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A sequence of positive linear operators related to powered Baskakov basis

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ABSTRACT. In this paper we study some approximation properties of a sequence of positive linear operators defined by means of the powered Baskakov basis. We prove that in the particular case of squared Baskakov basis the operators behave better than the classical Baskakov operators. For this particular case we give also a quantitative Voronovskaya type result.

1. INTRODUCTION

The operators defined by

$$V_n(f,x) = \sum_{k=0}^{\infty} v_{n,k}(x) f\left(\frac{k}{n}\right), \quad x \in [0,\infty),\ n = 1,2,\dots,$$

are called the Baskakov operators [2] and the functions $v_{n,k}$

$$v_{n,k}(x) = \binom{-n}{k} \cdot \frac{(-x)^k}{(1+x)^{n+k}} = \frac{n(n+1)\dots(n+k)}{(n+k)k!} \frac{x^k}{(1+x)^{n+k}}, \quad k = 0,1,\dots$$

form the Baskakov basis.

Motivated by [1], [4] and [6] we study the following operators

$$L_{n,r}(f,x) = \frac{\sum_{k=0}^{\infty} [v_{n,k}(x)]^r f\left(\frac{k}{n}\right)}{\sum_{k=0}^{\infty} [v_{n,k}(x)]^r}, \quad x \ge 0,\ n = 1,2,\dots \tag{1.1}$$

where r is a positive integer. For $r = 1$, because $\sum_{k=0}^{\infty} v_{n,k}(x) = 1$ we obtain the classical Baskakov operators.

Let us denote by $\psi_n^{[r]}$ the sum of the r-powered Baskakov functions

$$\psi_n^{[r]}(x) = \sum_{k=0}^{\infty} [v_{n,k}(x)]^r. \tag{1.2}$$

In this paper, we show that the study of the operators (1.1) is closely related to the study of the function (1.2). In the final part of the paper, we present some approximation properties of the operators $L_{n,r}$ for the particular case $r = 2$, including a quantitative Voronovskaya theorem.

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2. Some properties of the operators

Let us denote

$$\mu_{n,k}^{[r]}(x) = L_{n,r}((e_1 - x)^k, x), \quad k = 0, 1, 2, \ldots$$

the central moments of the operators $L_{n,r}$. Because $L_{n,r}$ preserve the constant functions we have $\mu_{n,0}^{[r]}(x) = 1$. There is a strong connection between the central moments of the operators, as the next two lemmas will reveal.

Lemma 2.1. *For every $x \geq 0$ and every $n \geq 1$ we have*

$$\mu_{n,r}^{[r]}(x) = -\left[1 - \left(\frac{x}{1+x}\right)^r\right]^{-1} \cdot \sum_{k=0}^{r-1} \binom{r}{k} x^{r-k} \left[1 - \left(\frac{x}{1+x}\right)^k\right] \mu_{n,k}^{[r]}(x). \tag{2.3}$$

Proof. From the relation

$$\begin{aligned}
\sum_{k=0}^{\infty} v_{n,k}^r(x) \cdot \frac{k^r}{n^r} &= \frac{1}{n^r(1+x)^{rn}} \sum_{k=1}^{\infty} \left(\frac{n(n+1)\ldots(n+k-1)}{(k-1)!}\right)^r \cdot \left(\frac{x}{1+x}\right)^{kr} \\
&= \frac{1}{n^r(1+x)^{rn}} \sum_{i=0}^{\infty} \left(\frac{n(n+1)\ldots(n+i)}{i!}\right)^r \cdot \left(\frac{x}{1+x}\right)^{(i+1)r} \\
&= \frac{x^r}{(1+x)^r} \sum_{i=0}^{\infty} \left(\frac{n(n+1)\ldots(n+i-1)}{i!} \frac{x^i}{(1+x)^{n+i}}\right)^r \left(1 + \frac{i}{n}\right)^r
\end{aligned}$$

we deduce that $L_{n,r}(e_r, x) = \frac{x^r}{(1+x)^r} L_{n,r}((1+e_1)^r, x)$, for every $x \geq 0$. Using this identity and the relations

$$L_{n,r}(e_r, x) = L_{n,r}((e_1 - x + x)^r, x) = \mu_{n,r}^{[r]}(x) + \sum_{k=0}^{r-1} \binom{r}{k} x^{r-k} \mu_{n,k}^{[r]}(x),$$

$$L_{n,r}((1+e_1)^r, x) = \mu_{n,r}^{[r]}(x) + \sum_{k=0}^{r-1} \binom{r}{k} (x+1)^{r-k} \mu_{n,k}^{[r]}(x),$$

the formula (2.3) is proved. □

Lemma 2.2. *For every $x > 0$ we have*

$$\left(\mu_{n,k}^{[r]}(x)\right)' + k \cdot \mu_{n,k-1}^{[r]}(x) = \frac{rn}{x(1+x)} \cdot \left[\mu_{n,k+1}^{[r]}(x) - \mu_{n,1}^{[r]}(x) \cdot \mu_{n,k}^{[r]}(x)\right]. \tag{2.4}$$

Proof. We use the equality $x(1+x)v_{n,k}'(x) = v_{n,k}(x)(k - nx)$ and we get

$$\begin{aligned}
\left(\frac{v_{n,k}^r(x)}{\sum_{i=0}^{\infty} v_{n,i}^r(x)}\right)' &= \frac{r v_{n,k}^{r-1}(x) v_{n,k}'(x)}{\sum_{i=0}^{\infty} v_{n,i}^r(x)} - \frac{r v_{n,k}^r(x) \sum_{i=0}^{\infty} v_{n,i}^{r-1}(x) v_{n,i}'(x)}{\left(\sum_{i=0}^{\infty} v_{n,i}^r(x)\right)^2} \\
&= \frac{r v_{n,k}^r(x)}{\sum_{i=0}^{\infty} v_{n,i}^r(x)} \cdot \left(\frac{k - nx}{x(1+x)} - \frac{\sum_{i=0}^{\infty} v_{n,i}^r(x) \frac{i-nx}{x(1+x)}}{\sum_{i=0}^{\infty} v_{n,i}^r(x)}\right) \\
&= \frac{r\,n}{x(1+x)} \cdot \frac{v_{n,k}^r(x)}{\sum_{i=0}^{\infty} v_{n,i}^r(x)} \cdot \left(\frac{k}{n} - \frac{\sum_{i=0}^{\infty} v_{n,i}^r(x) \frac{i}{n}}{\sum_{i=0}^{\infty} v_{n,i}^r(x)}\right).
\end{aligned}$$

We obtain

$$\begin{aligned}(L_{n,r}(f,x))' &= \frac{rn}{x(1+x)} \cdot L_{n,r}(f \cdot (e_1 - L_{n,r}(e_1,x)),x) \\ &= \frac{rn}{x(1+x)} \cdot [L_{n,r}(f \cdot (e_1 - x),x) + L_{n,r}(f \cdot (x - L_{n,r}(e_1,x)),x)] \\ &= \frac{rn}{x(1+x)} \cdot [L_{n,r}(f \cdot (e_1 - x),x) - L_{n,r}(e_1 - x,x) \cdot L_{n,r}(f,x)].\end{aligned}$$

We consider $f = (e_1 - x)^k$ in the last equality and we obtain (2.4). □

Remark 2.1. The recurrence relation (2.4) is similar to the relation (2.7) from [7] for the so called exponential type operators. Because the central moment of order 0 is known, we can use this relation to express every central moment only in terms of the first central moment $\mu_{n,1}^{[r]}$. Using the relation (2.3), we deduce that the first central moment verifies a differential equation of order $r-1$.

Lemma 2.3. *The following representation formula for the first central moment is true for every $x \geq 0$ and every $n \geq 1$*

$$\mu_{n,1}^{[r]}(x) = \frac{x(1+x)}{rn} \cdot \frac{\left(\psi_n^{[r]}(x)\right)'}{\psi_n^{[r]}(x)}. \tag{2.5}$$

Proof. Using again the well-known relation $x(1+x)v_{n,k}'(x) = v_{n,k}(x) \cdot (k - nx)$ we have

$$\begin{aligned}rn \cdot \mu_{n,1}^{[r]}(x) &= rn \left(\frac{\sum_{k=0}^{\infty} v_{n,k}^r(x) \cdot \frac{k}{n}}{\psi_n^{[r]}(x)} - x \right) = \frac{r \sum_{k=0}^{\infty} v_{n,k}^r(x)(k-nx)}{\psi_n^{[r]}(x)} \\ &= \frac{r\ x(1+x) \sum_{k=0}^{\infty} v_{n,k}^{r-1}(x) v_{n,k}'(x)}{\psi_n^{[r]}(x)} = \frac{x(1+x)\left(\psi_n^{[r]}(x)\right)'}{\psi_n^{[r]}(x)}.\end{aligned}$$

□

Remark 2.2. The first central moment is expressed using the function $\psi_n^{[r]}$. It is important to know as many properties as one can of this function since $\psi_n^{[r]}$ defines the first central moment, which defines the rest of the central moments. The function $\psi_n^{[r]}$ is conjectured in [1] to be completely monotonic. This function can be written in terms of a generalized hypergeometric function by

$$\psi_n^{[r]}(x) = \frac{1}{(1+x)^{rn}} \cdot {}_rF_{r-1}\left(n, \ldots, n; 1, \ldots, 1; \left(\frac{x}{1+x}\right)^r\right). \tag{2.6}$$

The function $\psi_n^{[r]}$ is also related to the Rényi entropy of order $r \geq 0$, $r \neq 1$, which is defined by the expression (see for example [10])

$$\mathcal{E}_{n,r}(x) = (1-r)^{-1} \cdot \ln \psi_n^{[r]}(x).$$

In [8] it is given the asymptotic expansion of the Rényi entropy of order r for a probability distribution.

$$\mathcal{E}_{n,r}(x) = \frac{1}{2}\ln n + \frac{1}{2}\ln(2\pi\sigma^2) + \frac{1}{2(r-1)}\ln r + \frac{1}{r}\left[\frac{\kappa_4(1-r)}{8\sigma^4} + \frac{\kappa_3^2(r-2)}{12\sigma^6}\right]\frac{1}{n} + \mathcal{O}\left(\frac{1}{n^2}\right).$$

In our case $\sigma^2 = V_1(e_2,x) - [V_1(e_1,x)]^2 = x(1+x)$. We deduce the following asymptotic expansion for the function $\psi_n^{[r]}$

$$\psi_n^{[r]}(x) \sim \frac{1}{\sqrt{r}} \cdot (2\pi n x(1+x))^{\frac{1-r}{2}} \cdot \exp\left(\frac{\theta_r(x)}{n} + \mathcal{O}\left(\frac{1}{n^2}\right)\right), \tag{2.7}$$

where θ_r is a function depending on x and r but not on n.

Lemma 2.4. *For any $x > 0$ we have*

$$\max_{k=0,1,\ldots} v_{n,k}(x) \sim \frac{1}{\sqrt{2\pi n x(1+x)}} \quad (n \to \infty). \tag{2.8}$$

Proof. Using Lemma 2.4 from [3] we know that $\max_{k=0,1,\ldots} v_{n,k}(x) = v_{n,[(n-1)x]}(x)$. Let $x > 0$. Denoting $n-1 = m$ and using Stirling's formula $n! = n^n e^{-n}\sqrt{2\pi n}\; e^{\frac{c_n}{12n}}$, with $c_n \in (0,1)$ we deduce that

$$\begin{aligned} v_{n,[(n-1)x]}(x) = v_{m+1,[mx]}(x) &= \frac{(m+[mx])!}{m!\,[mx]!} \cdot \left(\frac{x}{1+x}\right)^{[mx]} \cdot \frac{1}{(1+x)^{m+1}} \\ &= \frac{\sqrt{m+[mx]}}{(1+x)\sqrt{2\pi m\,[mx]}} \cdot \left(\frac{m+[mx]}{m(1+x)}\right)^m \cdot \left(\frac{(m+[mx])x}{[mx](1+x)}\right)^{[mx]} \cdot e^c, \end{aligned}$$

where $c = \frac{c_1}{12(m+[mx])} - \frac{c_2}{12m} - \frac{c_3}{12[mx]}$. Because

$$\frac{\sqrt{m+[mx]}}{(1+x)\sqrt{2\pi m\,[mx]}} \sim \frac{1}{\sqrt{2\pi m x(1+x)}} \quad (m \to \infty),$$

it remains to prove that

$$\lim_{m\to\infty} \left(\frac{m+[mx]}{m(1+x)}\right)^m \cdot \left(\frac{(m+[mx])x}{[mx](1+x)}\right)^{[mx]} = 1.$$

Denoting $\epsilon = mx - [mx] \in [0,1)$ the fractional part of mx and applying the logarithm

$$\ln\left[\left(\frac{m+[mx]}{m(1+x)}\right)^m \cdot \left(\frac{(m+[mx])x}{[mx](1+x)}\right)^{[mx]}\right] = m\ln\left(1 - \frac{\epsilon}{m(1+x)}\right) +$$

$$(mx-\epsilon)\ln\left(1 + \frac{\epsilon}{(mx-\epsilon)(1+x)}\right) = -\frac{m\epsilon}{m(1+x)} + \frac{(mx-\epsilon)\epsilon}{(mx-\epsilon)(1+x)} + \mathcal{O}(m^{-1}) = \mathcal{O}(m^{-1}).$$

□

Remark 2.3. Using (2.7) and (2.8) we obtain the following asymptotic result

$$\lim_{n\to\infty} \frac{\max_{k\ge 0}[v_{n,k}(x)]^{r-1}}{\psi_n^{[r]}(x)} = \sqrt{r}.$$

This suggests that for a fixed $x > 0$ and large n and for a positive function f the operator $L_{n,r}$ and the classical Baskakov operator V_n have the same behaviour

$$L_{n,r}(f,x) = \sum_{k=0}^{\infty} \frac{[v_{n,k}(x)]^{r-1}}{\psi_n^{[r]}(x)} \cdot v_{n,k}(x) f\left(\frac{k}{n}\right) \le (\sqrt{r}+\varepsilon) V_n(f,x).$$

We will prove in the next section that in the case $r = 2$ the operator $L_{n,r}$ behaves better from approximation point of view than the case $r = 1$.

3. The particular case $r = 2$

In the case $r = 2$ we can prove more results. The relation (2.3) becomes

$$\mu_{n,2}^{[2]}(x) = -\frac{2x(1+x)}{1+2x}\mu_{n,1}^{[2]}(x), \quad x \geq 0. \tag{3.9}$$

For $k = 1$ in (2.4) we obtain

$$\left(\mu_{n,1}^{[2]}(x)\right)' + 1 = \frac{2n}{x(1+x)} \cdot \left[\mu_{n,2}^{[2]}(x) - \mu_{n,1}^{[2]}(x) \cdot \mu_{n,1}^{[2]}(x)\right], \tag{3.10}$$

so the first central moment satisfies the following Riccati equation

$$\left(\mu_{n,1}^{[2]}(x)\right)' = -1 - \frac{4n}{1+2x}\mu_{n,1}^{[2]}(x) - \frac{2n}{x(1+x)}(\mu_{n,1}^{[2]}(x))^2. \tag{3.11}$$

Replacing (2.5) in (3.11) we deduce that $\psi_n^{[2]}(x)$ satisfies the following equation

$$x(1+x)(1+2x)y'' + [1 + 4(n+1)x(1+x)]y' + 2n(1+2x)y = 0,$$

which is equation (15) from [9].

Lemma 3.5. *We have for every $x \geq 0$*

$$\mu_{n,2}^{[2]}(x) \leq C \cdot \frac{x(1+x)}{n}, \text{ where } C \leq \frac{\pi^3}{32}. \tag{3.12}$$

Proof. Using the formula $\psi_n^{[2]}(x) = \frac{1}{\pi}\int_0^1 (t + (1-t)(1+2x)^2)^{-n} \left(t(1-t)\right)^{-1/2} \mathrm{d}t$ (see (10) from [9]) with the substitution $t = \cos^2 v$ we obtain

$$\psi_n^{[2]}(x) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} (1 + 4x(1+x)\sin^2 v)^{-n} \,\mathrm{d}v. \tag{3.13}$$

Using (3.9) and (2.5) the following representation holds true

$$\mu_{n,2}^{[2]}(x) = -\frac{x^2(1+x)^2}{n(1+2x)} \cdot \frac{\left(\psi_n^{[2]}(x)\right)'}{\psi_n^{[2]}(x)} = \frac{4x^2(1+x)^2 \int_0^{\frac{\pi}{2}} (1 + 4x(1+x)\sin^2 v)^{-n-1} \sin^2 v \,\mathrm{d}v}{\int_0^{\frac{\pi}{2}} (1 + 4x(1+x)\sin^2 v)^{-n} \,\mathrm{d}v}.$$

Let us denote $a = 4x(1+x) \geq 0$. We have to prove that

$$\frac{a\int_0^{\frac{\pi}{2}} (1 + a\sin^2 v)^{-n-1} \sin^2 v \,\mathrm{d}v}{\int_0^{\frac{\pi}{2}} (1 + a\sin^2 v)^{-n} \,\mathrm{d}v} \leq \frac{\pi^3}{32n}.$$

Using the inequalities $\frac{2v}{\pi} \leq \sin v \leq v$, for $v \in (0, \pi/2)$, and the substitution $u = 2v/\pi$

$$\int_0^{\frac{\pi}{2}} \frac{a\sin^2 v}{(1 + a\sin^2 v)^{n+1}}\mathrm{d}v \leq \int_0^{\frac{\pi}{2}} \frac{av^2}{(1 + \frac{4a}{\pi^2}v^2)^{n+1}}\mathrm{d}v = \frac{\pi^3}{8}\int_0^1 \frac{au^2}{(1 + au^2)^{n+1}}\mathrm{d}u.$$

Integrating by parts the last integral, we get

$$\int_0^1 u \cdot \frac{au}{(1+au^2)^{n+1}}\mathrm{d}u = \frac{1}{2n}\int_0^1 \frac{1}{(1+au^2)^n}\mathrm{d}u - \frac{1}{2n(1+a)^n} < \frac{1}{2n}\int_0^1 \frac{1}{(1+au^2)^n}\mathrm{d}u.$$

Using the inequality $1 \leq \frac{1}{2\sqrt{u(1-u)}}$, which holds true for every $u \in (0,1)$ and substituting $u = \sin^2 v$, we finally obtain

$$\frac{1}{2n}\int_0^1 \frac{1}{(1+au^2)^n}\mathrm{d}u < \frac{1}{4n}\int_0^1 \frac{1}{(1+au^2)^n}\frac{\mathrm{d}u}{\sqrt{u(1-u)}} = \frac{1}{4n}\int_0^{\frac{\pi}{2}} \frac{1}{(1+a\sin^2 v)^n}\,\mathrm{d}v.$$

□

Remark 3.4. Because $\mu_{n,2}^{[1]} = \frac{x(1+x)}{n}$ and $\pi^3 \approx 31$, inequality (3.12) shows that the second central moment of the operators $L_{n,2}$ is smaller than the corresponding one of the classical Baskakov operators. If we consider the following estimation in terms of the usual modulus of continuity

$$|L_{n,r}(f,x) - f(x)| \leq \left(1 + n\mu_{n,2}^{[r]}(x)\right) \cdot \omega\left(f, \frac{1}{\sqrt{n}}\right),$$

which is valid for every uniformly continuous function f, we deduce that the error by approximating f with $L_{n,2}f$ is smaller than the error of approximation by the classical Baskakov operators. The constant C from (3.12) is less than 1 and can be improved, but it cannot be less that $\frac{1}{2}$, as one can see from the next lemma.

Lemma 3.6. *For every x in $(0,\infty)$*

$$\lim_{n\to\infty} 4n \cdot \mu_{n,1}^{[2]}(x) = -(1+2x) \tag{3.14}$$

$$\lim_{n\to\infty} 2n \cdot \mu_{n,2}^{[2]}(x) = x(1+x). \tag{3.15}$$

Proof. Using the representation (3.13) it is not difficult to obtain that

$$\left(\psi_n^{[2]}(x)\right)' = -\frac{1+2x}{x(1+x)}\left[\psi_n^{[2]}(x) - \psi_{n+1}^{[2]}(x)\right]. \tag{3.16}$$

With this in (2.5) we get

$$\mu_{n,1}^{[2]}(x) = \frac{1+2x}{2}\left(\frac{\psi_{n+1}^{[2]}(x)}{\psi_n^{[2]}(x)} - 1\right) = \frac{1+2x}{2} \cdot H_n(x). \tag{3.17}$$

The asymptotic relation (2.7) gives us

$$H_n(x) = \frac{\psi_{n+1}^{[2]}(x)}{\psi_n^{[2]}(x)} - 1 = \left(1 + \frac{1}{n}\right)^{\frac{-1}{2}} e^{-\frac{\theta_2(x)}{n(n+1)} + \mathcal{O}\left(\frac{1}{n^3}\right)} - 1 = \frac{-1}{2n} + \frac{3/8 - \theta_2(x)}{n^2} + \mathcal{O}\left(\frac{1}{n^3}\right).$$

This proves (3.14). Using (3.9) the relation (3.15) is proved, too. □

Remark 3.5. It can be proved by induction that all the derivatives of $\mu_{n,1}^{[2]}$ satisfy

$$\lim_{n\to\infty} 4n \cdot \frac{\mathrm{d}^i}{\mathrm{d}x^i}\mu_{n,1}^{[2]}(x) = -\frac{\mathrm{d}^i}{\mathrm{d}x^i}(1+2x), \quad i = 0,1,2,\ldots \tag{3.18}$$

Using (3.17) it suffices to prove that $(nH_n^{(i)})$ tend to zero when n tends to infinity for all $i \in \mathbb{N}$. With the representation (3.16) we get

$$H_n'(x) = \frac{1+2x}{x(1+x)} \cdot [H_n(x) + 1] \cdot [H_{n+1}(x) - H_n(x)].$$

Because $n \cdot [H_{n+1}(x) - H_n(x)]$ tends to zero and all the higher derivatives of H_n can be expressed in terms of this forward difference, the relation (3.18) is proved.

Lemma 3.7. *For every $x > 0$*

$$\lim_{n\to\infty} (2n)^k \mu_{n,2k}^{[2]}(x) = 1 \cdot 3 \cdots (2k-1) \cdot [x(1+x)]^k, \quad k = 1,2,\ldots \tag{3.19}$$

Proof. We will prove this by induction. Actually, we will prove more than that. Denoting $g_k(x) = 1 \cdot 3 \cdots (2k-1) \cdot [x(1+x)]^k$ and $g_0(x) = 1$ we will prove that

$$\lim_{n\to\infty} (2n)^k \cdot \frac{\mathrm{d}^i}{\mathrm{d}x^i}\,\mu_{n,2k}^{[2]}(x) = \frac{\mathrm{d}^i}{\mathrm{d}x^i} g_k(x), \quad k = 0,1,2,\ldots \quad i = 0,1,2,\ldots \tag{3.20}$$

We will prove also that

$$\lim_{n\to\infty}(2n)^{k+1}\cdot\frac{\mathrm{d}^i}{\mathrm{d}x^i}\,\mu^{[2]}_{n,2k+1}(x)=\frac{\mathrm{d}^i}{\mathrm{d}x^i}\,h_k(x),\quad k=0,1,2,\dots\quad i=0,1,2,\dots \tag{3.21}$$

for some functions h_k (which can be given explicitly, but not useful in the sequel).

For $k=0$, the relation (3.21) holds true because of (3.18) and (3.20) because $\mu^{[2]}_{n,0}(x)=1$.

We use (2.4) for $r=2$. Using Leibniz rule, we apply the derivative i times to

$$2n\mu^{[2]}_{n,2k+2}(x)=2n\mu^{[2]}_{n,1}(x)\mu^{[2]}_{n,2k+1}(x)+x(1+x)\left(\mu^{[2]}_{n,2k+1}\right)'(x)+(2k+1)x(1+x)\mu^{[2]}_{n,2k}(x).$$

Then, we multiply the result with $(2n)^k$ and let n tend to infinity. We obtain

$$\lim_{n\to\infty}(2n)^{k+1}\frac{\mathrm{d}^i}{\mathrm{d}x^i}\,\mu^{[2]}_{n,2k+2}(x)=\frac{\mathrm{d}^i}{\mathrm{d}x^i}\left((2k+1)x(1+x)g_k(x)\right)=\frac{\mathrm{d}^i}{\mathrm{d}x^i}\,g_{k+1}(x).$$

Similarly, from

$$2n\mu^{[2]}_{n,2k+1}(x)=2n\mu^{[2]}_{n,1}(x)\mu^{[2]}_{n,2k}(x)+x(1+x)\left(\mu^{[2]}_{n,2k}\right)'(x)+2kx(1+x)\mu^{[2]}_{n,2k-1}(x)$$

we obtain $h_k(x)=h_0(x)g_k(x)+x(1+x)g_k'(x)+2kx(1+x)h_{k-1}(x)$. □

In order to give some approximation results for the operators $L_{n,2}$, let us denote by E_α, $\alpha\ge 0$, the space of all continuous functions $f:(0,\infty)\to\mathbb{R}$ with the property that exists a constant $M>0$ such that $|f(x)|\le Me^{\alpha x}$, for every $x>0$.

Let us observe that for n large the functions $L_{n,2}f$ exist for every $f\in E_\alpha$. We prove

Lemma 3.8. *The sequence $L_{n,2}(e^{\alpha t},x)$ converges pointwise to the function $e^{\alpha x}$.*

Proof. Using the definition, $L_{n,2}(e^{\alpha t},x)$ can be expressed as a quotient of two hypergeometric functions $L_{n,2}(e^{\alpha t},x)={}_2F_1(n,n;1;(\frac{x}{1+x})^2e^{\frac{\alpha}{n}})/{}_2F_1(n,n;1;(\frac{x}{1+x})^2)$. For $x>0$ we consider y_n such that $(\frac{x}{1+x})^2e^{\frac{\alpha}{n}}=(\frac{y_n}{1+y_n})^2$, i.e. $y_n=\frac{xe^{\frac{\alpha}{2n}}}{1+x-xe^{\frac{\alpha}{2n}}}\to x$. Using (2.6) we obtain

$$L_{n,2}(e^{\alpha t},x)=\frac{(1+y_n)^{2n}\psi^{[2]}_n(y_n)}{(1+x)^{2n}\psi^{[2]}_n(x)}.$$

Because

$$\frac{(1+y_n)^{2n}}{(1+x)^{2n}}=\left(1+x(1-e^{\frac{\alpha}{2n}})\right)^{-2n}\to e^{\alpha x}$$

it remains to prove that $\psi^{[2]}_n(y_n)/\psi^{[2]}_n(x)\to 1$. But this is true, because $\psi^{[2]}(x)$ is completely monotonic (see [1]) and using (2.5), (3.12) and (3.9) we have

$$\left|1-\frac{\psi^{[2]}_n(y_n)}{\psi^{[2]}_n(x)}\right|=\frac{\left(\psi^{[2]}_n\right)'(c)}{\psi^{[2]}_n(x)}(x-y_n)\le\frac{\left(\psi^{[2]}_n\right)'(x)}{\psi^{[2]}_n(x)}(x-y_n)\le\frac{\pi^3(1+2x)}{32x(1+x)}(y_n-x).$$

□

Remark 3.6. The Lemma 3.8 implies that for a fixed $x>0$ there is a constant $M_\alpha(x)$ not depending on n such that

$$L_{n,2}(\max(e^{\alpha t},e^{\alpha x}),x)\le M_\alpha(x),\quad n\in\mathbb{N}. \tag{3.22}$$

Indeed, for $x>0$, there is $n_0\in\mathbb{N}$ such that $|L_{n,2}(e^{\alpha t},x)-e^{\alpha x}|\le 1$, for every $n\ge n_0$. We obtain $L_{n,2}(\max(e^{\alpha t},e^{\alpha x}),x)\le L_{n,2}(e^{\alpha t}+e^{\alpha x},x)\le 1+2e^{\alpha x}$, for every $n\ge n_0$. For $n\le n_0$ we define $M_\alpha(x)$ to be the maximum of the values $L_{n,2}(\max(e^{\alpha t},e^{\alpha x}),x)$.

Theorem 3.1. *Let $f \in E_\alpha$ be such that f is twice continuously differentiable and $f'' \in E_\alpha$. Then*

$$\left| L_{n,2}(f,x) - f(x) - \mu_{n,1}(x) f'(x) - \frac{\mu_{n,2}(x)}{2} f''(x) \right| \le \frac{M(\alpha,x)}{n} \cdot \omega_\alpha \left(f'', \frac{1}{\sqrt{n}} \right),$$

for every $n \in \mathbb{N}$ and $x > 0$, where $M(\alpha,x)$ is a positive constant depending only on α and x.

Proof. Let $x > 0$ be fixed. We use Theorem 1 from [5] for $\rho(x) = \varphi(x) = x$ and $\delta_n = \frac{1}{\sqrt{n}}$ which states that

$$\left| L_{n,2}(f,x) - \sum_{k=0}^{2} \frac{f^{(k)}(x)}{k!} \cdot \mu_{n,k}(x) \right| \le \frac{1}{2} \left(A_{n,2}(x) + \frac{A_{n,3}(x)}{\delta_n} \right) \omega_\alpha \left(f'', \delta_n \right)$$

where $A_{n,k}(x) = L_n \left(\max\left(e^{\alpha t}, e^{\alpha x}\right) |t-x|^k, x \right)$ and $\omega_\alpha(f,\delta) = \sup\limits_{|t-x| \le \delta} \frac{|f(t)-f(x)|}{\max\left(e^{\alpha t}, e^{\alpha x}\right)}$.

Because of (3.19), is true that $(2n)^k \mu^{[2]}_{n,2k}(x) \le M_k(x)$, for some $M_k(x) > 0$. Using the Cauchy-Schwarz inequality for positive linear operators and condition (3.22)

$$A_{n,k}(x) \le \sqrt{L_{n,2}\left(\max\left(e^{2\alpha t}, e^{2\alpha x}\right), x\right)} \cdot \sqrt{L_{n,2}\left(|t-x|^{2k}, x\right)} \le \frac{\sqrt{M_{2\alpha}(x) M_k(x)}}{\sqrt{(2n)^k}}, \quad k = 2, 3.$$

This proves that $A_{n,2}(x) + \sqrt{n} A_{n,3}(x) \le \frac{M(\alpha,x)}{n}$, for every $n \in \mathbb{N}$, for some $M(\alpha,x) > 0$. □

Corollary 3.1. *For every $f \in E_\alpha$ such that $f'' \in E_\alpha$ and $g(x) = e^{-\alpha x} \cdot f''(x)$ is uniformly continuous on $(0,\infty)$ and for every $x > 0$ it holds true*

$$\lim_{n\to\infty} n[L_{n,2}(f,x) - f(x)] = -\frac{1+2x}{4} \cdot f'(x) + \frac{x(1+x)}{4} \cdot f''(x).$$

Conjecture 3.1. *In the same conditions as in the Corollary 3.1 we have*

$$\lim_{n\to\infty} n[L_{n,r}(f,x) - f(x)] = \frac{(1-r)(1+2x)}{2r} \cdot f'(x) + \frac{x(1+x)}{2r} \cdot f''(x).$$

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