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Dedicated to Professor Yeol Je Cho on the occasion of his retirement

Hybrid Bregman projection methods for fixed point and equilibrium problems

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ABSTRACT. The purpose of this article is to investigate a projection algorithm for solving a fixed point problem of a closed multi-valued Bregman quasi-strict pseudocontraction and an equilibrium problem of a bifunction. Strong convergence of the projection algorithm is obtained without any compact assumption in a reflexive Banach space. As applications, monotone variational inequality problems are considered. Finally, a numerical simulation example is presented for demonstrating the feasibility and convergence of the algorithm proposed in main result.

1. INTRODUCTION

Let *E* be a Banach space and *C* be a nonempty closed convex subset of *E*. N(C) and CB(C) stand for the family of nonempty subsets and nonempty closed bounded subsets of *C*, respectively. Let $H(\cdot, \cdot)$ be the Hausdorff metric on CB(C) defined as $H(A, B) = \max\{\sup_{y \in B} d(y, A), \sup_{x \in A} d(x, B)\}, \forall A, B \in CB(C)$, where $d(a, B) = \inf\{||a - b|| : b \in B\}$ is the distance from point *a* to subset *B*. Let $T : C \to CB(C)$ be a multi-valued mapping. $F(T) := \{p \in C : p = T(p)\}$ represents the fixed point set of *T*.

Lots of problems can be studied via fixed point techniques of multi-valued mappings, such as optimal control, signal processing, image reconstruction, which makes construction of iterative algorithms for approximating fixed points of multi-valued mappings become one of the main concerns of fixed point theory [5, 6, 7, 8, 10]. On the other hand, the "so-called" equilibrium problem with respect to a bifunction $g: C \times C \to \mathbb{R}$ is described as follows: find \tilde{x} such that $g(\tilde{x}, y) \geq 0$, for all $y \in C$. The set of solutions of the equilibrium problem is denoted as EP(g). To solve the equilibrium problem, the following assumptions hold: (A1) g(x, x) = 0 for all $x \in C$; (A2) g is monotone, i.e., $g(x, y) + g(y, x) \leq 0$ for all $x, y \in C$; (A3) for all $x, y, z \in C$, $\limsup_{t \downarrow 0} g(tz + (1 - t)x, y) \leq g(x, y)$; (A4) for all $x \in C$, $g(x, \cdot)$ is convex and lower semi-continuous.

In this paper, our main goal is to address the convergence of iterative algorithms for approximating a common element in the fixed points set of a multi-valued Bregman quasistrict pseudo-contraction and the solutions set of an equilibrium problem in a reflexive Banach space. The results presented in this paper improve some corresponding results announced in [9, 14, 15, 16].

2. PRELIMINARIES

In this section, we collect some preliminaries which are used in the following section. Unless mentioned otherwise, all throughout the paper, E is a real reflexive Banach space

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with the norm $\|\cdot\|$ and E^* is the dual space of E, C is a nonempty, closed, and convex subset of E. $f: E \to (-\infty, +\infty]$ is a proper, convex and lower semi-continuous function. Denote the domain of f by dom f, i.e., dom $f := \{x \in E : f(x) < +\infty\}$. \mathbb{N} and \mathbb{R} are denoted as the sets of positive integers and real numbers, respectively. Let any $x \in$ int dom f and $y \in E$, the right-hand derivative of f at x in the direction of y is defined by

$$f^{\circ}(x,y) = \lim_{t \to 0^+} \frac{f(x+ty) - f(x)}{t}.$$

Definition 2.1. The function f is said to be: (i) Gâteaux differentiable at x if the limit $f^{\circ}(x, y)$ exists for any y; (ii) Gâteaux differentiable if it is Gâteaux differentiable for any $x \in \text{int dom } f$; (iii) Fréchet differentiable at x if the limit $f^{\circ}(x, y)$ is attained uniformly in ||y|| = 1; (iv) uniformly Fréchet differentiable on a subset C of E if the limit $f^{\circ}(x, y)$ is attained uniformly for $x \in C$ and ||y|| = 1.

Remark 2.1. (i) If f is Gâteaux differentiable at x, then $f^{\circ}(x, y)$ coincides with $\nabla f(x)$, the value of the gradient ∇f of f at x; (ii) if a continuous convex function $f \to \mathbb{R}$ is Gâteaux differentiable, ∇f is norm-to-weak^{*} continuous; (iii) if f is Fréchet differentiable, ∇f is norm-to-norm continuous.

Let $x \in \text{int dom} f$, the subdifferential of f at x is the convex set defined by $\partial f(x) = \{x^* \in E^* : f(x) + \langle x^*, y - x \rangle \leq f(y), \forall y \in E\}$. The Fenchel conjugate of f is the function $f^* : E^* \to (-\infty, +\infty]$ defined by $f^*(x^*) = \sup\{\langle x^*, x \rangle - f(x) : x \in E\}$, where $x^* \in E^*$.

Definition 2.2. The function f is called: (i) essentially smooth if ∂f is both locally bounded and single-valued on its domain; (ii) essentially strictly convex if $(\partial f)^{-1}$ is locally bounded on its domain and f is strictly convex on every convex subset of dom ∂f ; (iii) Legendre, if it is both essentially smooth and essentially strictly convex.

Remark 2.2. Let *E* be a reflexive Banach space, the following conclusions hold: (i) *f* is essentially smooth if and only if f^* is essentially strictly convex; (ii) $(\partial f)^{-1} = \partial f^*$; (iii) *f* is Legendre if and only if f^* is Legendre; (iv)if *f* is Legendre, then ∇f is bijection satisfying $\nabla f = (\nabla f^*)^{-1}$, ran ∇f =dom ∇f^* =int dom f^* and ran ∇f^* =dom ∇f =int dom*f*, see [1].

Definition 2.3. Let $f : E \to (-\infty, +\infty]$ be a Gâteaux differentiable function. The Bregman distance with respect to f [4] is the function $D_f : \operatorname{dom} f \times \operatorname{int} \operatorname{dom} f \to [0, +\infty)$ defined by $D_f(y, x) := f(y) - \langle \nabla f(x), y - x \rangle - f(x).$

Recall that the bifunction $V_f : E \times E^* \to [0,\infty)$ associated with f is defined by $V_f(x,x^*) = f(x) - \langle x,x^* \rangle + f^*(x^*), \forall x \in E, x^* \in E^*$. Then V_f is nonnegative and satisfies $V_f(x,x^*) = D_f(x, \nabla f^*(x^*)), \forall x \in E, x^* \in E^*$. Although $D_f(\cdot, \cdot)$ does not normally satisfy the symmetry and the triangle inequality, it has the following important property, called "three point identity": for any $x \in \text{dom} f$ and $y, z \in \text{int dom} f, \langle \nabla f(z) - \nabla f(y), x - y \rangle = D_f(x,y) + D_f(y,z) - D_f(x,z)$.

Definition 2.4. If $f : E \to (-\infty, +\infty]$ is convex and Gâteaux differentiable, $C \subset \text{dom} f$ is a nonempty, closed, and convex set. The Bregman projection [11] $x \in \text{int dom} f$ onto C is the unique vector $P_C^f(x) \in C$ satisfying $D_f(P_C^f(x), x) = \inf\{D_f(y, x) : y \in C\}$.

In fact, the Bregman projection $P_C^f(x)$, which is more general than the generalized projection $\Pi_C(x)$ defined by $\Pi_C(x) = \arg \min_{y \in C} \phi(y, x)$ from *E* onto *C*, reduces to the generalized projection by taking $f(x) = ||x||^2$ for all $x \in E$.

Let $f : E \to (-\infty, +\infty]$ be Gâteaux differentiable. The modulus of total convexity of f at $x \in \text{dom} f$ is the function $\nu_f(x, \cdot) : [0, +\infty) \to [0, +\infty]$ defined by $\nu_f(x, t) :=$ $\inf\{D_f(y, x) : y \in \text{dom} f, ||y - x|| = t\}$. The modulus of total convexity of the function f on the set B is the function $\nu_f : \text{int dom} f \times [0, +\infty) \to [0, +\infty]$ defined by $\nu_f(B, t) :=$ $\inf\{\nu_f(x, t) : x \in B \cap \text{dom} f\}$. **Definition 2.5.** A function f is said to be: (i) totally convex at x if $\nu_f(x,t) > 0$, whenever t > 0; (ii) totally convex if it is totally convex at any point $x \in \text{int dom} f$; (iii) totally convex on bounded sets if $\nu_f(B, t) > 0$ for any nonempty bounded subset B of E and t > 0.

Definition 2.6. A function f is said to be: (i) strongly coercive if $\lim_{\|x\|\to\infty} f(x)/\|x\| = \infty$; (ii) sequentially consistent [3], if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that the first one is bounded, $\lim_{n\to\infty} D_f(y_n, x_n) = 0 \implies \lim_{n\to\infty} \|y_n - x_n\| = 0$.

Definition 2.7. A multi-valued mapping $T : C \to CB(C)$ is said to be multi-valued Bregman quasi-strictly pseudo-contractive with respect to f if $F(T) \neq \emptyset$ and

$$D_f(p,u) \le D_f(p,x) + kD_f(x,u), \forall u \in Tx, \ x \in C, \ p \in F(T).$$

In the following, we list some lemmas which are important in our proof.

Lemma 2.1. [3] Suppose that f is Gâteaux differentiable and totally convex on int dom f. For a nonempty, closed and convex set $C \subset int$ dom f, $x \in int$ dom f and $\hat{x} \in C$, then the following conditions are equivalent: (i) the vector \hat{x} is the Bregman projection of x onto C with respect to f, i.e., $z = P_C^f(x)$; (ii) the vector \hat{x} is the unique solution for $\langle \nabla f(x) - \nabla f(z), z - y \rangle \ge 0$, $\forall y \in C$; (iii) The vector \hat{x} is the unique solution for $D_f(y, z) + D_f(z, x) \le D_f(y, x)$, $\forall y \in C$.

Lemma 2.2. [1] Suppose $x \in E$ and $y \in int \text{ dom} f$. If f is essentially strictly convex, then $D_f(x, y) = 0 \Leftrightarrow x = y$.

Lemma 2.3. [2] *The function f is totally convex on bounded sets if and only if the function f is sequentially consistent.*

Lemma 2.4. [12] Suppose that $f : E \to \mathbb{R}$ is Gâteaux differentiable and totally convex. If $x_0 \in E$ and the sequence $\{D_f(x_n, x_0)\}$ is bounded, the sequence $\{x_n\}$ is bounded too.

Lemma 2.5. [11] Suppose that the convex function $f : E \to \mathbb{R}$ is bounded on bounded subsets of *E*. Then the following assertions are equivalent: (a) *f* is strongly coercive and uniformly convex on bounded subsets of *E*; (b) *f*^{*} is Fréchet differentiable and ∇f^* is uniformly norm-to-norm continuous on bounded subsets of dom $f^* = E^*$.

Lemma 2.6. [13] Suppose that the convex, continuous and strongly coercive function $f : E \to \mathbb{R}$ is bounded on bounded subsets and uniformly convex on bounded subsets of $E, g : C \times C \to \mathbb{R}$ is a bifunction satisfying (A1)-(A4), $\operatorname{Res}_r^g : E \to C$ is the resolvent operator defined by $\operatorname{Res}_r^g(x) =$ $\left\{z \in C : g(z, y) + \frac{1}{r}\langle y - z, \nabla f(z) - \nabla f(x) \rangle \ge 0, \forall y \in C\right\}$, where $r > 0, x \in E$. Then the following statements hold: (a) Res_r^g is single-valued; (b) $F(\operatorname{Res}_r^g) = EP(g)$; (c) EP(g) is closed and convex; (d) $D_f(p, \operatorname{Res}_r^g x) + D_f(\operatorname{Res}_r^g x, x) \le D_f(p, x), \forall p \in EP(g), \forall x \in E$.

3. MAIN RESULTS

In this section, we state and prove our main theorem.

Theorem 3.1. Suppose that $f : E \to \mathbb{R}$ is a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E, g is a bifunction from $C \times C$ to \mathbb{R} satisfying (A1)–(A4), $T : C \to CB(C)$ is a closed mapping defined as Definition 2.7 such that $F(T) \cap EP(g) \neq \emptyset$. For an arbitrary element $x_0 \in C$, let $C_0 = C$ and $\{x_n\}_{n \in \mathbb{N}}$ be a sequence generated by the following iterative algorithm:

(3.1)
$$\begin{cases} y_n = \nabla f^*[\alpha_n \nabla f(x_n) + (1 - \alpha_n) \nabla f(z_n)], & z_n \in Tx_n, \\ g(u_n, y) + \frac{1}{r_n} \langle y - u_n, \nabla f(u_n) - \nabla f(y_n) \rangle \ge 0, & \forall y \in C, \\ x_{n+1} = P^f_{C_{n+1}}(x_0), & n \in \mathbb{N} \cup \{0\}, \end{cases}$$

where
$$C_{n+1} = \left\{ z \in C_n : D_f(z, u_n) \le D_f(z, y_n) \le D_f(z, x_n) + \frac{\kappa}{1-\kappa} \langle x_n - z, \nabla f(x_n) - \nabla f(z_n) \rangle \right\}$$
, $\kappa \in [0, 1)$, $\liminf_{n \to \infty} r_n > 0$. Then the sequence $\{x_n\}$ converges strongly to $\hat{p} = P_{F(T) \cap EP(g)}^f(x_0)$, where $P_{F(T) \cap EP(g)}^f$ is the Bregman projection of E onto $F(T) \cap EP(g)$.
Proof. Due to the construction of C_n , one sees that C_n is closed for all $n \in \mathbb{N} \cup \{0\}$. Since $D_f(z, u_n) \le D_f(z, y_n)$ and $D_f(z, y_n) \le D_f(z, x_n) + \frac{k}{1-k} \langle x_n - z, \nabla f(x_n) - \nabla f(z_n) \rangle$
are equal to $\langle z, \nabla f(y_n) - \nabla f(u_n) \rangle \le f(u_n) - f(y_n) + \langle y_n, \nabla f(y_n) \rangle - \langle u_n, \nabla f(u_n) \rangle$, and $\left\langle z, \frac{1}{1-k} \nabla f(x_n) - \nabla f(y_n) - \frac{k}{1-k} \nabla f(z_n) \right\rangle \le f(y_n) - f(x_n) + \left\langle x_n, \frac{1}{1-k} \nabla f(x_n) \right\rangle - \left\langle x_n, \frac{k}{1-k} \nabla f(z_n) \right\rangle$

 $-\langle y_n, \nabla f(y_n) \rangle$ respectively, thus C_n is convex and closed for $n \in \mathbb{N} \cup \{0\}$.

Next, we show that $F(T) \cap EP(g) \subset C_n$ for all $n \in \mathbb{N} \cup \{0\}$. It is clear that $F(T) \cap EP(g) \subset C_0 = C$. Suppose that $F(T) \cap EP(g) \subset C_m$ for some $m \in \mathbb{N}$. For any $w \in F(T) \cap EP(g) \subset C_m$, since $u_m = \operatorname{Res}_{r_m}^g y_m$, one has from Lemma 2.6 (d) that

$$D_f(w, u_m) \le D_f(w, y_m) \le \alpha_m V(w, \nabla f(x_m)) + (1 - \alpha_m) V(w, \nabla f(z_m))$$

= $\alpha_m D_f(w, x_m) + (1 - \alpha_m) D_f(w, z_m)$
 $\le \alpha_m D_f(w, x_m) + (1 - \alpha_m) [D_f(w, x_m) + k D_f(x_m, z_m)]$
 $\le D_f(w, x_m) + \frac{k}{1 - k} \langle x_m - w, \nabla f(x_m) - \nabla f(z_m) \rangle.$

This implies that $w \in C_{m+1}$. Therefore, one has $F(T) \cap EP(g) \subset C_n$ for all $n \in \mathbb{N} \cup \{0\}$.

Now, we are in a position to show that $\lim_{n\to\infty} D_f(x_n, x_0)$ exists. In fact, since $x_n = P_{C_n}^f(x_0)$, from Lemma 2.1 (ii), one has $\langle y - x_n, \nabla f(x_0) - \nabla f(x_n) \rangle \leq 0$, $\forall y \in C_n$, and since $F(T) \cap EP(g) \subset C_n$ for all $n \in \mathbb{N} \cup \{0\}$, we arrive at

(3.2)
$$\langle w - x_n, \nabla f(x_0) - \nabla f(x_n) \rangle \le 0, \quad \forall \ w \in F(T) \cap EP(g).$$

From Lemma 2.1 (iii), one has $D_f(x_n, x_0) = D_f(P_{C_n}^f(x_0), x_0) \leq D_f(w, x_0) - D_f(w, P_{C_n}^f(x_0))$ $\leq D_f(w, x_0)$, for each $w \in F(T) \cap EP(g)$ and for each $n \in \mathbb{N} \cup \{0\}$. Therefore, $\{D_f(x_n, x_0)\}$ is bounded. From Lemma 2.4, one has $\{x_n\}$ is also bounded. Since $x_n = P_{C_n}^f(x_0)$ and $x_{n+1} = P_{C_{n+1}}^f(x_0) \in C_{n+1} \subset C_n$, one has $D_f(x_n, x_0) \leq D_f(x_{n+1}, x_0)$ for all $n \in \mathbb{N} \cup \{0\}$. This implies that $\{D_f(x_n, x_0)\}$ is a nondecreasing sequence. Therefore $\lim_{n\to\infty} D_f(x_n, x_0)$ exists. Since $\{x_n\}$ is bounded and E is reflexive, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \rightarrow \hat{p} \in C = C_1$. Since C_n is closed and convex and $C_{n+1} \subset C_n$, this implies that C_n is weakly closed and $\hat{p} \in C_n$ for all $n \in \mathbb{N} \cup \{0\}$. Hence $\hat{p} \in C_{n_i}$ for all $n_i \in \mathbb{N} \cup \{0\}$. In view of $x_{n_i} = P_{C_{n_i}}^f(x_0)$, one has from the definition of Bregman projection that $D_f(x_{n_i}, x_0) \leq D_f(\hat{p}, x_0)$, $\forall n_i \in \mathbb{N} \cup \{0\}$. Since f is a lower semi-continuous function on convex set C, it is weakly lower semi-continuous on C. Hence we have

$$\liminf_{i \to \infty} D_f(x_{n_i}, x_0) = \liminf_{i \to \infty} \{ f(x_{n_i}) - f(x_0) - \langle \nabla f(x_0), x_{n_i} - x_0 \rangle \}$$

$$\geq f(\hat{p}) - f(x_0) - \langle \nabla f(x_0), \hat{p} - x_0 \rangle = D_f(\hat{p}, x_0).$$

Thus, one has $D_f(\hat{p}, x_0) \leq \liminf_{i \to \infty} D_f(x_{n_i}, x_0) \leq \limsup_{i \to \infty} D_f(x_{n_i}, x_0) \leq D_f(\hat{p}, x_0)$, which implies that $\lim_{i \to \infty} D_f(x_{n_i}, x_0) = D_f(\hat{p}, x_0)$. Employing 2.1 (iii), one obtains that $D_f(\hat{p}, x_{n_i}) \leq D_f(\hat{p}, x_0) - D_f(x_{n_i}, x_0)$. When $n_i \to \infty$ in the above inequality, one obtains $\lim_{n_i \to \infty} D_f(\hat{p}, x_{n_i}) = 0$, which implies from Lemma 2.2 that $\lim_{n_i \to \infty} D_f(x_{n_i}, x_0) = \hat{p}$. Besides, noticing that $\{D_f(x_n, x_0)\}$ is convergent, hence, one gets $\lim_{n \to \infty} D_f(x_n, x_0) = \hat{p}$. $D_f(\widehat{p}, x_0)$. And since $x_n = P_{C_n}^f x_0$, from Lemma 2.1 (iii), one has $D_f(\widehat{p}, x_n) \leq D_f(\widehat{p}, x_0) - D_f(x_n, x_0)$. Similarly, one also obtains $\lim_{n_i \to \infty} x_n = \widehat{p}$. Since $x_{n+1} \in C_{n+1}$, from the construction of C_{n+1} , one has $D_f(x_{n+1}, u_n) \leq D_f(x_{n+1}, y_n) \leq D_f(x_{n+1}, x_n) + \frac{k}{1-k} \langle x_n - x_{n+1}, \nabla f(x_n) - \nabla f(z_n) \rangle$. Noticing that $\lim_{n_i \to \infty} x_n = \widehat{p}$, one has $\lim_{n \to \infty} D_f(x_{n+1}, y_n) = 0$, and

$$\begin{split} \lim_{n\to\infty} D_f(x_{n+1},u_n) &= 0. \text{ In view of Lemma 2.3 and Definition 2.6, one has} \\ \lim_{n\to\infty} \|x_{n+1}-y_n\| &= 0 \text{ and } \lim_{n\to\infty} \|x_{n+1}-u_n\| = 0, \text{ furthermore, } \lim_{n\to\infty} \|x_n-y_n\| = 0 \\ \text{ and } \lim_{n\to\infty} \|x_n-u_n\| &= 0. \text{ Since } \nabla f \text{ is uniformly continuous on each bounded subset of} \\ E, \text{ one has } \lim_{n\to\infty} \|\nabla f(x_n) - \nabla f(y_n)\| = 0. \text{ Due to } y_n = \nabla f^*[\alpha_n \nabla f(x_n) + (1-\alpha_n) \nabla f(z_n)], \\ \text{ one has } \lim_{n\to\infty} \|\nabla f(x_n) - \nabla f(z_n)\| = \lim_{n\to\infty} \frac{1}{1-\alpha_n} \|\nabla f(x_n) - \nabla f(y_n)\| = 0. \text{ From} \\ \text{ Lemma 2.5, one has } \lim_{n\to\infty} \|x_n - z_n\| = 0. \text{ Therefore } \lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n = \widehat{p}. \text{ In view of } z_n \in Tx_n, \text{ and from the closedness of } T, \text{ it follows } \widehat{p} \in T\widehat{p}, \text{ that is, } \widehat{p} \in F(T). \end{split}$$

Next, we prove $\hat{p} \in EP(g)$. Obviously, $\lim_{n\to\infty} \|u_n - y_n\| = 0$. Hence $\lim_{n\to\infty} \|\nabla f(u_n) - \nabla f(y_n)\| = 0$. By the assumption $\liminf_{n\to\infty} r_n > 0$, one has $\lim_{n\to\infty} \frac{\|\nabla f(u_n) - \nabla f(y_n)\|}{r_n} = 0$, which together with $u_n = T_{r_n}^g y_n$ implies that $g(u_n, y) + \frac{1}{r_n} \langle y - u_n, \nabla f(u_n) - \nabla f(y_n) \rangle \geq 0$, $\forall y \in C$. From (A2), we deduce that $\|y - u_n\| \frac{\|\nabla f(u_n) - \nabla f(y_n)\|}{r_n} \geq \frac{1}{r_n} \langle y - u_n, \nabla f(u_n) - \nabla f(u_n) - \nabla f(y_n) \rangle \geq -g(u_n, y) \geq g(y, u_n), \forall y \in C$. Letting $n \to \infty$ in the above inequality, one has from (A4) that $g(y, \hat{p}) \leq 0$, $\forall y \in C$. For $t \in (0, 1)$ and $y \in C$, let $y_t = ty + (1 - t)\hat{p}$. Then $y_t \in C$, which yields that $g(y_t, \hat{p}) \leq 0$. Therefore, from (A1) and (A4) one has $0 = g(y_t, y_t) \leq tg(y_t, y) + (1 - t)g(y_t, p) \leq tg(y_t, y)$. Dividing by t, one has $g(y_t, y) \geq 0$, $\forall y \in C$. Letting $t \downarrow 0$, from (A3), one has $g(\hat{p}, y) \geq 0$, $\forall y \in C$. Hence $\hat{p} \in EP(g)$.

Finally, we take $n \to \infty$ in (3.2) and obtain that $\langle w - \hat{p}, \nabla f(x_0) - \nabla f(\hat{p}) \rangle \leq 0, \forall w \in F(T) \cap EP(g)$. In view of Lemma 2.10 (i) and (ii), one has $\hat{p} = P_{F(T) \cap EP(g)}^f(x_0)$.

4. APPLICATIONS AND EXAMPLES

Let $A : C \subseteq E \to E^*$ be a nonlinear mapping. The variational inequality problem for a nonlinear mapping A and its domain C is to find $\bar{x} \in C$ such that $\langle A\bar{x}, y - \bar{x} \rangle \ge 0, \forall y \in C$. The set of solutions of the variational inequality problem is denoted by VI(C, A). Recall that a mapping $A : C \to E^*$ is called monotone if $\langle Ax - Ay, x - y \rangle \ge 0, \forall x, y \in C$.

Assume that *A* is a continuous and monotone mapping. For r > 0, define the resolvent operator $\operatorname{Res}_r^f : E \to C$ as follows: for all $x \in E$, $\operatorname{Res}_r^A := \{z \in C : \langle Az, y - z \rangle + \frac{1}{r} \langle \nabla f(z) - \nabla f(x), y - z \rangle \geq 0, \forall y \in C \}$. Similar to Lemma 2.15, the following conclusions hold: (1) Res_r^A is single-valued; (2) $F(\operatorname{Res}_r^A) = VI(C, A)$; (3) $D_f(p, \operatorname{Res}_r^A x) + D_f(\operatorname{Res}_r^A x, x) \leq D_f(p, x)$, for $p \in F(\operatorname{Res}_r^A)$; (4) VI(C, A) is closed and convex.

Theorem 4.2. Suppose that f and g are defined as Theorem 3.1, $A : C \to E^*$ is a continuous monotone mapping such that $VI(C, A) \cap EP(g) \neq \emptyset$. Reset $Tx_n = \operatorname{Res}_r^A x_n$ in the algorithm (3.1), let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence generated by the algorithm (3.1). Then the sequence $\{x_n\}$ converges strongly to $\hat{p} = P_{VI(C,A)\cap EP(g)}^f(x_0)$, where $P_{VI(C,A)\cap EP(g)}^f$ is the Bregman projection of E onto $VI(C, A) \cap EP(g)$.

Finally, a numerical experiment will be carried out to demonstrate the efficiency of the algorithm (3.1). Based on Example 5.1 of Wang and Wei [16] and Example 1 of Saewan, Cho, Kumam [13], the following example could be obtained easily.

Example 4.1. Let $E = \mathbb{R}$, $C = [-\pi, \pi]$, $f(x) = x^2$, $Tx = \sin\left(\frac{1}{2}x\right)$, $g(z, y) = y^2 + zy - 2z^2$. Then *T* is a closed Bregman quasi-strict pseudo-contraction with $EP(g) \cap F(T) = \{0\}$. Based on the assumption of Example 4.1, replace Tx_n and $g(u_n, y)$ by $Tx_n = \sin\left(\frac{1}{2}x_n\right)$, $g(u_n, y) = y^2 + u_n y - 2u_n^2$ in the algorithm (3.1). For the initial conditions $x_0 = -0.8, 1$, $r_n \equiv 1, \alpha_n = \frac{1}{n}$, the picture (a) in Fig.1 shows that the sequence $\{x_n\}$ converge to the same value for the different initial points. For the initial conditions $r_n = 10^{-4}, 1, n^2$, $x_0 = 1, \alpha_n = \frac{1}{n}$, the picture (b) in Fig.1 shows that the different values of parameter sequence $\{r_n\}$ do not significantly influence on the rates of convergence. Therefore, in a real world application, the parameter sequence $\{r_n\}$ of algorithm (3.1) can be regarded as the constant 1.



Fig. 1. the convergence process of the sequence $\{x_n\}$ with different initial conditions.

5. CONCLUSIONS

In the paper, we have investigated a fixed point problem of a closed multi-valued Bregman quasi-strict pseudocontraction and an equilibrium problem via hybrid Bregman projection methods, and obtained a strong convergence result. Furthermore, a kind of variational inequality problem has been solved as an application and a numerical example has been given to illustrate the effectiveness of the proposed algorithm.

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