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Dedicated to Professor Yeol Je Cho on the occasion of his retirement

Iterative algorithms for split common fixed points of demicontractive operators without priori knowledge of operator norms

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ABSTRACT. The split common fixed points problem for demicontractive operators has been studied in Hilbert spaces. An iterative algorithm is considered and the weak convergence result is given under some mild assumptions.

1. INTRODUCTION

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces equipped up their own inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let $\mathcal{S}\colon \mathcal{H}_1 \to \mathcal{H}_1$ and $\mathcal{T}\colon \mathcal{H}_2 \to \mathcal{H}_2$ be two nonlinear operators. Denote the fixed point sets of $\mathcal{S}$ and $\mathcal{T}$ by $\mathrm{Fix}(\mathcal{S})$ and $\mathrm{Fix}(\mathcal{T})$, respectively. Let $\mathcal{A}\colon \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator with its adjoint $\mathcal{A}^*$.

The present article focusses on the split common fixed point problem which is to find a point $u^\dagger \in \mathcal{H}_1$ such that

$$u^\dagger \in \mathrm{Fix}(\mathcal{S}) \quad \text{and} \quad Au^\dagger \in \mathrm{Fix}(\mathcal{T}). \tag{1.1}$$

The split common fixed point problem (1.1) is a generalization of the split feasibility problem arising from signal processing and image restoration ([4, 6, 17, 18, 19, 21]), which is to find a point $u^\dagger$ such that

$$u^\dagger \in \mathcal{C} \text{ and } \mathcal{A}u^\dagger \in \mathcal{Q}, \tag{1.2}$$

where $\mathcal{C} \subset \mathcal{H}_1$ and $\mathcal{Q} \subset \mathcal{H}_2$ are two nonempty closed convex sets. Problem (1.1) was firstly introduced by Censor and Segal [7]. Note that solving (1.1) can be translated to solve the fixed point equation

$$x^* = \mathcal{S}(x^* - \tau \mathcal{A}^*(I - \mathcal{T})\mathcal{A}x^*), \quad \tau > 0.$$

Whereafter, Censor and Segal proposed an algorithm for directed operators. Since then, there has been growing interest in the split common fixed point problem ([1, 3, 5, 8, 12, 13, 14, 20, 22]). In particular, Wang [16] introduced the following new iterative algorithm for the split common fixed point problem of firmly-nonexpansive mappings.

Algorithm 1.1. Choose an arbitrary initial guess x_0.

Step 1. Given x_n, compute the next iteration via the formula:

$$x_{n+1} = x_n - \rho_n[x_n - \mathcal{S}x_n + \mathcal{A}^*(I - \mathcal{T})\mathcal{A}x_n], \quad n \geq 0. \tag{1.3}$$

Step 2. If the following equality

$$\|x_{n+1} - \mathcal{S}x_{n+1} + \mathcal{A}^*(I - \mathcal{T})\mathcal{A}x_{n+1}\| = 0. \tag{1.4}$$

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holds, then stop; otherwise go to step 1.

Subsequently, Wang [16] demonstrated the convergence analysis of Algorithm 1.1.

Theorem 1.1. *Assume the following conditions are satisfied:*

$(A1)$ *$\mathcal{A}$ is a bounded linear operator;*
$(A2)$ *the solution set of problem* (1.1)*, denoted by Ω, is nonempty;*
$(A3)$ *both $\mathcal{S}$ and $\mathcal{T}$ are firmly nonexpansive operators.*

Let $\{x_n\}$ be the sequence generated by Algorithm 1.1*. If the sequence $\{\rho_n\}$ satisfies $\sum_{n=0}^{\infty} \rho_n = \infty$ and $\sum_{n=0}^{\infty} \rho_n^2 < \infty$, then $\{x_n\}$ converges weakly to a solution z of problem* (1.1)*, where $z = \lim_{n\to\infty} \mathrm{proj}_\Omega x_n$.*

At the same time, Wang[16] gave the following remark.

Remark 1.1. It is readily seen that, in Algorithm 1.1, the selection of the stepsize does not depend on the operator norm $\|\mathcal{A}\|$. It seems that the assumption $(A3)$ cannot weaken to directed operators.

Inspired by the work in the literature, the main purpose of this paper is to give an answer to Remark 1.1. We will extend Wang's [16] result from the firmly nonexpansive operators to a more general demicontractive operators. We demonstrate the convergence of Algorithm 1.1 for solving the split common fixed point problem (1.1). Weak convergence theorem is given under some mild assumptions.

2. Preliminaries

Let $\mathcal{C}$ be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$.

Definition 2.1. An operator $\mathcal{T}\colon \mathcal{C} \to \mathcal{C}$ is said to be nonexpansive if

$$\|\mathcal{T}u - \mathcal{T}u^\dagger\| \le \|u - u^\dagger\|$$

for all $u, u^\dagger \in \mathcal{C}$.

Definition 2.2. An operator $\mathcal{T}\colon \mathcal{C} \to \mathcal{C}$ is said to be firmly nonexpansive if

$$\|\mathcal{T}u - \mathcal{T}u^\dagger\|^2 \le \|u - u^\dagger\|^2 - \|(I - \mathcal{T}u - (I - \mathcal{T})u^\dagger\|^2$$

for all $u, u^\dagger \in \mathcal{C}$.

A typical example of firmly nonexpansive operators is an orthogonal projection $\mathrm{proj}_\mathcal{C}$ from $\mathcal{H}$ onto $\mathcal{C}$ defined by

$$\mathrm{proj}_\mathcal{C}(u) := \arg\min_{v\in\mathcal{C}}\{\|u - v\|\}, u \in \mathcal{H}.$$

The metric projection $\mathrm{proj}_\mathcal{C}$ of $\mathcal{H}$ onto $\mathcal{C}$ is characterized by

$$\langle u - \mathrm{proj}_\mathcal{C}(u), v - \mathrm{proj}_\mathcal{C}(u)\rangle \ge 0 \tag{2.5}$$

for all $u \in \mathcal{H}, v \in \mathcal{C}$.

Definition 2.3. An operator $\mathcal{T}\colon \mathcal{C} \to \mathcal{C}$ is said to be directed if

$$\|\mathcal{T}u - u^\dagger\|^2 \le \|u - u^\dagger\|^2 - \|\mathcal{T}u - u\|^2$$

for all $u \in \mathcal{C}$ and $u^\dagger \in \mathrm{Fix}(\mathcal{T})$.

The class of directed operators is an important class since it includes the orthogonal projections and the subgradient projectors which are fundamental in the convex optimization.

Definition 2.4. An operator $\mathcal{T}$ is called demicontractive if there exists a constant $\beta \in [0, 1)$ such that

$$\|\mathcal{T}u - v\|^2 \leq \|u - v\|^2 + \beta\|u - \mathcal{T}u\|^2,$$

or equivalently,

$$\langle u - \mathcal{T}u, u - v\rangle \geq \frac{1-\beta}{2}\|u - \mathcal{T}u\|^2, \tag{2.6}$$

for all $(u, v) \in \mathcal{H} \times \mathrm{Fix}(\mathcal{T})$.

Remark 2.2. It is clear that the demicontractive operators include the directed operators as special cases. The class of demicontractive operators is fundamental because many common types of operators arising in optimization belong to this class, see for example ?????????? **[?]** and references therein.

Definition 2.5. An operator $\mathcal{T}$ is said to be demiclosed if for any sequence $\{x_n\}$ which weakly converges to $\tilde{x}$, and if the sequence $\{\mathcal{T}(x_n)\}$ strongly converges to z, then $\mathcal{T}(\tilde{x}) = z$.

Definition 2.6. A sequence $\{x_n\}$ is called Fejér-monotone with respect to a given nonempty set Ω if for every $x \in \Omega$,

$$\|x_{n+1} - x\| \leq \|x_n - x\|, \quad \text{for all } n \geq 0.$$

Next we adopt the following notations:
· $x_n \rightharpoonup x$ means that x_n converges weakly to x;
· $\omega_w(x_n) := \{x : \exists x_{n_j} \rightharpoonup x\}$ is the weak ω-limit set of the sequence $\{x_n\}$.

Lemma 2.1. ([2]) *Let Ω be a nonempty closed convex subset in $\mathcal{H}$. If the sequence $\{x_n\}$ is Fejér monotone with respect to Ω, then we have the following conclusions:*

(i) $x_n \rightharpoonup x^\dagger \in \Omega$ *iff* $\omega_w(x_n) \subset \Omega$;
(ii) *the sequence* $\{\mathrm{proj}_\Omega(x_n)\}$ *converges strongly;*
(iii) *if* $x_n \rightharpoonup x^\dagger \in \Omega$, *then* $x^\dagger = \lim_{n\to\infty} \mathrm{proj}_\Omega(x_n)$.

Lemma 2.2. ([15]) *Let $\{a_n\}$ and $\{b_n\}$ be positive real sequences such that $\sum_{n=0}^{\infty} b_n < \infty$. If either $a_{n+1} \leq (1 + b_n)a_n$ or $a_{n+1} \leq a_n + b_n$, then $\lim_{n\to\infty} a_n$ exists.*

3. Main results

In this section, we will give the convergence analysis of Algorithm 1.1 for solving the split common fixed points problem (1.1). For the purpose, the following hypothesises are involved.

(HP1): $\mathcal{H}_1$ and $\mathcal{H}_2$ are two real Hilbert spaces;
(HP2): $\mathcal{S}\colon \mathcal{H}_1 \to \mathcal{H}_1$ and $\mathcal{T}\colon \mathcal{H}_2 \to \mathcal{H}_2$ are two demicontractive operators with coefficients $\beta \in [0, 1)$ and $\mu \in [0, 1)$, respectively;
(HP3): $\mathcal{S}\colon \mathcal{H}_1 \to \mathcal{H}_1$ and $\mathcal{T}\colon \mathcal{H}_2 \to \mathcal{H}_2$ are Lipschitz continuous with Lipschitz constant $L > 1$;
(HP4): $\mathcal{A}\colon \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator with its adjoint operator $\mathcal{A}^*$.

We use Ω to denote the solution set of problem (1.1), that is,

$$\Omega = \{\bar{z} : \bar{z} \in \mathrm{Fix}(\mathcal{S}) \text{ and } \mathcal{A}\bar{z} \in \mathrm{Fix}(\mathcal{T})\}.$$

Throughout, assume Ω is nonempty.

First, we have the following remark.

Remark 3.3. If (1.4) holds, then x_{n+1} solves problem (1.1). As a matter of fact, we have the following a more general conclusion.

Proposition 3.1. *$z^\dagger$ solves* (1.1) *iff* $\|z^\dagger - \mathcal{S}z^\dagger + \mathcal{A}^*(I-\mathcal{T})\mathcal{A}z^\dagger\| = 0$.

Proof. If $z^\dagger$ solves (1.1), then $z^\dagger = \mathcal{S}z^\dagger$ and $(I-\mathcal{T})\mathcal{A}z^\dagger = 0$. It is obvious that $\|z^\dagger - \mathcal{S}z^\dagger + \mathcal{A}^*(I-\mathcal{T})\mathcal{A}z^\dagger\| = 0$.

To see the converse, assume that $\|z^\dagger - \mathcal{S}z^\dagger + \mathcal{A}^*(I-\mathcal{T})\mathcal{A}z^\dagger\| = 0$, then for any $z \in \Omega$, we obtain

$$
\begin{aligned}
0 &= \|z^\dagger - \mathcal{S}z^\dagger + \mathcal{A}^*(I-\mathcal{T})\mathcal{A}z^\dagger\|\|z^\dagger - z\| \\
&\geq \langle z^\dagger - \mathcal{S}z^\dagger + \mathcal{A}^*(I-\mathcal{T})\mathcal{A}z^\dagger, z^\dagger - z\rangle \\
&= \langle z^\dagger - \mathcal{S}z^\dagger, z^\dagger - z\rangle + \langle \mathcal{A}^*(I-\mathcal{T})\mathcal{A}z^\dagger, z^\dagger - z\rangle \\
&= \langle z^\dagger - \mathcal{S}z^\dagger, z^\dagger - z\rangle + \langle (I-\mathcal{T})\mathcal{A}z^\dagger, \mathcal{A}z^\dagger - \mathcal{A}z\rangle.
\end{aligned} \tag{3.7}
$$

Since $\mathcal{S}$ and $\mathcal{T}$ are demicontractve, from (2.6), we deduce

$$
\langle z^\dagger - \mathcal{S}z^\dagger, z^\dagger - z\rangle \geq \frac{1-\beta}{2}\|z^\dagger - \mathcal{S}z^\dagger\|^2, \tag{3.8}
$$

and

$$
\langle (I-\mathcal{T})\mathcal{A}z^\dagger, \mathcal{A}z^\dagger - \mathcal{A}z\rangle \geq \frac{1-\mu}{2}\|(I-\mathcal{T})\mathcal{A}z^\dagger\|^2. \tag{3.9}
$$

By (3.7)-(3.9), we get

$$
\begin{aligned}
0 &\geq \langle z^\dagger - \mathcal{S}z^\dagger + \mathcal{A}^*(I-\mathcal{T})\mathcal{A}z^\dagger, z^\dagger - z\rangle \\
&\geq \frac{1-\beta}{2}\|z^\dagger - \mathcal{S}z^\dagger\|^2 + \frac{1-\mu}{2}\|(I-\mathcal{T})\mathcal{A}z^\dagger\|^2.
\end{aligned} \tag{3.10}
$$

Since $\beta, \mu \in [0,1)$, we deduce $z^\dagger \in \mathrm{Fix}(\mathcal{S})$ and $\mathcal{A}z^\dagger \in \mathrm{Fix}(\mathcal{T})$ by (3.10). Therefore, $z^\dagger$ solves problem (1.1). The proof is completed. □

We assume that the sequence $\{x_n\}$ generated by Algorithm 1.1 is infinite. In other words, Algorithm 1.1 does not terminate in a finite number of iterations. In this case, we demonstrate the convergence of the sequence $\{x_n\}$ generated by Algorithm 1.1.

Theorem 3.2. *Assume the following conditions are satisfied*

(i) $I - \mathcal{S}$ *and* $I - \mathcal{T}$ *are demiclosed at zero;*
(ii) $\sum_{n=0}^\infty \rho_n = \infty$ *and* $\sum_{n=0}^\infty \rho_n^2 < \infty$.

Then the sequence $\{x_n\}$ generated by Algorithm 1.1 converges weakly to a solution z^* of problem (1.1), *where* $z^* = \lim_{n\to\infty} \mathrm{proj}_\Omega(x_n)$.

Proof. Firstly, we prove that the sequence $\{x_n\}$ is bounded. Let $z \in \Omega$. Set $y_n = x_n - \mathcal{S}x_n + \mathcal{A}^*(I-\mathcal{T})\mathcal{A}x_n$ for all $n \geq 0$. From (2.6), we have

$$
\begin{aligned}
\langle y_n, x_n - z\rangle &= \langle x_n - \mathcal{S}x_n + \mathcal{A}^*(I-\mathcal{T})\mathcal{A}x_n, x_n - z\rangle \\
&= \langle x_n - \mathcal{S}x_n, x_n - z\rangle + \langle \mathcal{A}^*(I-\mathcal{T})\mathcal{A}x_n, x_n - z\rangle \\
&\geq \frac{1-\beta}{2}\|x_n - \mathcal{S}x_n\|^2 + \frac{1-\mu}{2}\|(I-\mathcal{T})\mathcal{A}x_n\|^2 \\
&= \frac{1-\beta}{2}\|x_n - \mathcal{S}x_n\|^2 + \frac{(1-\mu)\|\mathcal{A}\|^2}{2\|A\|^2}\|(I-\mathcal{T})\mathcal{A}x_n\|^2 \\
&\geq \frac{1-\beta}{2}\|x_n - \mathcal{S}x_n\|^2 + \frac{(1-\mu)}{2\|\mathcal{A}\|^2}\|A^*(I-\mathcal{T})\mathcal{A}x_n\|^2 \\
&\geq \frac{\min\{1-\beta, 1-\mu\}}{2\max\{1, \|\mathcal{A}\|^2\}}(\|x_n - \mathcal{S}x_n\|^2 + \|A^*(I-\mathcal{T})\mathcal{A}x_n\|^2) \qquad (3.11)\\
&\geq \frac{\min\{1-\beta, 1-\mu\}}{4\max\{1, \|\mathcal{A}\|^2\}}(\|x_n - \mathcal{S}x_n\| + \|A^*(I-\mathcal{T})\mathcal{A}x_n\|)^2 \\
&\geq \frac{\min\{1-\beta, 1-\mu\}}{4\max\{1, \|\mathcal{A}\|^2\}}\|x_n - \mathcal{S}x_n + A^*(I-\mathcal{T})\mathcal{A}x_n\|^2 \\
&= \frac{\min\{1-\beta, 1-\mu\}}{4\max\{1, \|\mathcal{A}\|^2\}}\|y_n\|^2 \\
&= \tau\|y_n\|^2,
\end{aligned}
$$

where $\tau = \dfrac{\min\{1-\beta, 1-\mu\}}{4\max\{1, \|\mathcal{A}\|^2\}}$.

According to (1.3) and (3.11), we derive

$$
\begin{aligned}
\|x_{n+1} - z\|^2 &= \|x_n - z + \rho_n y_n\|^2 \\
&= \|x_n - z\|^2 - 2\rho_n\langle y_n, x_n - z\rangle + \rho_n^2\|y_n\|^2 \qquad (3.12)\\
&\leq \|x_n - z\|^2 - 2\rho_n\tau\|y_n\|^2 + \rho_n^2\|y_n\|^2.
\end{aligned}
$$

Since S and T are L-Lipschitzian, we have

$$
\begin{aligned}
\|y_n\| &= \|x_n - \mathcal{S}x_n + \mathcal{A}^*(I-\mathcal{T})\mathcal{A}x_n - [z - \mathcal{S}z + \mathcal{A}^*(I-\mathcal{T})\mathcal{A}z]\| \\
&\leq \|(I-\mathcal{S})x_n - (I-\mathcal{S})z\| + \|\mathcal{A}^*(I-\mathcal{T})\mathcal{A}x_n - \mathcal{A}^*(I-\mathcal{T})\mathcal{A}z\| \qquad (3.13)\\
&\leq (L+1)\|x_n - z\| + (L+1)\|\mathcal{A}\|^2\|x_n - z\| \\
&= (L+1)(1+\|\mathcal{A}\|^2)\|x_n - z\|.
\end{aligned}
$$

By (3.12) and (3.13), we get

$$
\begin{aligned}
\|x_{n+1} - z\|^2 &\leq \|x_n - z\|^2 - 2\rho_n\tau\|y_n\|^2 + \rho_n^2\|y_n\|^2 \\
&\leq \|x_n - z\|^2 + \rho_n^2(L+1)^2(1+\|\mathcal{A}\|^2)^2\|x_n - z\|^2 - 2\rho_n\tau\|y_n\|^2 \qquad (3.14)\\
&\leq \|x_n - z\|^2 + \rho_n^2(L+1)^2(1+\|\mathcal{A}\|^2)^2\|x_n - z\|^2.
\end{aligned}
$$

Noting that $\sum_{n=0}^{\infty}\rho_n^2 < \infty$, from Lemma 2.2 and (3.14), we deduce $\lim_{n\to\infty}\|x_n - z\|$ exists. Hence, $\{x_n\}$ is bounded and so is $\{y_n\}$.

Next, we show that every weak cluster point of the sequence $\{x_n\}$ belongs to the solution set of problem (1.1), i.e., $\omega_w(x_n) \subset \Omega$.

From (3.12), we obtain

$$
\begin{aligned}
2\tau\rho_n\|y_n\|^2 &\leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + \rho_n^2\|y_n\|^2 \\
&\leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + \rho_n^2 M,
\end{aligned}
$$

where $M = \sup_n\{\|y_n\|^2\}$.

An induction induces that

$$2\tau \sum_{n=0}^{\infty} \rho_n \|y_n\|^2 \leq \|x_0 - z\|^2 + M \sum_{n=0}^{\infty} \rho_n^2 < \infty, \tag{3.15}$$

which implies that

$$\liminf_{n\to\infty} \|y_n\| = 0 \tag{3.16}$$

due to the fact that $\sum_{n=0}^{\infty} \rho_n = \infty$.

Observe that

$$\begin{aligned} \|y_{n+1} - y_n\| &= \|x_{n+1} - \mathcal{S}x_{n+1} + \mathcal{A}^*(I - \mathcal{T})\mathcal{A}x_{n+1} \\ &\quad - [x_n - \mathcal{S}x_n + \mathcal{A}^*(I - \mathcal{T})\mathcal{A}x_n]\| \\ &\leq \|x_{n+1} - \mathcal{S}x_{n+1} - [x_n - \mathcal{S}x_n]\| \\ &\quad + \|\mathcal{A}^*(I - \mathcal{T})\mathcal{A}\S_{n+1} - \mathcal{A}^*(I - \mathcal{T})\mathcal{A}x_n\| \\ &\leq (L+1)\|x_{n+1} - x_n\| + (L+1)\|\mathcal{A}\|^2\|x_{n+1} - x_n\| \\ &= (L+1)(1 + \|\mathcal{A}\|^2)\|x_{n+1} - x_n\| \\ &= (L+1)(1 + \|\mathcal{A}\|^2)\rho_n\|y_n\|. \end{aligned} \tag{3.17}$$

It follows that

$$\begin{aligned} \|y_{n+1}\|^2 &= \|y_n\|^2 + 2\langle y_n, y_{n+1} - y_n\rangle + \|y_{n+1} - y_n\|^2 \\ &\leq \|y_n\|^2 + 2\|y_n\|\|y_{n+1} - y_n\| + \|y_{n+1} - y_n\|^2 \\ &\leq \|y_n\|^2 + 2(L+1)(1 + \|\mathcal{A}\|^2)\rho_n\|y_n\|^2 + (L+1)^2(1 + \|\mathcal{A}\|^2)^2\rho_n^2\|y_n\|^2. \end{aligned} \tag{3.18}$$

From (3.15), we know that $\sum_{n=0}^{\infty} \rho_n\|y_n\|^2 < \infty$. At the same time, $\sum_{n=0}^{\infty} \rho_n^2 < \infty$ by the assumption. Thus, we can apply Lemma 2.2 to (3.18) to deduce $\lim_{n\to\infty} \|y_n\|$ exists. This together with (3.16) implies that

$$\lim_{n\to\infty} \|y_n\| = \lim_{n\to\infty} \|x_n - \mathcal{S}x_n + \mathcal{A}^*(I - \mathcal{T})\mathcal{A}x_n\| = 0. \tag{3.19}$$

Observe that

$$\begin{aligned} &\frac{1-\beta}{2}\|x_n - \mathcal{S}x_n\|^2 + \frac{1-\mu}{2}\|(I - \mathcal{T})\mathcal{A}x_n\|^2 \\ &\leq \langle x_n - \mathcal{S}x_n, x_n - z\rangle + \langle \mathcal{A}^*(I - \mathcal{T})\mathcal{A}x_n, x_n - z\rangle \\ &= \langle x_n - \mathcal{S}x_n + \mathcal{A}^*(I - \mathcal{T})\mathcal{A}x_n, x_n - z\rangle \\ &\leq \|x_n - \mathcal{S}x_n + \mathcal{A}^*(I - \mathcal{T})\mathcal{A}x_n\|\|x_n - z\|. \end{aligned} \tag{3.20}$$

In view of (3.19) and (3.20), we have

$$\lim_{n\to\infty} \|x_n - \mathcal{S}x_n\| = 0 \text{ and } \lim_{n\to\infty} \|(I - \mathcal{T})\mathcal{A}x_n\| = 0. \tag{3.21}$$

By the demiclosedness (at zero) of $I - \mathcal{S}$ and $I - \mathcal{T}$, we deduce immediately $\omega_w(x_n) \subset \Omega$. To this end, the conditions of Lemma 2.1 are all satisfied. Consequently, $x_n \rightharpoonup z^* = \lim_{n\to\infty} \mathrm{proj}_\Omega(x_n)$. The proof is completed. □

Since the demicontractive operators include the directed operators and the firmly-nonexpansive mappings operators as special cases, we have immediately the following corollaries.

Corollary 3.1. *Assume the following conditions are satisfied*

(i) $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *are two real Hilbert spaces;*

(ii) $\mathcal{S}\colon \mathcal{H}_1 \to \mathcal{H}_1$ *and* $\mathcal{T} : \mathcal{H}_2 \to \mathcal{H}_2$ *are two directed operators;*

(iii) $\mathcal{S}\colon \mathcal{H}_1 \to \mathcal{H}_1$ *and* $\mathcal{T}: \mathcal{H}_2 \to \mathcal{H}_2$ *are Lipschitz continuous with Lipschitz constant* $L > 1$*;*
(iv) $I - \mathcal{S}$ *and* $I - \mathcal{T}$ *are demiclosed at zero;*
(v) $\sum_{n=0}^{\infty} \rho = \infty$ *and* $\sum_{n=0}^{\infty} \rho_n^2 < \infty$*.*

Then the sequence $\{x_n\}$ *generated by Algorithm 1.1 converges weakly to a solution* z^* *of problem* (1.1)*, where* $z^* = \lim_{n\to\infty} \mathrm{proj}_{\Omega}(x_n)$*.*

Corollary 3.2. *Assume the following conditions are satisfied*

(i) $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *are two real Hilbert spaces;*
(ii) $\mathcal{S}\colon \mathcal{H}_1 \to \mathcal{H}_1$ *and* $\mathcal{T}\colon \mathcal{H}_2 \to \mathcal{H}_2$ *are two firmly-nonexpansive mappings;*
(iii) $\sum_{n=0}^{\infty} \rho_n = \infty$ *and* $\sum_{n=0}^{\infty} \rho_n^2 < \infty$*.*

Then the sequence $\{x_n\}$ *generated by Algorithm 1.1 converges weakly to a solution* z^* *of problem* (1.1)*, where* $z^* = \lim_{n\to\infty} \mathrm{proj}_{\Omega}(x_n)$*.*

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