

Bounds for the skew Laplacian spectral radius of oriented graphs

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ABSTRACT. We consider the skew Laplacian matrix of a digraph \vec{G} obtained by giving an arbitrary direction to the edges of a graph G having n vertices and m edges. We obtain an upper bound for the skew Laplacian spectral radius in terms of the adjacency and the signless Laplacian spectral radius of the underlying graph G . We also obtain upper bounds for the skew Laplacian spectral radius and skew spectral radius, in terms of various parameters associated with the structure of the digraph \vec{G} and characterize the extremal graphs.

1. INTRODUCTION

Consider a simple graph G with n vertices and m edges and having the vertex set $V = \{v_1, v_2, \dots, v_n\}$. Let \vec{G} be a digraph obtained by assigning arbitrarily a direction to each of the edges of G . The digraph \vec{G} is called an orientation of G or oriented graph corresponding to G . Also the graph G is called the underlying graph of \vec{G} . Let $d_i^+ = d^+(v_i)$, $d_i^- = d^-(v_i)$ and $d_i = d_i^+ + d_i^-$, $i = 1, 2, \dots, n$ be respectively the out-degree, in-degree and degree of the vertices of \vec{G} . The out-adjacency matrix of the digraph \vec{G} is the $n \times n$ matrix $A^+ = A^+(\vec{G}) = (a_{ij})$, where $a_{ij} = 1$, if (v_i, v_j) is an arc and $a_{ij} = 0$, otherwise. The in-adjacency matrix of the digraph \vec{G} is the $n \times n$ matrix $A^- = A^-(\vec{G}) = (a_{ij})$, where $a_{ij} = 1$, if (v_j, v_i) is an arc and $a_{ij} = 0$, otherwise. We note that $A^- = (A^+)^t$. The skew adjacency matrix of a digraph \vec{G} is the $n \times n$ matrix $S = S(\vec{G}) = (s_{ij})$, where

$$s_{ij} = \begin{cases} 1, & \text{if there is an arc from } v_i \text{ to } v_j, \\ -1, & \text{if there is an arc from } v_j \text{ to } v_i, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly $S(\vec{G})$ is a skew symmetric matrix, so all its eigenvalues are zero or purely imaginary. For recent developments on the theory of skew spectrum, we refer to the papers [2, 14, 16, 23, 26, 28]. Let $D^+ = D^+(\vec{G}) = \text{diag}(d_1^+, d_2^+, \dots, d_n^+)$, $D^- = D^-(\vec{G}) = \text{diag}(d_1^-, d_2^-, \dots, d_n^-)$ and $D(\vec{G}) = \text{diag}(d_1, d_2, \dots, d_n)$ be respectively the diagonal matrix of vertex out-degrees, vertex in-degrees and vertex degrees of \vec{G} . Further, let A^+ and A^- be respectively the out-adjacency and in-adjacency matrix of a digraph \vec{G} . If $S(\vec{G})$ is the skew adjacency matrix of \vec{G} and $A(G)$ is the adjacency matrix of the underlying graph G of the digraph \vec{G} , clearly $A(G) = A^+ + A^-$ and $S(\vec{G}) = A^+ - A^-$. Analogous to the definition of Laplacian matrix of a graph, Cai et al. [4] called the matrix $\tilde{S}L(\vec{G}) = \tilde{D}(\vec{G}) - S(\vec{G})$, where $\tilde{D}(\vec{G}) = D^+(\vec{G}) - D^-(\vec{G})$, as the *skew Laplacian matrix* of the digraph

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\vec{G} . Clearly the matrix $\widetilde{SL}(\vec{G})$ is not symmetric and so its eigenvalues need not be real. The characteristic polynomial

$$P_{sl}(\vec{G}, x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n,$$

of the matrix $\widetilde{SL}(\vec{G})$ is called the *skew Laplacian characteristic polynomial* of the digraph \vec{G} . The zeros of the polynomial $P_{sl}(\vec{G}, x)$, that is, the eigenvalues of the matrix $\widetilde{SL}(\vec{G})$ are the skew Laplacian eigenvalues of the digraph \vec{G} and are denoted by $\nu_1, \nu_2, \dots, \nu_n$. The skew Laplacian spectrum of the digraph \vec{G} is denoted by $\text{Spect}_{sl}(\vec{G})$. The sign of the even cycle $C_k = u_1u_2 \dots u_ku_1$, denoted by $\text{sgn}(C_k)$, is defined as $\text{sgn}(C_k) = s_{12}s_{23} \dots s_{k-1k}s_{k1}$. An even oriented cycle C_k is called *evenly-oriented* (oddly-oriented) if its sign is positive (negative). If every even cycle in \vec{G} is evenly-oriented, \vec{G} is called *evenly-oriented*. An even oriented cycle C_{2k} is said to be *uniformly oriented* if $\text{sgn}(C_{2k}) = (-1)^k$. The following observations are immediate from the definition of \widetilde{SL} .

Theorem 1.1. [4]

- (i) If $\nu_1, \nu_2, \dots, \nu_n$ are the eigenvalues of $\widetilde{SL}(\vec{G})$, then $\sum_{i=1}^n \nu_i = 0$.
- (ii) 0 is an eigenvalue of $\widetilde{SL}(\vec{G})$ with multiplicity at least p , where p is the number of components of \vec{G} with all ones vector $(1, 1, \dots, 1)$ as the corresponding eigenvector.
- (iii) If $P_{sl}(\vec{G}, x) = x^n + \sum_{i=1}^n a_i x^{n-i}$ is the skew Laplacian characteristic polynomial of the digraph \vec{G} , then $a_1 = 0$, $a_2 = m + \sum_{i < j} (d_i^+ - d_i^-)(d_j^+ - d_j^-)$, $a_n = 0$.

Evidently a good amount of research work has been done on spectral theory of skew matrices of oriented graphs, see [16], but the work on the skew Laplacian spectrum of a digraph \vec{G} has been recently started and it will be of interest to develop the theory in this direction. For some recent work, see [10, 12] and the references therein. Although the skew Laplacian matrix of a digraph was so defined that it uses the structure of the digraph and at the same time enjoys the same characteristics as possessed by the Laplacian matrix of a graph, it seems that the definition of \widetilde{SL} uses the structure of the digraph, but not all the properties of $L(G)$ are possessed by \widetilde{SL} . It is well-known that 0 is an eigenvalue of $L(G)$ with multiplicity equal to the number of components of G . In fact, the eigenvalue 0 in the spectrum of $L(G)$ decides the connectedness of the graph G . This need not be true for the matrix \widetilde{SL} , as is clear from the following observation, the proof of which follows from Theorem 2.1 in [27].

Theorem 1.2. Let G be a bipartite graph and let \vec{G} be the corresponding digraph of G . If \vec{G} is an Eulerian digraph such that each even cycle of G is oriented uniformly in \vec{G} , then the multiplicity of 0 in the spectrum of \widetilde{SL} is same as the multiplicity of 0 in the spectrum of $A(G)$.

As usual, we denote the complete graph on n vertices by K_n , the complete bipartite graph on $s + t$ vertices by $K_{s,t}$, the cycle on n vertices by C_n . For other undefined notations and terminology from graphs and spectral graph theory, the readers are referred to [6, 22]. Let $K_{r,s}$ be the complete bipartite graph with both r and s even. Orient the edges of $K_{r,s}$ in such a way that in the resulting digraph \vec{G} all the even cycles are oriented uniformly. Since 0 is an adjacency eigenvalue of $K_{r,s}$ of multiplicity $r + s - 2$, from Theorem 1.2, it follows that 0 is the skew Laplacian eigenvalue of \vec{G} of multiplicity $r + s - 2$.

2. Skew Laplacian spectrum of an oriented graph

Let \widetilde{SL} be the skew Laplacian matrix of the digraph \vec{G} . If we reverse the direction of all the edges of \vec{G} , we obtain a new digraph \overleftarrow{G} , which we call the *converse digraph* of \vec{G} . Clearly $-\widetilde{SL}$ is the skew Laplacian matrix of \overleftarrow{G} . Therefore, we have the following observation.

Theorem 2.3. *If \overleftarrow{G} is the converse digraph of the digraph \vec{G} , then $\text{Spect}_{sl}(\overleftarrow{G}) = -\text{Spect}_{sl}(\vec{G})$.*

Let \vec{H} be an induced subdigraph of \vec{G} corresponding to the induced subgraph H of G and let $\vec{H}^* = \vec{H} \cup (n - n(H))K_1$, that is, \vec{H} together with $n - n(H)$ isolated vertices. Let $\vec{G} - E(\vec{H})$ be the subdigraph obtained by removing the arcs of \vec{H} in \vec{G} and $\vec{G} - \vec{H}$ be the subdigraph obtained by deleting the vertices of \vec{H} and the arcs incident at the vertices of \vec{H} . With out loss of generality, we can choose a labelling of the vertices of \vec{G} , so that

$$S(\vec{G}) = \begin{pmatrix} S(\vec{H}^*) & X \\ -X^t & S(\vec{G} - \vec{H}) \end{pmatrix} = S(\vec{H}^*) + S(\vec{G} - E(\vec{H})),$$

$$\text{and } \tilde{D}(\vec{G}) = \tilde{D}(\vec{H}^*) + \tilde{D}(\vec{G} - E(\vec{H})),$$

where X corresponds to the arcs connecting \vec{H} and $\vec{G} - \vec{H}$. Therefore,

$$\widetilde{SL}(\vec{G}) = \tilde{D} - S(\vec{G}) = \left(\tilde{D}(\vec{H}^*) - S(\vec{H}^*) \right) + \left(\tilde{D}(\vec{G} - E(\vec{H})) - S(\vec{G} - E(\vec{H})) \right).$$

Suppose both \vec{H} and $\vec{G} - \vec{H}$ are Eulerian subdigraphs of \vec{G} . Let \vec{G}_1 be the digraph obtained from \vec{G} by reversing the direction of all the arcs in \vec{H} and keeping the other arcs unchanged. We have

$$\begin{aligned} \widetilde{SL}(\vec{G}_1) &= \left(\tilde{D}(\overleftarrow{H}^*) - S(\overleftarrow{H}^*) \right) + \left(\tilde{D}(\vec{G} - E(\vec{H})) - S(\vec{G} - E(\vec{H})) \right) \\ &= \left(\tilde{D}(\vec{H}^*) + S(\vec{H}^*) \right) + \left(\tilde{D}(\vec{G} - E(\vec{H})) - S(\vec{G} - E(\vec{H})) \right), \end{aligned}$$

as $-S(\vec{H}^*) = S(\overleftarrow{H}^*)$ and $\tilde{D}(\overleftarrow{H}^*) = \tilde{D}(\vec{H}^*)$. The last equality is due to the fact that the only non-zero contribution to the $(i, i)^{th}$ element $d_i^+ - d_i^-$ of the matrix $\tilde{D}(\vec{H}^*)$ is due to the arcs connecting the vertices in \vec{H} and $\vec{G} - \vec{H}$. Let \vec{G}_2 be the digraph obtained from \vec{G} by reversing the direction of all arcs in $\vec{G} - E(\vec{H})$ and keeping other arcs unchanged. Since $-S(\vec{G} - E(\vec{H})) = S(\overleftarrow{G} - E(\overleftarrow{H}))$ and the only non-zero contribution to the $(i, i)^{th}$ element $d_i^+ - d_i^-$ of the matrix $\tilde{D}(\overleftarrow{G} - E(\overleftarrow{H}))$ is due to the arcs connecting the vertices in \vec{H} and $\vec{G} - \vec{H}$, it follows that

$$\begin{aligned} \widetilde{SL}(\vec{G}_2) &= \left(-\tilde{D}(\vec{H}^*) - S(\vec{H}^*) \right) + \left(\tilde{D}(\overleftarrow{G} - E(\overleftarrow{H})) - S(\overleftarrow{G} - E(\overleftarrow{H})) \right) \\ &= \left(-\tilde{D}(\vec{H}^*) - S(\vec{H}^*) \right) + \left(-\tilde{D}(\vec{G} - E(\vec{H})) + S(\vec{G} - E(\vec{H})) \right) \\ &= -\left[\left(\tilde{D}(\vec{H}^*) + S(\vec{H}^*) \right) + \left(\tilde{D}(\vec{G} - E(\vec{H})) - S(\vec{G} - E(\vec{H})) \right) \right] \\ &= -\widetilde{SL}(\vec{G}_1). \end{aligned}$$

Therefore, it follows that the skew Laplacian spectrum of \vec{G}_2 is negative of the skew Laplacian spectrum of \vec{G}_1 .

Again, if both \vec{H} and $\vec{G} - \vec{H}$ are Eulerian subdigraphs of \vec{G} , and \vec{G}_3 is the digraph

obtained from \vec{G} by reversing the direction of the arcs having one end in \vec{H} and other end in $\vec{G} - \vec{H}$, then proceeding similarly as above, it can be seen that $\widetilde{SL}(\vec{G}_3) = -\widetilde{SL}(\vec{G}_4)$, where \vec{G}_4 is the digraph obtained from \vec{G} by reversing the direction of arcs in both \vec{H} and $\vec{G} - \vec{H}$ and keeping other arcs unchanged. From this, it follows that the skew Laplacian spectrum of \vec{G}_4 is negative of the skew Laplacian spectrum of \vec{G}_3 . Thus, we have proved the following.

Theorem 2.4. *Let \vec{G} be an orientation of a graph G and let \vec{H} be an induced subdigraph of \vec{G} corresponding to the subgraph H of G . If the subdigraphs \vec{H} and $\vec{G} - \vec{H}$ of \vec{G} are Eulerian, then*

$$(i) \text{Spect}_{sl}(\vec{G}_1) = -\text{Spect}_{sl}(\vec{G}_2), \quad (ii) \text{Spect}_{sl}(\vec{G}_3) = -\text{Spect}_{sl}(\vec{G}_4),$$

where $\vec{G}_1, \vec{G}_2, \vec{G}_3$ and \vec{G}_4 are the digraphs defined above.

If \vec{G} is itself an Eulerian digraph, the conclusion of Theorem 2.4 holds for all induced subdigraphs. A subset W of the vertex set $V(\vec{G})$ is said to be independent if the induced subdigraph $\langle W \rangle$ is an empty digraph. In other words, W is an independent subset of $V(\vec{G})$ if the vertices in W are mutually non-adjacent. We have the following observation.

Theorem 2.5. *Let \vec{G} be an orientation of a graph G and let \vec{H} be an induced subdigraph of \vec{G} corresponding to the subgraph H of G . If the subdigraph \vec{H} is Eulerian and the subdigraph $\vec{G} - \vec{H}$ is independent, then*

$$(i) \text{Spect}_{sl}(\vec{G}_1) = -\text{Spect}_{sl}(\vec{G}_2), \quad (ii) \text{Spect}_{sl}(\vec{G}_3) = -\text{Spect}_{sl}(\vec{G}_4),$$

where $\vec{G}_1, \vec{G}_2, \vec{G}_3$ and \vec{G}_4 are the digraphs defined as above.

The skew Laplacian spectral radius of the digraph \vec{G} is denoted by $\rho_{sl}(\vec{G})$ and is defined as

$$\rho_{sl}(\vec{G}) = \max_i \{|\nu_i| : i = 1, 2, \dots, n\}.$$

The singular values of a square matrix X of order n are defined as the positive square roots of the eigenvalues of the matrix X^*X . If $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n$ are the singular values and $|\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_n|$ are the absolute values of the eigenvalues of X , then it is well known [15] that $|\lambda_1| \leq \sigma_1$, with equality if and only if X is a normal matrix (a matrix X is said to be normal if $XX^* = X^*X$). This observation implies that any upper bound for the largest singular value σ_1 gives an upper bound for the spectral radius.

In this paper, we obtain upper bounds for the skew Laplacian spectral radius of \vec{G} in terms of various parameters associated with the structure of the digraph \vec{G} and the underlying graph G . The spectral radius of the matrices has been discussed in general for all matrices real or complex. But when restricted to a special kind of a matrix associated to a graph or a digraph, it is always interesting to find estimates for the spectral radius in terms of the structure of the graph or the digraph. Also, when restricted to a particular class of graphs or digraphs, it is of interest to characterize the graphs or digraphs which attain the extremal values in that class. A reasonable amount of work has been done in these directions and various research articles can be found in the literature regarding the spectral radius of a graph with respect to different matrices, like adjacency matrix $A(G)$, Laplacian matrix $L(G)$, signless Laplacian matrix $Q(G)$, distance matrix, etc, for example, see the recent articles [3, 8, 9, 11, 21]. Similarly, various papers can be found in the literature regarding the spectral radius of a digraph with respect to different matrices like adjacency matrix $A(\vec{G})$, Laplacian matrix $L(\vec{G})$, skew matrix $S(\vec{G})$, etc, associated

to the digraph \vec{G} . Here, we consider spectral radius of the digraph \vec{G} with respect to its skew Laplacian matrix $\widetilde{SL}(\vec{G})$. The following observation is due to Perron and Frobenius [15].

Lemma 2.1. *Let $S = (s_{ij})$ be a complex matrix and X be an irreducible matrix of the same order. Let $|S|$ denote the matrix whose (i, j) -entry is $|s_{ij}|$. If $|S| \leq X$ and S has t as an eigenvalue, then $|t| \leq \lambda_1(X)$. If the equality holds, then $|S| = X$, and there is a diagonal matrix E with diagonal entries of absolute value 1 and a constant c of absolute value 1, such that $S = cEXE^{-1}$.*

Now, we obtain an upper bound for $\rho_{sl}(\vec{G})$ in terms of the adjacency spectral radius and signless Laplacian spectral radius of the underlying graph G .

Theorem 2.6. *Let \vec{G} be an orientation of a connected graph G of order n and let $\rho_{sl}(\vec{G})$ be the skew Laplacian spectral radius of \vec{G} . Let λ_1 and q_1 be respectively the largest adjacency eigenvalue and the largest signless Laplacian eigenvalue of the graph G . Then*

$$\rho_{sl}(\vec{G}) \leq \begin{cases} \lambda_1, & \text{if } \vec{G} \text{ is Eulerian} \\ q_1, & \text{if } \vec{G} \text{ is non-Eulerian.} \end{cases}$$

If \vec{G} is Eulerian, equality occurs if and only if G is a bipartite graph such that each even cycle of G is oriented uniformly in \vec{G} .

Proof. Let G be a connected graph of order n and let \vec{G} be an orientation of G . Further, let $Q(G) = L(G) + A(G)$ be the signless Laplacian matrix of the graph G and let $\widetilde{SL}(\vec{G})$ be the skew Laplacian matrix of the digraph \vec{G} . With out loss of generality, suppose that the vertices of \vec{G} and G are labelled in the same order. If \vec{G} is Eulerian, then $\widetilde{SL}(\vec{G}) = -S(\vec{G})$ and so $|\widetilde{SL}(\vec{G})| = |-S(G)| = A(G)$. Since the matrix $A(G)$ is irreducible, by Lemma 2.1, it follows that $\rho_{sl}(\vec{G}) \leq \lambda_1$. On the other hand, if \vec{G} is non-Eulerian, then for any orientation \vec{G} of G , we always have $|\widetilde{SL}(\vec{G})| \leq Q(G)$. Since the matrix $Q(G)$ is irreducible, by Lemma 2.1, it follows that $\rho_{sl}(\vec{G}) \leq q_1$.

If \vec{G} is Eulerian, then $\widetilde{SL}(\vec{G}) = -S(\vec{G})$ and so by Theorem 2.1 of [27] equality occurs if and only if G is a bipartite graph such that each even cycle of G is oriented uniformly in \vec{G} . On the other hand if \vec{G} is non-Eulerian, then G is either a bipartite graph or a non-bipartite graph. If G is a non-bipartite graph, the equality can never occur. This is due to the fact that for non-bipartite graphs the smallest eigenvalue for the matrix $Q(G)$ and so for the matrix $cEQ(G)E^{-1}$ is positive while as the smallest eigenvalue of $\widetilde{SL}(\vec{G})$ is always zero. So, assume that G is a bipartite graph and \vec{G} is a non-Eulerian graph. Let (V_1, V_2) be the bipartition of the vertex set $V(G)$ of the graph G . If we choose an orientation \vec{G} such that all the arcs are directed from V_1 to V_2 , then with out loss of generality we can label the vertices of \vec{G} so that its skew Laplacian matrix is $\widetilde{SL}(\vec{G}) = \begin{pmatrix} D_1 & -X \\ X^t & -D_2 \end{pmatrix}$, where $\tilde{D} = \begin{pmatrix} D_1 & 0 \\ 0 & -D_2 \end{pmatrix}$ and $S(G) = \begin{pmatrix} 0 & X \\ -X^t & 0 \end{pmatrix}$. Clearly $|\widetilde{SL}| = Q(G)$, but equality can not occur in this case. If $E = \text{diag}(\alpha_1, \alpha_2, \dots, \alpha_n)$, $|\alpha_i| = 1$, $Q(G) = (q_{ij})$, $\widetilde{SL}(\vec{G}) = (s_{ij})$, it can be seen that $cEQ(G)E^{-1} = \widetilde{SL}(\vec{G})$ gives $c\alpha_i\alpha_j^{-1}a_{ij} = s_{ij}$, for all $i, j = 1, 2, \dots, n$. Since $a_{ii} \neq 0$ and $s_{ii} \neq 0$, there is no such c , $|c| = 1$, for which this equality occurs. This completes the proof. \square

Remark 2.1. From Theorem 2.6, we observe that if the orientation \vec{G} is an Eulerian digraph, then any upper bound for $\lambda_1(G)$ gives an upper bound for $\rho_{sl}(\vec{G})$ and if the orientation \vec{G} is a non-Eulerian digraph, then any upper bound for $q_1(G)$ gives an upper bound for $\rho_{sl}(\vec{G})$. For example, if the digraph \vec{G} of the graph G is Eulerian, then $\rho_{sl}(\vec{G}) \leq \Delta = \max_i \{d_i^+ + d_i^-\}$, this is because $\lambda_1(G) \leq \Delta$. If \vec{G} is a non-Eulerian digraph, then $\rho_{sl}(\vec{G}) \leq \max_{v_i \sim v_j} \{d_i + d_j\}$, because $q_1(G) \leq \max_{v_i \sim v_j} \{d_i + d_j\}$.

The following observation is immediate from Theorem 2.6.

Corollary 2.1. *Among all the Eulerian digraphs, the bipartite Eulerian digraphs with each even cycle of G oriented uniformly in \vec{G} has the maximum skew Laplacian spectral radius.*

For Eulerian digraphs \vec{G} , since the skew Laplacian spectral radius $\rho_{sl}(\vec{G})$ and skew spectral radius $\rho_s(\vec{G})$ are same, we have the following observation, the proof of which follows from Theorem 2.1 of [27] and Corollary 2.1.

Corollary 2.2. *Among all the bipartite Eulerian digraphs, the complete bipartite digraph with partite sets of even cardinality and with each even cycle oriented uniformly has the maximum skew Laplacian spectral radius.*

It will be interesting to determine the oriented graphs among the oriented trees, oriented unicyclic graphs, oriented bicyclic graphs, oriented non-Eulerian graphs, etc, which attain the extremal values for the skew Laplacian spectral radius. Now, we obtain an upper bound for $\rho_{sl}(\vec{G})$ in terms of the out-degree d_i^+ and in-degree d_i^- of the vertex v_i of the digraph \vec{G} .

Theorem 2.7. *Let $\rho_{sl}(\vec{G})$ be the skew Laplacian spectral radius of the digraph \vec{G} . If d_i^+ and d_i^- are the out and in-degrees of the vertices of \vec{G} , then*

$$\rho_{sl}(\vec{G}) \leq \max_i \{|d_i^+ - d_i^-| + d_i\}.$$

If \vec{G} is Eulerian, equality occurs if and only if G is a regular bipartite graph such that each even cycle of G is oriented uniformly in \vec{G} .

Proof. It is well known [15] that the spectral radius $\rho(X)$ of an $n \times n$ matrix $X = (x_{ij})$ always satisfies

$$\rho(X) \leq \min_i \{R_i, C_i\},$$

where

$$R_i = \max_i \left\{ \sum_{k=1}^n |x_{ik}| : 1 \leq i \leq n \right\}$$

and

$$C_i = \max_i \left\{ \sum_{k=1}^n |x_{ki}| : 1 \leq i \leq n \right\}.$$

Taking $X = \widetilde{SL}(\vec{G})$ and using the fact that $R_i = C_i = \max_i \{|d_i^+ - d_i^-| + d_i\}$, the result follows. By Theorem 2.6, $\rho_{sl}(\vec{G}) = \lambda_1$, where λ_1 is the adjacency spectral radius of G , if and only if G is a bipartite graph such that each even cycle of G is oriented uniformly in \vec{G} . The result now follows by using the fact that the adjacency matrix of G is irreducible and non-negative. \square

If $\rho_s(\vec{G})$ is the skew spectral radius of the digraph \vec{G} , we have the following observation, the proof of which is similar to that of Theorem 2.7.

Theorem 2.8. *Let $\rho_s(\vec{G})$ be the skew spectral of the digraph \vec{G} . If d_i^+ and d_i^- , are the out and in-degrees of the vertices of \vec{G} , then*

$$\rho_s(\vec{G}) \leq \max_i \{d_i\}.$$

Equality occurs if and only if G is a regular bipartite graph such that each even cycle of G is oriented uniformly in \vec{G} .

For each vertex $v_i \in V(\vec{G})$, let $\alpha_i = d_i^+ - d_i^-$ be its oriented degree. Let t_i be the sum of the absolute values of the oriented degrees of the vertices which are adjacent to v_i , that is, $t_i = \sum_{v_j, v_i v_j \in E(G)} |\alpha_j|$. We call t_i the 2-oriented degree and $m(v_i) = \frac{t_i}{|\alpha_i|}$ as the average oriented degree of the vertex v_i . Let $m^+(v_i) = \frac{t_i^+}{d_i^+}$, where $t_i^+ = \sum_{v_j, v_i v_j \in E(G)} d_j^+$, be the average positive degree of the vertex v_i . The next result gives an upper bound for $\rho_{sl}(\vec{G})$ in terms of oriented degrees α_i and the average oriented degrees $m(v_i)$ of the vertices of the digraph \vec{G} .

Theorem 2.9. *Let \vec{G} be an orientation of a connected graph G . If α_i is the oriented degree and $m(v_i)$ is the average oriented degree of the vertex v_i , then*

$$\rho_{sl}(\vec{G}) \leq \begin{cases} \max_i \{|\alpha_i| + m(v_i)\}, & \text{if } \alpha_i \neq 0, \text{ for all } i \\ \max_i \{\delta_i^k\}, & \text{if } \alpha_i = 0, \text{ for } 1 \leq i \leq k \\ \max_i \{\delta_i\}, & \text{if } \alpha_i = 0, \text{ for } 1 \leq i \leq n, \end{cases}$$

where for $1 \leq i \leq k$, we have

$$\delta_i^k = \sum_{\substack{v_j, v_i v_j \in E(G) \\ j \leq k}} \beta_i^{-1} \beta_j + \sum_{\substack{v_j, v_i v_j \in E(G) \\ j \geq k+1}} \beta_i^{-1} |\alpha_j|,$$

and for $k+1 \leq i \leq n$, we have

$$\delta_i^k = \sum_{\substack{v_j, v_i v_j \in E(G) \\ j \leq k}} \beta_j |\alpha_i^{-1}| + \sum_{\substack{v_j, v_i v_j \in E(G) \\ j \geq k+1}} |\alpha_i^{-1} \alpha_j| + |\alpha_i|$$

and

$$\delta_i = \sum_{v_j, v_i v_j \in E(G)} \beta_i^{-1} \beta_j.$$

If \vec{G} is Eulerian, equality occurs if and only if G is a regular bipartite graph such that each even cycle of G is oriented uniformly in \vec{G} .

Proof. Let G be a connected graph of order n having m edges and let \vec{G} be an orientation of G . First suppose that the oriented degrees α_i of each of the vertices v_i of the orientation \vec{G} are non-zero. If $\tilde{D} = \text{diag}(\alpha_1, \alpha_2, \dots, \alpha_n)$, then $\tilde{D}^{-1} = \text{diag}(\alpha_1^{-1}, \alpha_2^{-1}, \dots, \alpha_n^{-1})$ exists. Consider the matrix $\tilde{D}^{-1} \tilde{S} \tilde{L} \tilde{D}$. It is easy to see that the $(i, i)^{\text{th}}$ entry of this matrix is $\alpha_i = d_i^+ - d_i^-$ and the $(i, j)^{\text{th}}$ entry is $(d_i^+ - d_i^-)^{-1} (d_j^+ - d_j^-) s_{ij}$, where $s_{ij} = -1$ or 0 or -1 . Using the fact that the matrices $\tilde{S} \tilde{L}$ and $\tilde{D}^{-1} \tilde{S} \tilde{L} \tilde{D}$ are similar, it follows that $\rho_{sl}(\vec{G}) = \rho(\tilde{D}^{-1} \tilde{S} \tilde{L} \tilde{D})$. Now, proceeding similarly as in Theorem 2.7, and by taking $X = \tilde{D}^{-1} \tilde{S} \tilde{L} \tilde{D}$, the result follows in this case.

Let some k , $1 \leq k \leq n-1$, vertices of \vec{G} have oriented degree zero. Without loss of

generality, suppose these vertices are v_1, v_2, \dots, v_k . Label the vertices of \vec{G} in such a way that the first k rows and columns of the matrix $\widetilde{S}L$ correspond to the vertices v_1, v_2, \dots, v_k . Let $D_1 = \text{diag}(\beta_1, \beta_2, \dots, \beta_k, \alpha_{k+1}, \dots, \alpha_n)$, where $\beta_i = \min_i\{d_i^+, d_i^-\}$. Clearly, the matrix $\widetilde{D}^{-1} = \text{diag}(\beta_1^{-1}, \beta_2^{-1}, \dots, \beta_k^{-1}, \alpha_{k+1}^{-1}, \dots, \alpha_n^{-1})$ exists. Consider the matrix $D_1^{-1}\widetilde{S}LD_1$. It is easy to see that the $(i, i)^{\text{th}}$ entry of this matrix is equal to 0, for $1 \leq i \leq k$, and equal to $\alpha_i = d_i^+ - d_i^-$, for $k+1 \leq i \leq n$. Also, for $1 \leq i \leq k$, its $(i, j)^{\text{th}}$ entry is equal to $\beta_i^{-1}\beta_j$, for $1 \leq j \leq k$ and equal to $\beta_i^{-1}(d_j^+ - d_j^-)s_{ij}$, for $k+1 \leq j \leq n$; and for $k+1 \leq i \leq n$ its $(i, j)^{\text{th}}$ entry is equal to $\beta_i(d_j^+ - d_j^-)^{-1}s_{ij}$, for $1 \leq j \leq k$ and equal to $(d_i^+ - d_i^-)^{-1}(d_j^+ - d_j^-)s_{ij}$, for $k+1 \leq j \leq n$, where $s_{ij} = -1$ or 0 or 1 . Since the matrices $\widetilde{S}L$ and $D_1^{-1}\widetilde{S}LD_1$ are similar, the result follows by taking $X = \widetilde{D}^{-1}\widetilde{S}L\widetilde{D}$, in Theorem 2.7.

Lastly, we suppose that the oriented degrees of all the vertices of \vec{G} are zero. Let $D_2 = \text{diag}(\beta_1, \beta_2, \dots, \beta_n)$. Clearly, the matrix $\widetilde{D}^{-1} = \text{diag}(\beta_1^{-1}, \beta_2^{-1}, \dots, \beta_n^{-1})$ exists. Now, consider the matrix $D_2^{-1}\widetilde{S}LD_2$ and proceeding similarly as above, it can be seen that the result follows in this case as well. If λ_1 is the adjacency spectral radius of G , then as shown in Theorem 2.6, $\rho_{sl}(\vec{G}) = \lambda_1$, if and only if G is a bipartite graph such that each even cycle of G is oriented uniformly in \vec{G} . Therefore, if G is a regular graph, equality follows by using the fact that the adjacency matrix of G is irreducible and non-negative. This completes the proof. \square

Proceeding similarly as in Theorem 2.9, we obtain the following upper bound for the skew spectral radius $\rho_s(\vec{G})$ of the digraph \vec{G} .

Theorem 2.10. *Let \vec{G} be an orientation of a connected graph G . Then*

$$\rho_s(\vec{G}) \leq \max_i \left\{ \sum_{v_j, v_i v_j \in E(G)} \beta_i^{-1} \beta_j \right\},$$

where $\beta_i = \min_i\{d_i^+, d_i^-\}$, if $d_i^+ d_i^- \neq 0$ and $\beta_i = \max_i\{d_i^+, d_i^-\}$, if $d_i^+ d_i^- = 0$. Equality occurs if and only if G is a regular bipartite graph such that each even cycle of G is oriented uniformly in \vec{G} .

The following observation can be found in [15].

Lemma 2.2. *Let A and B be square complex matrices of order n with singular values*

$$\sigma_1(A) \geq \dots \geq \sigma_n(A)$$

and

$$\sigma_1(B) \geq \dots \geq \sigma_n(B).$$

Then

$$(2.1) \quad \sigma_{i+j-1}(A+B) \leq \sigma_i(A) + \sigma_j(B),$$

where

$$\sigma_1(A+B) \geq \dots \geq \sigma_n(A+B)$$

are the singular values of $A+B$.

The following result gives a relation between $\rho_{sl}(\vec{G})$ and $\rho_s(\vec{G})$ for a digraph \vec{G} .

Theorem 2.11. *Let $\rho_{sl}(\vec{G})$ and $\rho_s(\vec{G})$ be respectively, the skew Laplacian spectral radius and the skew spectral radius of the digraph \vec{G} . Then*

$$\rho_{sl}(\vec{G}) \leq \max_i \{|d_i^+ - d_i^-|\} + \rho_s(\vec{G}),$$

with equality if \vec{G} is Eulerian.

Proof. Let G be a connected graph of order n having m edges and let \vec{G} be an orientation of G . Since $\widetilde{SL}(\vec{G}) = \tilde{D} - S(\vec{G})$, from Lemma 2.2, it follows that

$$(2.2) \quad \sigma_1(\widetilde{SL}) \leq \sigma_1(\tilde{D}) + \sigma_1(S(\vec{G})).$$

For any complex matrix X , it is well known [15], that the spectral radius $\rho(X)$ satisfies

$$\rho(X) \leq \sigma_1(X),$$

with equality if and only if X is a normal matrix. Since the matrices \tilde{D} and $S(\vec{G})$ are normal, from (1), it follows that

$$\rho(\widetilde{SL}) \leq \sigma_1(\widetilde{SL}) \leq \rho(\tilde{D}) + \rho(S(\vec{G})),$$

which implies that

$$\rho_{sl}(\vec{G}) \leq \max_i \{|d_i^+ - d_i^-|\} + \rho_s(\vec{G}).$$

If \vec{G} is an Eulerian digraph, then $\tilde{D} = 0$ and so

$$\widetilde{SL}(\vec{G}) = -S(\vec{G}).$$

Therefore equality occurs in this case. □

Remark 2.2. If \vec{G} is an Eulerian digraph, the upper bound given by Theorem 2.11, becomes $\rho_{sl}(\vec{G}) \leq \rho_s(\vec{G})$. Since $\rho_s(\vec{G}) \leq \lambda_1$, for Eulerian digraphs, it follows that the upper bound given by Theorem 2.11 is better than the upper bound given by Theorem 2.6.

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