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Dedicated to Professor Yeol Je Cho on the occasion of his retirement

On the Krein-Milman theorem in CAT(κ) spaces

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ABSTRACT. Let $\kappa > 0$ and (X, ρ) be a complete CAT(κ) space whose diameter smaller than $\frac{\pi}{2\sqrt{\kappa}}$. It is shown that if K is a nonempty compact convex subset of X, then K is the closed convex hull of its set of extreme points. This is an extension of the Krein-Milman theorem to the general setting of CAT(κ) spaces.

1. INTRODUCTION AND PRELIMINARIES

One of the fundamental and celebrated results in functional analysis related to extreme points is the Krein-Milman theorem. In [5], the authors proved that every compact convex subset of a locally convex Hausdorff space is the closed convex hull of its set of extreme points. This result was extended to a special class of metric spaces, namely, CAT(0) spaces, by Niculescu [6] in 2007. Notice that Niculescu's result can be applied to CAT(κ) spaces with $\kappa \leq 0$ since any CAT(κ) space is a CAT(κ') space for $\kappa' \geq \kappa$ (see e.g., [1]). However, the result for $\kappa > 0$ is still unknown. In this paper, we extend Niculescu's result to the setting of CAT(κ) spaces with $\kappa > 0$.

Let $(\mathcal{P}, \preceq)$ be a partially ordered set. An element $p_0 \in \mathcal{P}$ is *maximal* in $\mathcal{P}$ if for each $p \in \mathcal{P}$, the following implication holds:

$$p_0 \preceq p \implies p_0 = p.$$

Similarly, an element $q_0 \in \mathcal{P}$ is *minimal* in $\mathcal{P}$ if for each $p \in \mathcal{P}$, the following implication holds:

$$p \preceq q_0 \implies p = q_0.$$

An upper bound (resp. *A lower bound*) of a nonempty subset $\mathcal{Q}$ of $\mathcal{P}$ is an element $p \in \mathcal{P}$ such that $q \preceq p$ (resp. $p \preceq q$) for all $q \in \mathcal{Q}$. A nonempty subset $\mathcal{C}$ of $\mathcal{P}$ is called a *chain* in $\mathcal{P}$ if any two elements p and q in $\mathcal{C}$ are comparable, that is, $p \preceq q$ or $q \preceq p$.

Lemma 1.1. (Zorn) *If $(\mathcal{P}, \preceq)$ is a partially ordered set such that every chain in $\mathcal{P}$ has an upper (resp. lower) bound in $\mathcal{P}$, then $\mathcal{P}$ contains a maximal (resp. minimal) element.*

Let (X, ρ) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ is a function ξ from the closed interval $[0, \rho(x, y)]$ to X such that $\xi(0) = x, \xi(l) = y$, and $\rho(\xi(t), \xi(t')) = |t - t'|$ for all $t, t' \in [0, \rho(x, y)]$. The image of ξ is called a *geodesic segment* joining x and y which is unique, denoted by $[x, y]$. This means that $z \in [x, y]$ if and only if there exists $\alpha \in [0, 1]$ such that

$$\rho(x, z) = (1 - \alpha)\rho(x, y) \text{ and } \rho(y, z) = \alpha\rho(x, y).$$

In this case, we write $z = \alpha x \oplus (1 - \alpha)y$. The space (X, ρ) is said to be a *geodesic space* (resp. *$D-$geodesic space*) if every two points of X (resp. every two points of distance smaller than

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D) are joined by a geodesic path. A subset C of X is said to be *convex* if C includes every geodesic segment joining any two of its points.

Now we introduce the model spaces M_κ^2, for more details on these spaces the reader is referred to [1, 3, 4, 8, 9]. We denote by $\langle \cdot, \cdot \rangle$ the Euclidean scalar product in $\mathbb{R}^3$. By $\mathbb{S}^2$ we denote the unit sphere in $\mathbb{R}^3$, that is the set $\left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\right\}$. The *spherical distance* on $\mathbb{S}^2$ is defined by

$$d_{\mathbb{S}^2}(x, y) := \arccos\langle x, y \rangle \text{ for all } x, y \in \mathbb{S}^2.$$

Definition 1.2. Given $\kappa \geq 0$, we denote by M_κ^2 the following metric spaces:

(i) if $\kappa = 0$ then M_κ^2 is the Euclidean space $\mathbb{E}^2$;

(ii) if $\kappa > 0$ then M_κ^2 is obtained from the spherical space $\mathbb{S}^2$ by multiplying the distance function by $1/\sqrt{\kappa}$.

A *geodesic triangle* $\triangle(x, y, z)$ in a geodesic space (X, ρ) consists of three points x, y, z in X (the *vertices* of $\triangle$) and three geodesic segments between each pair of vertices (the *edges* of $\triangle$). A *comparison triangle* for a geodesic triangle $\triangle(x, y, z)$ in (X, ρ) is a triangle $\overline{\triangle}(\bar{x}, \bar{y}, \bar{z})$ in M_κ^2 such that

$$\rho(x, y) = d_{M_\kappa^2}(\bar{x}, \bar{y}),\ \rho(y, z) = d_{M_\kappa^2}(\bar{y}, \bar{z}),\ \text{and } \rho(z, x) = d_{M_\kappa^2}(\bar{z}, \bar{x}).$$

It is well known that such a comparison triangle exists if $\rho(x, y) + \rho(y, z) + \rho(z, x) < 2D_\kappa$, where $D_\kappa = \pi/\sqrt{\kappa}$ for $\kappa > 0$ and $D_0 = \infty$. Notice also that the comparison triangle is unique up to isometry. A point $\bar{u} \in [\bar{x}, \bar{y}]$ is called a *comparison point* for $u \in [x, y]$ if $\rho(x, u) = d_{M_\kappa^2}(\bar{x}, \bar{u})$.

A metric space (X, ρ) is said to be a *CAT(κ) space* if it is $D_\kappa-$geodesic and for each two points u, v of any geodesic triangle $\triangle(x, y, z)$ in X with $\rho(x, y) + \rho(y, z) + \rho(z, x) < 2D_\kappa$ and for their comparison points $\bar{u}, \bar{v}$ in $\overline{\triangle}(\bar{x}, \bar{y}, \bar{z})$ the *CAT(κ) inequality*

$$\rho(u, v) \leq d_{M_\kappa^2}(\bar{u}, \bar{v}),$$

holds. Notice also that Pre-Hilbert spaces, $\mathbb{R}-$trees, Euclidean buildings are examples of CAT(κ) spaces (see [1, 2]).

Recall that a geodesic space (X, ρ) is said to be *$R-$convex* for $R \in (0, 2]$ ([7]) if for any three points $x, y, z \in X$, we have

$$\rho^2(x, (1-\alpha)y \oplus \alpha z) \leq (1-\alpha)\rho^2(x, y) + \alpha\rho^2(x, z) - \frac{R}{2}\alpha(1-\alpha)\rho^2(y, z). \tag{1.1}$$

The following lemmas will be needed.

Lemma 1.3. ([7]) *Let $\kappa > 0$ and (X, ρ) be a complete CAT(κ) space with* $\operatorname{diam}(X) \leq \dfrac{\pi/2 - \varepsilon}{\sqrt{\kappa}}$ *for some $\varepsilon \in (0, \pi/2)$. Then (X, ρ) is $R-$convex for $R = (\pi - 2\varepsilon)\tan(\varepsilon)$.*

Lemma 1.4. ([1]) *Let $\kappa > 0$ and (X, ρ) be a complete CAT(κ) space with* $\operatorname{diam}(X) \leq \dfrac{\pi/2 - \varepsilon}{\sqrt{\kappa}}$ *for some $\varepsilon \in (0, \pi/2)$. Then*

$$\rho((1-\alpha)x \oplus \alpha y, z) \leq (1-\alpha)\rho(x, z) + \alpha\rho(y, z),$$

for all $x, y, z \in X$ and $\alpha \in [0, 1]$.

Let (X, ρ) be a geodesic space. The *distance* from a point x in X to a subset C of X is defined by

$$\operatorname{dist}(x, C) := \inf\{\rho(x, y) : y \in C\}.$$

The set C is *bounded* if $\operatorname{diam}(C) := \sup\{\rho(x, y) : x, y \in C\} < \infty$.

Definition 1.5. Let $f : C \to \mathbb{R}$ be a function. Then

(i) f is said to be *convex* if $f(\alpha x \oplus (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)$ for all $\alpha \in [0,1]$ and $x, y \in C$;

(ii) f is said to be *strictly convex* if $f(\alpha x \oplus (1-\alpha)y) < \alpha f(x) + (1-\alpha)f(y)$ for all $\alpha \in (0,1)$ and $x, y \in C$ with $x \neq y$.

Let A be a nonempty subset of X. The *closed convex hull* of A is defined by

$$\overline{\operatorname{conv}}(A) := \bigcap \{B \subseteq X : A \subseteq B \text{ and } B \text{ is closed and convex}\}.$$

Let C be a convex subset of X. A subset A of C is called an *extremal subset* if it is nonempty, closed and satisfies the following property: If $x, y \in C$ and $\alpha x \oplus (1-\alpha)y \in A$ for some $\alpha \in (0,1)$, then $x, y \in A$. Notice that if A is an extremal subset of B and B is an extremal subset of C, then A is an extremal subset of C. A point z in C is called an *extreme point* of C if $\{z\}$ is an extremal subset of C. We denote by $\operatorname{Ext}(C)$ the set of all extreme points of C.

Example 1.6. In the Euclidean space $\mathbb{R}^2$, the square $A := \{(x,y) : |x| \leq 1, |y| \leq 1\}$ has four extreme points while the strip $B := \{(x,y) : 0 \leq x \leq 1, y \in \mathbb{R}\}$ does not have an extreme point.

2. Main results

We begin this section by proving a crucial lemma.

Lemma 2.1. *Let $\kappa > 0$ and (X, ρ) be a complete CAT(κ) space with $\operatorname{diam}(X) \leq \dfrac{\pi/2 - \varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. If K is a nonempty compact convex subset of X, then every extremal subset of K has an extreme point.*

Proof. Let $\mathcal{C}$ be the family of all nonempty extremal subset of K. Since $K \in \mathcal{C}$, it follows that $\mathcal{C}$ is nonempty and it can be partially ordered by set inclusion. By Zorn's Lemma, $\mathcal{C}$ has a minimal element, say M. It is enough to show that M consists of exactly one point. Suppose that it contains at least two points, say x_0 and y_0. Let $f : M \to \mathbb{R}$ be defined by

$$f(x) := \rho^2(x_0, x) \text{ for all } x \in M.$$

Since $x_0 \neq y_0$, f is not a constant function. By (1.1), f is strictly convex. Let $M_0 := \{x \in M : f(x) = \sup_{y \in M} f(y)\}$. Since f is continuous and K is compact, M_0 is nonempty. Notice also that it is a closed proper subset of M. Next, we show that M_0 is an extremal subset of M. Let $x', x'' \in M$ and M_0 contains a point $(1-\alpha)x' \oplus \alpha x''$ for some $\alpha \in (0,1)$. By (1.1), we have

$$\begin{aligned}
\sup_{y \in M} f(y) &= f((1-\alpha)x' \oplus \alpha x'') \\
&\leq (1-\alpha)f(x') + \alpha f(x'') - \alpha(1-\alpha)\rho^2(x', x'') \\
&\leq \sup_{y \in M} f(y) - \alpha(1-\alpha)\rho^2(x', x''),
\end{aligned}$$

which implies that $x' = x'' \in M_0$. Thus $M_0 \in \mathcal{C}$ which contradicts to the minimality of M and hence the proof is complete. □

Theorem 2.2. *Let $\kappa > 0$ and (X, ρ) be a complete CAT(κ) space with $\operatorname{diam}(X) \leq \dfrac{\pi/2 - \varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$. If K is a nonempty compact convex subset of X, then $\overline{conv}(Ext(K)) = K$.*

Proof. (This proof is patterned after the proof of Theorem 3.1 in [6]). By Lemma 2.1, $\mathrm{Ext}(K) \neq \emptyset$. Clearly, $\overline{\mathrm{conv}}(\mathrm{Ext}(K)) \subseteq K$. Suppose that $\overline{\mathrm{conv}}(\mathrm{Ext}(K)) \neq K$. Let $g : K \to \mathbb{R}$ be defined by $g(x) := \mathrm{dist}(x, \overline{\mathrm{conv}}(\mathrm{Ext}(K)))$ and let $K_0 := \{x \in K : g(x) = \sup_{y \in K} g(y)\}$. Since g is continuous and K is compact, K_0 is nonempty. Notice also that it is a closed subset of K. Since $\overline{\mathrm{conv}}(\mathrm{Ext}(K)) \neq K$, we get that $\sup\{g(y) : y \in K\} > 0$. By Lemma 1.4 for $x, y \in K$, $\alpha \in [0,1]$ and $z \in \overline{\mathrm{conv}}(\mathrm{Ext}(K))$ we have

$$\rho((1-\alpha)x \oplus \alpha y, z) \leq (1-\alpha)\rho(x,z) + \alpha\rho(y,z),$$

which implies that g is convex. Notice also that K_0 is an extremal subset of K. Again, by Lemma 2.1 there is a point z in $K_0 \cap \mathrm{Ext}(K)$. Thus $0 = g(z) = \sup\{g(y) : y \in K\}$ which is a contradiction. Hence $\overline{\mathrm{conv}}(\mathrm{Ext}(K)) = K$. □

As a consequence of Theorem 2.2, we obtain the following corollary.

Theorem 2.3. ([6, Theorem 1]) *Let (X, ρ) be a complete CAT(0) space and K be a nonempty compact convex subset of X. Then $\overline{conv}(Ext(K)) = K$.*

Proof. It is well known that every convex subset of a CAT(0) space, equipped with the induced metric, is a CAT(0) space (see e.g., [1]). Thus (K, ρ) is a CAT(0) space and hence it is a CAT(κ) space for all $\kappa > 0$. Notice also that it is $R-$convex for $R = 2$. Since K is bounded, we can choose $\varepsilon \in (0, \pi/2)$ and $\kappa > 0$ such that $\mathrm{diam}(K) \leq \dfrac{\pi/2 - \varepsilon}{\sqrt{\kappa}}$. The conclusion follows from Theorem 2.2. □

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