3.1. Since the co-domain of the considered functions is $\mathbb{R}$, in this situation, both f and g are subdifferentiable. This means Assumption 3.2 (II) guarantees that $\partial f(x) \neq \emptyset$ and $\partial g(y) \neq \emptyset$ for every $x \in \mathcal{H}_1$ and $y \in \mathcal{H}_2$. Subsequently, these assure the well-definedness of Algorithm 3.1.

We start with an important key inequality.

Lemma 3.1. *Suppose that* Ω *is a nonempty set. Then, the following statement holds*

$$\begin{aligned} \|x_{k+1} - q\|^2 \leq\ & \|x_k - q\|^2 - \gamma_k(2 - \gamma_k\|A\|^2)\|z_k - Ax_k\|^2 + 2\alpha_k\gamma_k\|d_k\|\|z_k - Ax_k\| \\ & + \alpha_k^2\|c_k\|^2 + 2\alpha_k\gamma_k\left(g(Aq) - g(SAx_k)\right) + 2\alpha_k\left(f(q) - f(Ty_k)\right), \end{aligned} \tag{3.3}$$

for all $k \geq 1$ *and* $q \in \Omega$

Proof. For each $x \in \mathcal{H}_1$ and $k \geq 1$, we note that

$$\|y_k - x\|^2 \leq \|x_k - x\|^2 + \gamma_k^2\|A\|^2\|z_k - Ax_k\|^2 + 2\gamma_k\langle Ax_k - Ax, z_k - Ax_k\rangle. \tag{3.4}$$

Now, let $q \in \Omega$ be given. We note that, for every $k \geq 1$

$$\begin{aligned} \langle Ax_k - Aq, z_k - Ax_k\rangle &= \langle Ax_k - z_k, z_k - Ax_k\rangle + \langle z_k - Aq, z_k - Ax_k\rangle \\ &\leq -\|z_k - Ax_k\|^2 + \alpha_k\left(g(Aq) - g(SAx_k)\right) + \alpha_k\langle d_k, Ax_k - z_k\rangle, \end{aligned} \tag{3.5}$$

which holds by the fact that S is a cutter operator and $d_k \in \partial g(SAx_k)$. Replacing x by q in (3.4), and using (3.5), we have for every $k \geq 1$

$$\begin{aligned} \|y_k - q\|^2 \leq\ & \|x_k - q\|^2 - \gamma_k(2 - \gamma_k\|A\|^2)\|z_k - Ax_k\|^2 \\ & + 2\alpha_k\gamma_k\langle d_k, Ax_k - z_k\rangle + 2\alpha_k\gamma_k\left(g(Aq) - g(SAx_k)\right). \end{aligned} \tag{3.6}$$

On the other hand, by the quasi-nonexpansiveness of T, $c_k \in \partial f(Ty_k)$ and (3.6), we have

$$\begin{aligned} \|x_{k+1} - q\|^2 &\leq \|y_k - q\|^2 + \alpha_k^2\|c_k\|^2 - 2\alpha_k\langle c_k, Ty_k - q\rangle \\ &\leq \|x_k - q\|^2 - \gamma_k(2 - \gamma_k\|A\|^2)\|z_k - Ax_k\|^2 + 2\alpha_k\gamma_k\|d_k\|\|Ax_k - z_k\| \\ &\quad + \alpha_k^2\|c_k\|^2 + 2\alpha_k\left(f(q) - f(Ty_k)\right) + 2\alpha_k\gamma_k\left(g(Aq) - g(SAx_k)\right), \end{aligned}$$

for all $k \geq 1$. This completes the proof. □

In order to guarantee the convergence of the generated sequence in Algorithm 3.1, we propose the following assumptions.

Assumption 3.3. *For every bounded subsets $B \subset \mathcal{H}_1$ and $C \subset \mathcal{H}_2$, we have $\bigcup_{x\in B} \partial f(x)$ and $\bigcup_{y\in C} \partial g(y)$ are bounded sets.*

Remark 3.2. If we assume that the objective functions f and g are continuous instead of the lower semicontinuity, as we proposed in Assumption 3.2 (II), then Assumption 3.3 is automatically satisfied, see [16, Theorem 2.4.13]. In particular, if the whole spaces are finite dimensional, Assumption 3.3 is also satisfied because any convex function is continuous in this setting.

Assumption 3.4. *The following inclusion holds:*

$$\Gamma \subset \{z \in \Omega : f(z) \leq f(Tx) \text{ and } g(Az) \leq g(SAx), \forall x \in \mathcal{H}_1\}.$$

Remark 3.3. Assumption 3.4 relates the minimality of a function value of a point in Γ to function values of images of T and S. Let $C \subset \mathcal{H}_1$ and $Q \subset \mathcal{H}_2$ be given. We notice that if $Q \subset \mathrm{Ran}(A)$ and we set $T := \mathrm{proj}_C$ and $S := \mathrm{proj}_Q$, then Assumption 3.4 is automatically satisfied.

Condition 3.5. *The sequences $\{\gamma_k\}_{k\in\mathbb{N}}$ and $\{\alpha_k\}_{k\in\mathbb{N}}$ are satisfying*

(C-1) $0 < \underline{\gamma} := \inf_{k\in\mathbb{N}} \gamma_k \leq \bar{\gamma} := \sup_{k\in\mathbb{N}} \gamma_k < \frac{1}{\|A\|^2}$.

(C-2) $\sum_{k\in\mathbb{N}} \alpha_k < +\infty$.

Remark 3.4.

(i) Note that Condition (C-1) implies $\sum_{k\in\mathbb{N}} \gamma_k = +\infty$ and the inequalities $0 < \underline{\gamma}(2 - \bar{\gamma}\|A\|^2) \leq \gamma_k(2 - \gamma_k\|A\|^2)$ and $1 \leq 2 - \gamma_k\|A\|^2$ hold for all $k \geq 1$.

(ii) If the generated sequence $\{x_k\}_{k\in\mathbb{N}}$ is bounded and $\{\alpha_k\}_{k\in\mathbb{N}}$ is bounded from above, then by using Assumptions 3.3 and Condition (C-1), we can obtain the boundedness of the sequences $\{c_k\}_{k\in\mathbb{N}}$ and $\{d_k\}_{k\in\mathbb{N}}$.

Next, we give some important convergence behavior of the generated sequences.

Lemma 3.2. *Suppose that $\Gamma \neq \emptyset$ and Assumptions 3.2, 3.3, and 3.4 and Condition 3.5 hold. If any sequence $\{x_k\}_{k\in\mathbb{N}}$ generated by Algorithm 3.1 is bounded, then*

(i) *$\{x_k\}_{k\in\mathbb{N}}$ is quasi-Fejér monotone with respect to Γ, and $\lim_{k\to+\infty} \|x_k - q\|$ exists for all $q \in \Gamma$.*

(ii) $\lim\limits_{k\to+\infty} \|z_k - Ax_k\| = 0$.

(iii) $\lim\limits_{k\to+\infty} \|x_k - y_k\| = 0$.

(iv) $\lim\limits_{k\to+\infty} \|SAx_k - Ax_k\| = 0$.

(v) $\lim\limits_{k\to+\infty} \|Ty_k - y_k\| = 0$.

Proof. (i) Let $q \in \Gamma$ be arbitrary. From Lemma 3.1, we have

$$\begin{aligned}\|x_{k+1} - q\|^2 \leq\ & \|x_k - q\|^2 + 2\alpha_k\gamma_k\|d_k\|\|z_k - Ax_k\| + \alpha_k^2\|c_k\|^2 \\ & +2\alpha_k\gamma_k\left(g(Aq) - g(SAx_k)\right) + 2\alpha_k\left(f(q) - f(Ty_k)\right),\end{aligned}$$

for all $k \geq 1$. Subsequently, by using Assumption 3.4, we have

$$\|x_{k+1} - q\|^2 \leq \|x_k - q\|^2 + 2\alpha_k\overline{\gamma}\|d_k\|\|z_k - Ax_k\| + \alpha_k^2\|c_k\|^2, \quad \forall k \geq 1.$$

This implies that $\{x_k\}_{k\in\mathbb{N}}$ is a quasi-Fejér monotone sequence with respect to Γ. Subsequently, the second one is immediately followed by Proposition 2.1(i).

(ii) Let $q \in \Gamma$ be given. Using Lemma 3.1, Assumption 3.4 and Remark 3.4(i), we see that for all $k \geq 1$

$$\|x_{k+1} - q\|^2 + \gamma_k\|z_k - Ax_k\|^2 \leq \|x_k - q\|^2 + \alpha_k^2\|c_k\|^2 + 2\overline{\gamma}\alpha_k\|d_k\|\|z_k - Ax_k\|,$$

and then

$$\begin{aligned}(\|x_{k+1} - q\|^2 - \|z_{k+1} - Ax_{k+1}\|) + \gamma_k\|z_k - Ax_k\|^2 \\ \leq\ & (\|x_k - q\|^2 - \|z_k - Ax_k\|) + \alpha_k^2\|c_k\|^2 + 2\overline{\gamma}\alpha_k\|d_k\|\|z_k - Ax_k\| \\ & +\|z_k - Ax_k\| - \|z_{k+1} - Ax_{k+1}\|, \quad \forall k \geq 1.\end{aligned}$$

Since $\{\|z_k - Ax_k\|\}_{k\in\mathbb{N}}$ is bounded, say by $M > 0$, we have $\|x_k - q\|^2 - \|z_k - Ax_k\| \geq -M, \forall k \geq 1$, which means that the sequence $\{\|x_k - q\|^2 - \|z_k - Ax_k\|\}_{k\in\mathbb{N}}$ is bounded from below. Further, we also note that $\sum_{k\in\mathbb{N}} [\|z_k - Ax_k\| - \|z_{k+1} - Ax_{k+1}\|] < +\infty$. These together with Proposition 2.2 imply that $\lim_{k\to+\infty}(\|x_k - q\|^2 - \|z_k - Ax_k\|)$ exists and $\sum_{k\in\mathbb{N}} \gamma_k\|z_k - Ax_k\|^2 < +\infty$. Since we know that $\lim_{k\to+\infty} \|x_k - q\|$ exists, we can also obtain that $\lim_{k\to+\infty} \|z_k - Ax_k\|$ exists. Using this one together with $\sum_{k\in\mathbb{N}} \gamma_k = +\infty$, we get $\lim_{k\to+\infty} \|z_k - Ax_k\| = 0$.

(iii) It is an immediate consequence of the definition of y_k and (ii).

(iv) Observes that $\|SAx_k - Ax_k\| \leq \|z_k - Ax_k\| + \alpha_k\|d_k\|$ for all $k \geq 1$. Using this inequality together with (ii) and Condition (C-2), we reach the required result.

(v) Let $q \in \Gamma$. We note that, for all $k \geq 1$

$$\|Ty_k - q\|^2 = \|x_{k+1} - q\|^2 + \alpha_k^2\|c_k\|^2 + 2\alpha_k\langle c_k, x_{k+1} - q\rangle.$$

Using this one together with the cutter of T and (3.6), we have

$$\begin{aligned}\|Ty_k - y_k\|^2 &\leq \|y_k - q\|^2 - \|Ty_k - q\|^2 \\ &\leq \|y_k - q\|^2 - \|x_{k+1} - q\|^2 + \alpha_k^2\|c_k\|^2 + 2\alpha_k\|c_k\|\|x_{k+1} - q\| \\ &\leq \|x_k - q\|^2 - \|x_{k+1} - q\|^2 + 2\overline{\gamma}\alpha_k\|d_k\|\|z_k - Ax_k\| \\ &\quad +2\alpha_k\|c_k\|\|x_{k+1} - q\|, \quad \forall k \geq 1,\end{aligned}$$

and hence $\lim_{k\to+\infty} \|Ty_k - y_k\| = 0$, as required. □

Building on all above materials, it is now possible to obtain a convergence result for the sequence generated by the subgradient-splitting algorithm as the follow theorem.

Theorem 3.1. *Suppose that $\Gamma \neq \emptyset$ and Assumptions 3.2, 3.3 and 3.4 hold and Condition 3.5 are satisfied. If any sequence $\{x_k\}_{k\in\mathbb{N}}$ generated by Algorithm 3.1 is bounded and it is satisfying $\lim_{k\to+\infty} \dfrac{\|x_{k+1} - x_k\|}{\alpha_k} = 0$, then $\{x_k\}_{k\in\mathbb{N}}$ converges weakly to an element in Γ.*

Proof. In order to get the conclusion, in view of Theorem 2.1 and Lemma 3.2(i), it suffices to prove that every weak cluster point of $\{x_k\}_{k\in\mathbb{N}}$ lies in Γ. Now, let $p \in \mathcal{H}_1$ be a weak cluster point of $\{x_k\}_{k\in\mathbb{N}}$. Then, there exists a subsequence $\{x_{k_j}\}_{j\in\mathbb{N}}$ of $\{x_k\}_{k\in\mathbb{N}}$ such that $x_{k_j} \rightharpoonup p$ as $j \to +\infty$. Subsequently, by using Lemma 3.2 (iii), there exists a corresponding subsequence $\{y_{k_j}\}_{j\in\mathbb{N}}$ of $\{y_k\}_{k\in\mathbb{N}}$ such that $y_{k_j} \rightharpoonup p$ as $j \to +\infty$. Thus, by Lemma 3.2 (v) and the demiclosedness of T, we have $p \in \mathrm{Fix}(T)$. Note that, since $Ax_{k_j} \rightharpoonup Ap \in \mathcal{H}_2$ as $j \to +\infty$ together with Lemma 3.2 (iv) and the demiclosed principle of S, we also obtain $Ap \in \mathrm{Fix}(S)$. These yield that $p \in \Omega$. We now show that $p \in \Gamma$.

Let $q \in \Gamma$ be an arbitrary. In view of Lemma 3.1, we have

$$\begin{aligned}\|x_{k+1} - q\|^2 \leq\ & \|x_k - q\|^2 + \alpha_k^2\|c_k\|^2 + 2\alpha_k\gamma_k\|d_k\|\|z_k - Ax_k\| \\ & +2\alpha_k\left(f(q) - f(Ty_k)\right) + 2\alpha_k\gamma_k\left(g(Aq) - g(SAx_k)\right), \forall k \geq 1.\end{aligned}$$

Note that, by Assumption 3.4, both $f(Ty_k) - f(q)$ and $g(SAx_k) - g(Aq)$ are nonnegative, for all $k \geq 1$. This implies that, for every $q \in \Gamma$ and $j \geq 1$

$$\begin{aligned} 2\left(f(Ty_{k_j}) - f(q)\right) + 2\underline{\gamma}\left(g(SAx_{k_j}) - g(Aq)\right) &\leq \frac{\|x_{k_j} - q\|^2 - \|x_{k_j+1} - q\|^2}{\alpha_{k_j}} \\ &\quad + \alpha_{k_j}\|c_{k_j}\|^2 + 2\overline{\gamma}\|d_{k_j}\|\|z_{k_j} - Ax_{k_j}\| \\ &\leq \frac{\|x_{k_j+1} - x_{k_j}\|}{\alpha_{k_j}}\left(\|x_{k_j} - q\| + \|x_{k_j+1} - q\|\right) \\ &\quad + \alpha_{k_j}\|c_{k_j}\|^2 + 2\overline{\gamma}\|d_{k_j}\|\|z_{k_j} - Ax_{k_j}\|. \end{aligned}$$

By approaching the inferior limit as $j \to +\infty$, we have

$$\liminf_{j\to+\infty} \left[\left(f(Ty_{k_j}) - f(q)\right) + \underline{\gamma}\left(g(SAx_{k_j}) - g(Aq)\right)\right] \leq 0,$$

and then

$$\liminf_{j\to+\infty} f(Ty_{k_j}) = f(q) \quad \text{and} \quad \liminf_{j\to+\infty} g(SAx_{k_j}) = g(Aq).$$

Note that, by Lemma 3.2 (iv)-(v), we have $Ty_{k_j} \rightharpoonup p$ and $SAx_{k_j} \rightharpoonup Ap$ as $j \to +\infty$. Invoking the weak lower semicontinuity of f and g, we obtain that

$$f(p) \leq \liminf_{j\to+\infty} f(Ty_{k_j}) = f(q) \quad \text{and} \quad g(Ap) \leq \liminf_{j\to+\infty} g(SAx_{k_j}) = g(Aq),$$

for every $q \in \Gamma$. This implies $p \in \Gamma$, the proof is completed. □

Remark 3.5. (i) After carfully considering the lines proof of Lemma 3.2 (ii)-(v), we can see that if we assume that the sequence $\{x_k\}_{k\in\mathbb{N}}$ is bounded and quasi-Fejér monotone with respect to Γ and $\lim_{k\to+\infty} \alpha_k = 0$, instead of all assumptions which were proposed in Lemma 3.2, then we can also show that Lemma 3.2 (ii)-(v) are still true.

(ii) If we can choose a simple bounded closed convex set X such that $X \supset \mathrm{Fix}(T)$, e.g., a closed ball with a large enough radius, and compute the iterate $x_{k+1} \in \mathcal{H}_1$ by

$$x_{k+1} := \mathrm{proj}_X[Ty_k - \alpha_k c_k], \qquad \text{where } c_k \in \partial f(Ty_k), \qquad \forall k \geq 1,$$

where proj_X is the metric projection onto the set X, instead of **Step 3** of Algorithm 3.1. Then, the boundedness of the sequence $\{x_k\}_{k\in\mathbb{N}}$ can be guaranteed and Theorem 3.1 is still true.

4. Numerical example

In this section, we present a numerical example for Algorithm 3.1 with different choices of chosen parameters. We will show and discuss the behavior of the generated sequence which approximating a solution of SHOP.

Let $\mathcal{H}_1 = \mathbb{R}^2, \mathcal{H}_2 = \mathbb{R}^3$ and $A : \mathcal{H}_1 \to \mathcal{H}_2$ be defined by $A(x_1, x_2) := (x_1, -x_1, -x_2)$ for all $(x_1, x_2) \in \mathcal{H}_1$. Note that A is a bounded linear operator with $\|A\| = 1$, and its adjoint $A^* : \mathcal{H}_2 \to \mathcal{H}_1$ is given by $A^*(x_1, x_2, x_3) = (x_1 - x_2, -x_3)$ for all $(x_1, x_2, x_3) \in \mathcal{H}_2$.

Define $f : \mathcal{H}_1 \to \mathbb{R}$ and $g : \mathcal{H}_2 \to \mathbb{R}$ by $f(x_1, x_2) := x_2^2$ for all $(x_1, x_2) \in \mathcal{H}_1$ and $g(x_1, x_2, x_3) := |x_1| + x_2^2 + 2$ for all $(x_1, x_2, x_3) \in \mathcal{H}_2$. We can check that both f and g are convex (continuous) functions. Note that $\nabla f(x_1, x_2) = (0, 2x_2)$ for all $(x_1, x_2) \in \mathcal{H}_1$ and

$$\partial g(x_1, x_2, x_3) = \begin{cases} (1, 2x_2, 0) & \text{if } x_1 > 0, \\ \{(a, 2x_2, 0) : a \in [-1, 1]\} & \text{if } x_1 = 0, \\ (-1, 2x_2, 0) & \text{if } x_1 < 0, \end{cases}$$

Now, let us consider the sets $C_1 := \{(x_1, x_2) \in \mathcal{H}_1 : (x_1 - 1)^2 + (x_2 - 1)^2 \leq 4\}, C_2 := \{(x_1, x_2) \in \mathcal{H}_1 : (x_1 - 1)^2 + (x_2 + 1)^2 \leq 2\}$ and $C := \{(x_1, x_2 \in \mathcal{H}_1 : x_1^2 + x_2^2 \leq 20\}$. In this case, we have $C_1 \cap C_2$ and C are nonempty closed convex sets and $C_1 \cap C_2 \subset C$.

We are focusing on the problem:

Problem 4.2. *Find a point* $x^* \in \mathcal{H}_1$ *such that*

$$x^* \in \arg\min_{x \in C_1 \cap C_2} f(x),$$

and

$$Ax^* \in \arg\min_{y \in \mathrm{Ran}(A)} g(y).$$

Note that the computations of metric projections proj_{C_1} and proj_{C_2} have closed-form expressions, however, the computation of $\mathrm{proj}_{C_1 \cap C_2}$ is not easy. To overcome this difficulty, let us define an operator $T : \mathcal{H}_1 \to \mathcal{H}_1$ by $T := 0.4\mathrm{proj}_{C_1} + 0.6\mathrm{proj}_{C_2}$. It follows that T is a cutter operator satisfying the demiclosed principle with $\mathrm{Fix}(T) = C_1 \cap C_2$. Note that, since C is a compact set, the boundedness of the generated sequence $\{x_k\}_{k \in \mathbb{N}}$ can be ensured. Moreover, we can see that $(0, 0)$ is a solution of the Problem 4.2. And, here we note that $(0, 0)$ is a solution of this problem. We will use the estimate $\|x_k\|$ to show the convergence of the sequence $\{x_k\}_{k \in \mathbb{N}}$ to the solution.

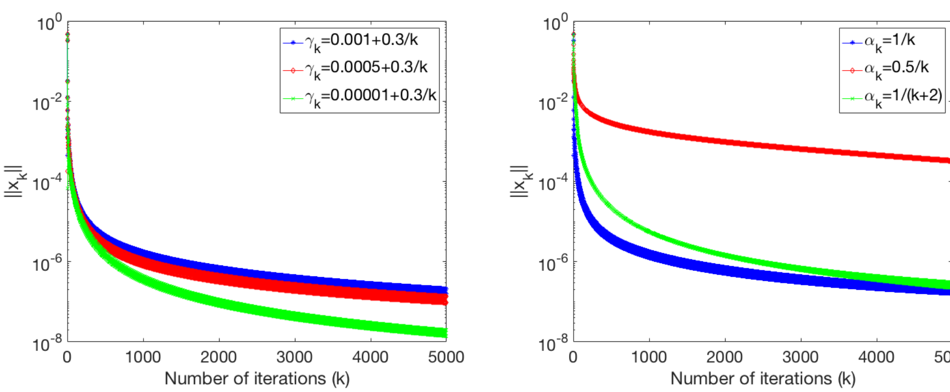

FIGURE 1. Illustration the behavior of $\|x_k\|$ when $\alpha_k = 1/k$ with different γ_k and when $\gamma_k = 0.001 + 0.3/k$ with different α_k, respectively.

In the left of Figure 1 shows the behavior of $\|x_k\|$, with $\alpha_k = 1/k$ where $k = 1, \ldots, 5,000$ and the chosen parameter γ_k is given by $0.001 + 0.3/k, 0.0005 + 0.3/k$ and $0.00001 + 0.3/k$, respectively. In this situation, we observe from Figure 1 that $\|x_k\|$ is closed to 0, when k is increasing for all types of γ_k. Further, we may observe that the parameter $\gamma_k = 0.00001 + 0.3/k$ gives the fastest convergence rate. Also, in the right of Figure 1 shows the behavior of $\|x_k\|$, with $\gamma_k = 0.001 + 0.3/k$ where $k = 1, \ldots, 5,000$ and the chosen parameter α_k is given by $1/k, 0.5/k$ and $1/(k+2)$. Similarly, in this situation, the parameter $\alpha_k = 1/k$ gives the fastest convergence rate.

5. Conclusion

This paper introduced and discussed the split hierarchical (nonsmooth) optimization problem over the fixed point sets of cutter operators. To solve the problem, we proposed an algorithm, which we call it by the subgradient-splitting method. We also considered its convergence results and gives a numerical example.

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