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A new class of fractional type set-valued functions

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ABSTRACT. The so-called ratios of affine functions, introduced by Rothblum (1985) in the framework of finite-dimensional Euclidean spaces, represent a special class of fractional type vector-valued functions, which transform convex sets into convex sets. The aim of this paper is to show that a similar convexity preserving property holds within a new class of fractional type set-valued functions, acting between any real linear spaces.

1. INTRODUCTION

Several classes of fractional type real-valued functions, such as ratios between a convex function and a concave one (in particular, a quadratic function and an affine one, or two affine functions), are known to play an important role in scalar optimization. Also, vector-valued functions having fractional type scalar components have been studied intensively within vector optimization (see, e.g., Cambini and Martein [4], Göpfert *et al.* [7], Schaible [11] and Stancu-Minasian [12], and the references therein). It seems that, although the set-valued optimization is an important field (see, e.g., Khan, Tammer and Zălinescu [8]), only a few concepts of fractional type set-valued functions have been introduced so far in the literature (see, e.g., Bhatia and Mehra [2], or the recent paper by Das and Nahak [5]).

An interesting class of fractional type vector-valued functions has been introduced by Rothblum [10] within finite-dimensional Euclidean spaces. We present here a slightly modified version.

Definition 1.1. A vector-valued function $f : D \to \mathbb{R}^m$, defined on a nonempty convex set $D \subseteq \mathbb{R}^n$, is said to be a ratio of affine functions if there exist a vector-valued affine function $g : \mathbb{R}^n \to \mathbb{R}^m$ and a real-valued affine function $h : \mathbb{R}^n \to \mathbb{R}$, such that

$$D \subseteq \{x \in \mathbb{R}^n \mid h(x) > 0\}$$

and

$$f(x) = \frac{g(x)}{h(x)},\ \forall\, x \in D. \tag{1.1}$$

As shown in [10], these functions have several important properties, among which two are of special interest for our purposes:

(P1) $\operatorname{conv} f(S) = f(\operatorname{conv} S)$ for any set $S \subseteq D$;
(P2) $f(A)$ is convex for any convex set $A \subseteq D$.

The principal aim of our paper is to generalize (P1) and (P2) within a special class of fractional type set-valued functions, defined similarly to (1.1), this time by means of an appropriate concept of affine set-valued function brought in the literature by Tan [13].

In the preliminary Section 2 we recall a few notions of set-valued analysis and we state some useful properties of affine set-valued functions. Then, in Section 3 we introduce a new concept of set-valued ratio of affine functions and establish our main results.

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2. Preliminaries

For any real linear space E, we denote by $\mathcal{P}(E)$ the collection of all subsets of E. Then, for any $S, S' \in \mathcal{P}(E)$ and $\lambda \in \mathbb{R}$, we adopt the following notational conventions:

$$S + S' = \{x \in E \mid \exists\, (s, s') \in S \times S' \ : \ x = s + s'\};$$
$$\lambda S = \{x \in E \mid \exists\, s \in S \ : \ x = \lambda s\};$$
$$\frac{S}{\lambda} = \frac{1}{\lambda} S, \ \text{ whenever } \lambda \neq 0.$$

The convex hull of S is denoted by $\mathrm{conv} S$. In order to use convex combinations, in the sequel it will be convenient to consider, for any $k \in \mathbb{N} = \{1, 2, \dots\}$, the standard simplex of the Euclidean space $\mathbb{R}^k$, namely

$$\Delta_{k-1} = \{(t_1, ..., t_k) \in \mathbb{R}^k \mid t_1 + ... + t_k = 1, \ t_1, ..., t_k \geq 0\}.$$

In what follows we consider two real linear spaces X and Y. As usual in set-valued analysis (see, e.g., Aubin and Frankowska [1] or Berge [3]), for any set-valued function $F : X \to \mathcal{P}(Y)$ we denote by

$$\mathrm{dom}\, F = \{x \in X \mid F(x) \neq \emptyset\}$$

the domain of F. The image of any set $A \in \mathcal{P}(X)$ by F is defined as

$$F(A) = \bigcup_{x \in A} F(x).$$

Remark 2.1. Every vector-valued function $f : D \to Y$, defined on a nonempty set $D \subseteq X$, can be identified with a set-valued function $F : X \to \mathcal{P}(Y)$, given by

$$F(x) = \begin{cases} \{f(x)\} & \text{if } x \in D \\ \emptyset & \text{if } x \in X \setminus D. \end{cases} \tag{2.2}$$

It is easy to see that $\mathrm{dom}\, F = D$ and, for any set $A \in \mathcal{P}(X)$, we have

$$f(A) = \{f(x) \mid x \in A\} = F(A).$$

Among different notions of affine set-valued functions known in the literature (see, e.g., Deutsch and Singer [6] or Nikodem and Popa [9], and the references therein) the following one is appropriate for the purpose of our paper. It is a particular instance of the original definition proposed by Tan [13, Def. 2] (where the functions were defined on some affine subset of X, not necessarily the whole space X).

Definition 2.2. A set-valued function $G : X \to \mathcal{P}(Y)$ is said to be affine if

$$G(tx_1 + (1-t)x_2) = tG(x_1) + (1-t)G(x_2) \tag{2.3}$$

for all $x_1, x_2 \in X$ and $t \in \mathbb{R}$.

Remark 2.2. According to Nikodem and Popa [9, Prop. 2.11], if $G : X \to \mathcal{P}(Y)$ is a set-valued function such that $\mathrm{dom}\, G = X$, then G is affine if and only if

$$G(tx_1 + (1-t)x_2) \supseteq tG(x_1) + (1-t)G(x_2)$$

for all $x_1, x_2 \in X$ and $t \in \mathbb{R}$. In other words, if $\mathrm{dom}\, G = X$, then G is affine in the sense of Definition 2.2 if and only if G is affine in the sense of Deutsch and Singer [6, Def. 1.1].

Remark 2.3. It is easy to see that a set-valued function $G : X \to \mathcal{P}(Y)$ is affine if and only if for any $k \in \mathbb{N}$, $x_1, ..., x_k \in X$ and $t_1, ..., t_k \in \mathbb{R}$ with $t_1 + ... + t_k = 1$, we have

$$G(t_1 x_1 + ... + t_k x_k) = t_1 G(x_1) + ... + t_k G(x_k).$$

3. Ratios of affine functions

Definition 3.3. Let $D \subseteq X$ be a nonempty convex set. We say that $F : X \to \mathcal{P}(Y)$ is a set-valued ratio of affine functions with respect to D if there exist a set-valued affine function $G : X \to \mathcal{P}(Y)$ with $\operatorname{dom} G = X$ and a real-valued affine function $h : X \to \mathbb{R}$, such that

$$D \subseteq \{x \in X \mid h(x) > 0\} \tag{3.4}$$

and

$$F(x) = \begin{cases} \dfrac{G(x)}{h(x)} & \text{if } x \in D \\ \emptyset & \text{if } x \in X \setminus D. \end{cases} \tag{3.5}$$

Remark 3.4. If $F : X \to \mathcal{P}(Y)$ is a set-valued ratio of affine functions w.r.t. D, then $\operatorname{dom} F = D$, since $\operatorname{dom} G = X$.

Example 3.1. Let $L : X \to Y$ be a linear operator and $M \subseteq Y$ a nonempty affine set. It is easy to see that the set-valued function $G : X \to \mathcal{P}(Y)$, defined by

$$G(x) = L(x) + M, \ \forall x \in X,$$

is affine and $\operatorname{dom} G = X$. Let $h : X \to \mathbb{R}$ be an affine function, other than the null functional. Consider any nonempty convex set $D \subseteq X$ satisfying (3.4), as for instance $D = \{x \in X \mid h(x) > 0\}$. Then, the set-valued function $F : X \to \mathcal{P}(Y)$, defined by (3.5) is a ratio of affine functions w.r.t. D.

Example 3.2. We have seen that Definition 1.1 was formulated within finite-dimensional Euclidean spaces. Naturally, it can be adapted to our general framework. Let $g : X \to Y$ be an affine vector-valued function. As in the previous example, let $h : X \to \mathbb{R}$ be an affine function, other than the null functional, and let $D \subseteq X$ be a nonempty convex set satisfying (3.4). Then, the function $f : D \to Y$, defined by

$$f(x) = \frac{g(x)}{h(x)}, \ \forall x \in D,$$

is a vector-valued ratio of affine functions, which can be identified, in view of Remark 2.1, with the set-valued function $F : X \to \mathcal{P}(Y)$ given by (2.2), which actually is a ratio of affine functions w.r.t. D of type (3.5), where the set-valued affine function $G : X \to Y$ is given by $G(x) = \{g(x)\}$ for all $x \in X$.

The following result is a generalization of property (P1) mentioned in the Introduction.

Theorem 3.1. *Let $D \subseteq X$ be a nonempty convex set. If $F : X \to \mathcal{P}(Y)$ is a set-valued ratio of affine functions w.r.t. D, then for any set $S \subseteq D$ we have*

$$\operatorname{conv} F(S) = F(\operatorname{conv} S). \tag{3.6}$$

Proof. Assuming that F is a ratio of affine functions w.r.t. D, we can choose $G : X \to \mathcal{P}(Y)$ and $h : X \to \mathbb{R}$ satisfying the conditions of Definition 3.3.

Consider an arbitrary set $S \subseteq D$. First we prove the inclusion

$$\operatorname{conv} F(S) \subseteq F(\operatorname{conv} S). \tag{3.7}$$

In order to do this, let $y \in \operatorname{conv} F(S)$. Then there exist $k \in \mathbb{N}$, $(t_1, \ldots, t_k) \in \Delta_{k-1}$ and $y_1, \ldots, y_k \in F(S)$ such that $y = \sum_{j=1}^{k} t_j y_j$. Since for every $j \in \{1, ..., k\}$ we have $y_j \in F(S)$, we can find $s_j \in S$ such that $y_j \in F(s_j)$. Taking into account that $s_1, \ldots, s_k \in S \subseteq D$, we infer by (3.5) that

$$y \in \sum_{j=1}^{k} t_j F(s_j) = \sum_{j=1}^{k} \frac{t_j}{h(s_j)} G(s_j). \tag{3.8}$$

Consider the numbers

$$u = \left(\sum_{j=1}^{k} \frac{t_j}{h(s_j)} \right)^{-1}$$

and

$$v_j = \frac{ut_j}{h(s_j)}, \ \forall\, j \in \{1, ..., k\}.$$

Define the point

$$x = \sum_{j=1}^{k} v_j s_j$$

and notice that $x \in \operatorname{conv} S$, since $(v_1, \ldots, v_k) \in \Delta_{k-1}$. By affinity of h, we deduce that

$$h(x) = h(\sum_{j=1}^{k} v_j s_j) = \sum_{j=1}^{k} v_j h(s_j) = \sum_{j=1}^{k} ut_j = u. \tag{3.9}$$

On the other hand, since G is affine, in view of Remark 2.3 we have

$$G(x) = G(\sum_{j=1}^{k} v_j s_j) = \sum_{j=1}^{k} v_j G(s_j) = \sum_{j=1}^{k} \frac{ut_j}{h(s_j)} G(s_j). \tag{3.10}$$

From (3.8) and (3.10) it follows that $uy \in G(x)$. By (3.9), we obtain

$$y \in \frac{G(x)}{h(x)} = F(x) \subseteq F(\operatorname{conv} S),$$

where the equality is due to (3.5) and the fact that $x \in \operatorname{conv} S \subseteq D$, by convexity of D. Thus (3.7) holds true.

Now we are going to prove the inclusion

$$F(\operatorname{conv} S) \subseteq \operatorname{conv} F(S). \tag{3.11}$$

To this aim, let $y \in F(\operatorname{conv} S)$. Then, we can choose a point $x \in \operatorname{conv} S$ such that $y \in F(x)$. Since $S \subseteq D$ and D is convex, we deduce that $x \in D$. Therefore, by (3.5) we get

$$y \in F(x) = \frac{G(x)}{h(x)}.$$

More precisely, there exist $k \in \mathbb{N}$, $(t_1, \ldots, t_k) \in \Delta_{k-1}$ and $s_1, \ldots, s_k \in S$ such that

$$x = \sum_{j=1}^{k} t_j s_j.$$

Denote $u = h(x)$ and notice that $u > 0$ by (3.4), since $x \in D$. From the above relations and the affinity of G, it follows that

$$y \in \frac{G(x)}{h(x)} = u^{-1} G(x) = u^{-1} G(\sum_{j=1}^{k} t_j s_j) = u^{-1} \sum_{j=1}^{k} t_j G(s_j). \tag{3.12}$$

Now, for any $j \in \{1, ..., k\}$, denote $v_j = u^{-1} t_j h(s_j)$. Since $s_1, \ldots, s_k \in S \subseteq D$, it follows by (3.4) and (3.5) that $(v_1, \ldots, v_k) \in \Delta_{k-1}$ and, respectively,

$$u^{-1} \sum_{j=1}^{k} t_j G(s_j) = \sum_{j=1}^{k} u^{-1} t_j G(s_j) = \sum_{j=1}^{k} v_j \frac{G(s_j)}{h(s_j)} = \sum_{j=1}^{k} v_j F(s_j) \subseteq \operatorname{conv} F(S). \tag{3.13}$$

By (3.13) and (3.12) we infer that $y \in \operatorname{conv} F(S)$, hence (3.11) is also true.

From relations (3.7) and (3.11) we conclude (3.6). □

As a consequence of the previous theorem, we establish now a generalization of property (P2) mentioned in the Introduction.

Corollary 3.1. *Let $D \subseteq X$ be a nonempty convex set. If $F : X \to \mathcal{P}(Y)$ is a set-valued ratio of affine functions w.r.t. D, then for any nonempty convex set $A \subseteq D$ the set $F(A)$ is convex.*

Proof. Let $A \subseteq D$ be a convex set. Since F is a ratio of affine functions, by Theorem 3.1 it follows that $\operatorname{conv} F(A) = F(\operatorname{conv} A) = F(A)$, hence the conclusion is true. □

We conclude the paper by illustrating how the classical results by Rothblum [10], mentioned in the Introduction, can be recovered from our general approach.

Example 3.3. Let $D \subseteq \mathbb{R}^n$ be a nonempty convex set. According to Rothblum [10], a function $f : D \to \mathbb{R}^m$ is a ratio of affine functions if there exist a vector-valued function $g = (g_1, \ldots, g_m) : D \to \mathbb{R}^m$ and a real-valued function $h : D \to \mathbb{R}$, which are affine on D in the sense that for any $k \in \mathbb{N}$, $x_1, ..., x_k \in D$ and $t_1, ..., t_k \in [0, +\infty)$ with $t_1 + ... + t_k = 1$, we have

$$g(t_1 x_1 + ... + t_k x_k) = t_1 g(x_1) + ... + t_k g(x_k);$$
$$h(t_1 x_1 + ... + t_k x_k) = t_1 h(x_1) + ... + t_k h(x_k).$$

Notice that, g and h are affine on D if and only if all real-valued functions $g_1, \ldots, g_m$ and h are both convex and concave on D. It is easy to see that every ratio of affine functions in the sense of Rothblum actually is a ratio of affine functions in the sense of Definition 1.1, where $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$. Therefore, in view of Example 3.2, f can be identified with a set-valued ratio of affine functions $F : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^m)$, and consequently, Theorem 3.1 and Corollary 3.1 generalize two classical results of Rothblum, namely Propositions 1 and 2 in [10], respectively.

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